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DR. PHIL. JULIUS WEISBACH,

OBERBERGRATH AND PROFESSOR AT THE ROYAL MINING ACADEMY AT FREIBERG;
MEMBER OF THE IMPERIAL ACADEMY OF SCIENCES
AT ST. PETERSBURG, ETC.

# A MANUAL
OF THE
# MECHANICS OF ENGINEERING
AND OF THE
# CONSTRUCTION OF MACHINES.

DESIGNED AS A TEXT-BOOK FOR TECHNICAL SCHOOLS AND COLLEGES,
AND FOR THE USE OF ENGINEERS, ARCHITECTS, ETC.

BY

DR. PHIL. JULIUS WEISBACH,

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*IN THREE VOLUMES.*

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## VOL. II.

SECTION II.—APPLICATION OF MECHANICS TO MACHINES.

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## Part I.—HYDRAULICS AND HYDRAULIC MOTORS.

WITH 380 WOOD-CUTS IN THE TEXT.

*TRANSLATED FROM THE FOURTH AUGMENTED AND IMPROVED
GERMAN EDITION BY*

A. JAY DU BOIS, PH.D.,

*Higgin Professor of Dynamic Engineering in the Sheffield Scientific School of Yale College.
Late Professor of Civil and Mechanical Engineering in Lehigh University,
Bethlehem, Pa.. Author of Elements of Graphical Statics, Etc.*

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AUTHORIZED TRANSLATION.

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NEW YORK:
JOHN WILEY & SONS, PUBLISHERS,
15 ASTOR PLACE.
1878.

**JOHN S. PRELL**
*Civil & Mechanical Engineer*
SAN FRANCISCO, CAL.

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S. W. GREEN,
PRINTER AND ELECTROTYPER,
18 Jacob Street,
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The present translation of the 2d. Vol. of my late father's „Lehrbuch der Ing. u. Masch. Mechanik" has been made bei Prof. Du Bois with my full approval, consent and authorisation.

Dr. Albin Weisbach
Prof. of Mineralogy.

Freiberg,
8 October 1877

# TRANSLATOR'S PREFACE.

THE story of the gradual development of the German original of the present work is told in the prefaces of the author to the successive editions through which it has passed. The number of these editions shows sufficiently the favor with which the work has been received at home, and the prefaces themselves point out the many changes, additions, and improvements, and indicate the untiring labor and energy which have brought it to its present state of completion.

The complete work consists of three volumes. The first volume contains an introduction to the Calculus, comprising all those mathematical processes afterwards made use of, thus rendering the work complete in itself, and is further divided into seven sections, treating respectively of, (1) Phoronomics, or the mathemathical theory of motion; (2) Mechanics, or the physical science of motion; (3) Statics of rigid bodies; (4) Application of Statics to the elasticity and strength of bodies; (5) Dynamics of rigid bodies; (6) Statics of Fluids; (7) Dynamics of Fluids—the whole treated with that thoroughness, simplicity, and clearness so characteristic of the distinguished author and teacher, and forming a fit introduction to the succeeding volumes of the course. This first volume has gone through three English editions. The first was issued in London, 1848, by Hippolyte Bailliere, translated by Prof. L. Gordon, of Glasgow College, a former student at Freiberg and pupil of the author, from the first German edition. The second was an American reprint of the same, by Lea & Blanchard, of Philadelphia, 1849, edited by Prof. Walter R. Johnson. The third was an entirely new translation from the fourth and last German edition, by Mr. Eckley B. Coxe, also an old pupil and friend of the author, published by D. Van Nostrand, New York, 1870. This translation was made with the approval of the author, and it was the original intention of the translator to complete the whole work. It contained about twice as much matter as the first English translations, and was in every other respect so enlarged and improved as to form substantially a new book. It is most deservedly well known to the American profession, and has passed through several editions.

A portion only of the *second* volume of the original work, of which the present volume is a first instalment, including only hydraulics and hydraulic motors, has likewise been through the two first mentioned English and American editions, and it may be said of it also that the present translation of this portion from the fourth and last German edition contains twice as much matter as those above mentioned, which were made from the first. The writer had for some time made liberal use of the original work in the preparation of his lectures, and for some time had looked forward to a translation of the second volume by Mr. Coxe, in continuation of the project announced by him. Such translation not appearing, the writer communicated with Mr. Coxe, learned that, owing to lack of time, he had been unable to complete the work, that he had therefore arranged with a professional friend to continue it, and that the latter had been prevented by sickness in his family from doing so. Having thus failed in this attempt, he was very glad to hear that the present translator was willing to undertake the work, as he considered it important that it should be published as soon as possible. Communicating with the professional friend referred to above, it was found that he also had definitely abandoned the project. The present instalment of the second volume is the result, and may therefore be considered as the continuation of Mr. Coxe's original plan, undertaken with his full approval and consent. The remainder of this second volume will soon appear.

The entire volume (1150 pages) treats of the application of the general mechanical principles set forth in the first, and is divided into two sections. The first of these treats of the applications of mechanics to structures of stability, such as bridge and roof trusses, arches and retaining walls. This section, of 250 pages, we have omitted, deeming it a subject already over-written, and in which many able works in English are easily procurable. With the second section our work therefore begins. This section treats of the application of mechanics to machinery, and is divided into two parts. Part I. discusses the various motive powers and their recipient machines, hydraulic and air motors. The above forms, therefore, the contents of the present volume. Part II. treats of heat, steam, and the steam-engine. This portion is now in print and will soon be issued. These two parts, therefore, will complete, with the exception of the first section above, the entire second volume of the original work. The translation has been made with care, all the calculations have been thoroughly revised, and the Prussian weights and measures reduced to English. All references to the first volume refer to Coxe's translation.

The third volume (1350 pages) completes the course. It treats of machines and machinery for the performance and transmission of work and motion, and of the component parts of such machines and machinery. It is divided into two sections. The first section discusses machinery of transmission, such as wheels, axles, chains, pulleys, cranks, eccentrics, screws, etc., and the proper proportions of these parts. The second section discusses machines for the raising of loads, for the transportation of loads, for the raising of water, for the compression or expansion of air, for changing the shape of bodies, etc. The principles for proportioning the various parts are deduced for each case, and the volume hence comprises the practical application of the general mechanical laws and principles set forth in the two preceding volumes.

The completion of the second part of the present volume, now in print, will thus, in connection with Coxe's translation of the first volume, render accessible in English the first two volumes of this valuable work. It is hoped that the third, and most valuable of all for the constructor and mechanic, will soon follow. Many may be inclined to consider the discussion of hydraulic motors given in the present volume as too voluminous. It need hardly be said, however, that the work need not be read through in sequence by any class. The intelligent teacher, by judicious selection, can give his class a course of such length and thoroughness as he considers desirable, and as the time at disposal will allow, and the work itself will then serve as a valuable book of reference and consultation long after it has ceased to be used as a text-book. The books we have studied and mastered in school and college are the ones to which we afterwards, in practice, most readily refer for advice and assistance. It is surely then no objection that such books should give a complete view of the whole ground, rather than a short and meagre presentation, which can have but a transitory value in merely introducing the student to the subject, and which often lead him to suppose that there is but little more of it worth knowing. This is almost the sole objection which can be brought against Weisbach's work—viz.: its fulness. As to the rest, it will be found the work of a skilful and experienced teacher, written expressly for young students, clear, simple, and thorough, and especially adapted for class instruction, while at the same time it forms a standard book of reference for the mechanic and constructor. The lack of any such book upon the subjects treated of, in English, encourages us to believe that in the present translation we are supplying a real need and performing a real service to the cause of technical education in this country, and we indulge the hope that the present second volume will be found worthy of the favor so justly accorded to the first.

We have added an introduction, which, we trust, will not be without interest to the student, and have incorporated in it those articles of the first volume to which most frequent reference is made in the text. The present volume is thus rendered complete in itself. Articles 269 to 297 were translated by Mr. Charles W. Kettell, of Charlestown, Mass. As his translation was of such character that a careful revision could find nothing to alter or to add, we take pleasure in thus acknowledging his assistance. We are also indebted to Mr. Morris B. Belknap, of the Sheffield Scientific School, for much valuable assistance in the correction of the proofs.

A. JAY DU BOIS.

SHEFFIELD SCIENTIFIC SCHOOL OF YALE COLLEGE,
October 21, 1877.

# PREFACE TO THE FIRST EDITION.

IN the preparation of this, the second volume of my "Mechanics of Engineering and of the Construction of Machines," I have adhered as closely as possible to the plan and views set forth in the preface to the first volume. I am well aware that these views are not held by all, and that many will especially deplore the absence in this work of a more general presentation and a more learned treatment, but, on the other hand, I feel assured that the elementary and popular method here pursued can be easily comprehended by those not possessed of very extended mathematical knowledge, and will therefore contribute to the more general acceptance and application of the science of mechanics, and thus increase its practical worth and value.

We find still very frequently, that practical men, in their constructions, ignore the application of scientific mechanics, and prefer to follow empirical methods; such men either lack sufficient confidence in the deductions of science, or they find the various treatises very unsatisfactory guides and advisers in the calculation and execution of their constructions. Let it only be recognized how much, in every respect, depends upon the execution of machines and structures in full accordance with all demands, and that this can only be accomplished by the correct application of correct and well-founded scientific conclusions, and the endeavor of the author to supply the above want cannot fail of recognition also.

Sound premises and simplicity are certainly the chief requisites for a work designed as a teacher and guide for the practical man. The lack of both is the main reason why the practical applications of mechanics have been thus far so little recognized. If, in the deduction of rules, unsafe or untenable assumptions are made, if the essential is not properly distinguished from that which is non-essential, that which is most important from that which is only subsidiary—if, finally, important relations and influences are disregarded, while those which are subordinate are too fully considered, then, no matter how correct the method of deduction, the rules thus deduced cannot possess practical value. Unfortunately, authors often fail in just these respects, and it is, therefore, little wonder if practical men often find theoretical rules unsafe, or at least unsatisfactory. Hence it is that such men often speak of an incorrect theory, while really referring only to its insufficient foundation and application. It is, moreover, by no means easy to deduce both practical and correct rules and formulæ for the use of the practical mechanic. For this is necessary, not only an exact knowledge of the nature of the subject which often is only to be obtained by special observations and experiments, but also special study, and even a special mental capability. The author, in the preparation of the present second volume, has aimed especially to deduce practically useful theories; he has at least endeavored, to the full extent of his ability, to give to the practical man a reliable guide and adviser, but acknowledges that his entire aim has not been perfectly attained.

The lack of simplicity, and the great generality of scientific treatment, as well as of the formulæ deduced, are also great hindrances to the more general practical acceptance of mechanics. It is by no means rare to find professional men of mathematical attainments disregarding, for these reasons, the aid of scientific mechanics, while those whose mathematical knowledge is less extensive, or who are less expert in its use, reject such aid entirely. To win for scientific mechanics a wider practical acceptance, it is therefore necessary to popularize its methods and simplify its rules. Thus, for example, to give, in place of involved general formulæ, several simpler special ones, or still further simplified approximate formulæ, to attain simplicity by the use of coefficients, etc.

The present second volume of my mechanics comprises two sections, the first of which treats of structures of stability, and the second of machines, containing especially the theory and description of the so-called prime movers. Many will perhaps find the first section too short, and the second too long. With reference to the first, I must admit my regret at not having gone deeper into the theory of wood and stone bridges, since these subjects, in view of many railroad projects, are at present of special importance. As to the second section, however, I consider that only in very few articles is a greater brevity possible without impairing the value of the work. It may be that many readers will find the chapter upon water-pressure engines too extended, in view of the limited application of such machines in mining. In the preparation of this chapter, I have had in mind that here was a vacant space in literature to be filled up, since all treatises upon mechanics contain little or nothing concerning these machines; I entertained also the hope of at the same time being of service to mining engineering. The chapter upon turbines may also perhaps be regarded by some as too extended, especially as it contains a monograph upon the older pressure and impact wheels. It seemed to me, nevertheless, that any omissions or contractions would be inadvisable, for the reason that, in estimating the value of a perfect machine, it is necessary to be familiar with the theory as well as with the defects of other similar but less perfect ones. The imperfect machines also will always be used to some extent, inasmuch as there are often places and circumstances where economy of power is of much less account than cheapness of construction. In the chapter upon the measurement of motive powers, etc., I should perhaps have treated at greater length the dynamometer, and this I should certainly have done had Morin's "Leçons de Mécanique Pratique" come into my hands sooner. The preparation of the second section has cost me the greatest pains, especially the chapter upon the steam-engine, and I still fear that this chapter will not fully answer the requirements of the reader. I might perhaps have shortened the chapter upon heat considerably, or even entirely omitted it, since the subject belongs more especially to the domain of physics. Considering, however, that I have given only that which is of direct importance and application to machines, and especially to the steam-engine, it would seem that the subject finds, not without right, a place in these pages. In the preparation of the chapter upon the steam-engine, I have made use of the Poncelet-Morin coefficient theory, as well as of the later theory of Pambour. At the same time, I have incorporated my own labors, and hope that this chapter will not be considered as a mere compilation only.

The results of my hydraulic experiments have been of great service in the preparation of this work. By the aid of the coefficients of resistance thus determined, I have been able to estimate the losses of work due to the hydraulic resistances in turbines, water-pressure engines, and steam-engines. I can assert that the development of useful theories of these machines has thus been materially assisted.

It only remains to me to announce to the reader that the entire work is not, as was originally intended, brought to a close with this second volume, and that I purpose adding still another volume. Indeed, I committed an error in underestimating the extent of material at disposal. As soon, however, as I had fairly entered upon the work, and had received from so many sides proofs of approval, there remained nothing for it but to proceed, and either abridge the plan of the work or to extend its limits. The first I could not resolve to do, because the still remaining subjects—viz., machinery of transmission, etc.—have been treated thus far very imperfectly, and a complete work is still lacking. I hope, then, in the third volume to supply a need.

In the press revision, I have been greatly assisted by Profs. Bornemann and Röting, and the accuracy of the book is largely due to them, for which I wish here to express my thanks.

JULIUS WEISBACH.

FREIBERG, December 1, 1847.

# PREFACE TO THE SECOND EDITION.

THIS second edition of the second volume of my "Mechanics of Engineering and Machine Construction" contains many emendations and improvements. In the first section, the statics of structures, the subject of bridges is more thoroughly treated than in the first edition, and tubular bridges also find a place. In the chapter upon vertical water-wheels, especially the overshot water-wheel, many enlargements and improvements have been made, and also the chapter upon reaction-wheels and turbines has been enlarged, and now contains the results of the latest experiments upon these wheels. Finally, the theory of heat and steam has undergone complete revision, and the latest experiments of Regnault (" Mémoire de l'académie royale des sciences de l'institut de France," T. XXI.) have been made use of. The value of the work has been enhanced by the addition of good illustrations of the Gölzschthal bridge and the Britannia bridge, as well as of a tangential wheel, a Sims steam-engine, etc. As to the rest, this second edition corresponds to the first, both as a whole and in its method of treatment.

JULIUS WEISBACH.

FREIBERG, May 24, 1851.

# PREFACE TO THE THIRD EDITION.

In the present edition also, all necessary corrections and improvements have been made; the work has been brought up to the present stand-point of science; much that was out of date and superfluous has been omitted and much entirely rewritten. The preparation of this edition has thus been one of much labor and care, and if still I have not completely met the needs of my public, I would remind the reader that the choice, arrangement, and preparation of the most important subjects from the department of practical mechanics, a department now hardly to be surveyed, were laborious, wearisome, and tedious in the extreme.

In the first section, comprising the statics of structures, the theory of earth pressure, as well as of the arch, has received considerable additions, and the theory of structures of wood and iron has been for the most part entirely rewritten. In the second section, which comprises the mechanics of machines, the chapter upon the dynamometer will be found much fuller, the chapter upon vertical water-wheels is also in part rewritten, and that upon horizontal water-wheels completed by the description and theoretical discussion of more recent turbines. The chapter upon water-pressure engines has also been increased in value by the description and discussion of new machines. Essential changes and improvements will also be found in the chapters upon the steam-engine, which I should have treated at still greater length had the space at disposal permitted. The new theory of heat and its application to the steam-engine has not been especially noticed, as it has not yet arrived at that degree of development which would enable the theory of the steam-engine to be based upon it with safety and advantage.

The illustrations of this edition are in great part drawn and engraved anew. Their quality and accuracy leave little to be desired.

JULIUS WEISBACH.

FREIBERG, April 24, 1859.

# PREFACE TO THE FOURTH EDITION.

THE fourth edition of the second volume of my "Mechanics of Engineering and Machine Construction," which I now offer to the public, does not indeed differ from the preceding in its plan and arrangement, but will be found superior not only in its appearance, but in its additions, corrections, and enlargements. As to the appearance of this new edition, the illustrations are new and black on a white ground, instead of white on a black ground as formerly. With reference to the contents, I would call attention to the following. The present second volume consists, as before, of two sections—the one comprising the statics of structures, the other the mechanics of machines. In both sections, the literature has of late years notably increased, especially in the chapters upon structures of wood and iron, and in those upon heat, steam, steam-boilers, and steam-engines. If, in the preparation of this new edition, I have not made as much use of this literature as many may have wished, the reason is that I consider it best, in an elementary work such as this, to give only such theories as have already found universal acceptance, have been used in investigations in many directions, or have proved themselves in practice reliable and satisfactory. It is indeed often better to relegate new theoretical views and doctrines at first to journals and monographs, and only to adopt them in technical text-books when they have been approved by practice. As in oral, so also in written teaching, the endeavor should be to attract the hearer or reader by simplicity and brevity, and not by discursive, wearisome, and unpractical speculations, to impair the desire for study of a science most important in practical life. For these reasons, I have retained, in the determination of earth pressure and thrust of arch, the old theory of Coulomb, and only made use, in structures of wood and iron, of approved new advances in the theory of elasticity, and of the theories of a few new bridge systems, such as Charnier's, Pauli's, etc. In the mechanics of machines, I have given a short theory of Schönemann's horizontal dynamometer, as well as the theory of the curve of backwater and of the determination of the pressure line in pipe systems, given by me thirty years ago. In the two chapters upon hydraulic motors will be found the jet turbine of the author, the turbines of Hänel and of Schiele, and the water-pressure engine at the "Grube Centrum." I have made no use of the new treatise of Pambour upon the theory of water-wheels, published in the *Comptes rendus*, because it contains nothing essentially new. In the chapters upon heat and steam will only be found numerous emendations, additions, and improvements, as an entire reworking of them, according to the mechanical theory of heat, does not yet seem justified. In the calculation of the theoretical delivery of the steam-engine, not only the old formulæ of Poncelet, Morin, and Pambour are made use of, but also the approximate formulæ of the mechanical heat theory given by Rankine and Zeuner. I have, moreover, given an outline of the mechanical heat theory, according to Zeuner, and its application to the steam-engine (§ 484 to § 487), as also, in conclusion, a comparison of the steam-engine with caloric and gas engines.

JULIUS WEISBACH.

FREIBERG, November, 1868.

JOHN S. PRELL
Civil & Mechanical Engineer.
SAN FRANCISCO, CAL.

# INTRODUCTION.*

## FUNDAMENTAL IDEAS AND DEFINITIONS.

BEFORE taking up the subject proper of the present treatise, we shall devote a few pages to the explanation of those mechanical principles, terms, and ideas, of which we shall make constant use in what follows, and which it is therefore necessary that the student should clearly understand. Of course, before taking up the subject of hydraulic motors, he should already have had, and it is here presumed that he has had, a thorough preparatory course, and is already conversant with the principles of the mechanics of motion, with hydrostatics and hydrodynamics. A short review, however, of the more important ideas and principles of which he is now about to make special application, will not be without advantage, especially as, in the opinion of the writer, some of these ideas and principles are not presented in the current text-books with sufficient clearness and in sufficient detail. We therefore commend to the student a careful perusal of the following preliminary remarks.

**1.—Work, Effective Delivery, etc.**—A machine may be defined as a body or collection of bodies or "pieces," which receive at certain points certain forces, and exert at other points certain other forces, which generally differ from the first in their intensity, direction, and the velocities of their points of application.

The point or points at which the original or "motive" forces are applied are the "receiving points," while those points to which, through the intervention of the machine, these forces are transmitted, in whole or in part, are the "working points."

*Work* is pressure into distance. If a machine is perfect, or completely utilizes the work of the motive forces, then the work performed at the working point is exactly equal to that performed at the receiving point. Thus if the intensity of the pressure at this latter point is greater than that at the receiving point, then, in order that the equality of work may be preserved, its velocity, or the distance passed through in the same time, must be proportionally less. Hence the theorem so familiar to all mechanics, that "what is gained in pressure is lost in speed." Inversely, of course, what is gained in speed is lost in pressure. This is nothing more than the statement in ordinary language of the well-known principle of "virtual velocities."

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* By the translator.

In a perfect machine, the product then of the pressure into the distance through which the pressure acts must always be the same, both for the point of application and for the working point. The office and object of a machine, therefore, is to transmit work from one point to another, and, it need scarcely be added, with as little loss of work on the way as may be. The object of such transmission is to produce certain desired results, and while thus transmitting as perfectly as possible the original work imparted, the machine must so change the direction, intensities, or velocities of the motive forces as may be necessary for the attainment of the end in view; as, for instance, the conversion of continuous rotary motion into alternating parallel motion, etc. Upon the completeness with which the machine transmits the work of the original forces, depends its "efficiency." If it transmits all the work without loss, the machine is perfect. This in practice is never the case. During transmission, a certain amount of work is always consumed in overcoming the friction of the pieces composing the machine, as well as the influence of other so-called "prejudicial resistances," and hence in practice the work actually transmitted to the working point is always less than the work of the applied forces. The "efficiency" of a machine is then measured by the ratio of the work actually "delivered" or performed at the working point to the total "available" work, or the work performed by the motive forces. This ratio is therefore in practice a fraction always less than unity, except for a theoretically perfect machine, for which it equals unity.

The work performed at the working point of a machine is called its "effective delivery." That performed at the receiving point by the motive forces is the "available work." The ratio of the first to the second gives them the measure of the "efficiency" of the machine. The work performed at the working point is also called the useful work—*i.e.*, the work actually transmitted and used for the attainment of the desired result, and which is therefore utilized. The work performed by the motive forces at the receiving point is the total work placed at disposal for the attainment of this result, of which some is always lost, and of which the machine should utilize as much as possible.

**2.—Unit of Work—Rate of Work—Horse-Power.**—Work, as already defined, is pressure overcome, or resistance, through distance. The unit of pressure is, as is well known, one pound. The unit of distance is one foot. Hence, the work performed in lifting one pound through a distance of one foot, is the unit by means of which different quantities of work are measured. This unit is called a "*foot-pound*." Work, by its definition, is, then, independent of time; the work implied in lifting one pound through a distance of one foot, is the same, whether it take place in one second or twelve hours, or in any other length of time. Thus the work done in lifting a weight of twenty pounds through a height of one hundred feet, is $20 \times 100 = 2000$ ft. lbs., or is the same as one pound raised 2000 ft., whatever may be the time in which the work is performed. Resistance or force in general is measured in pounds pressure, and hence the work performed by a force or expended in overcoming a resistance is equal to the intensity of the force or resistance measured in pounds, multiplied by the distance in feet through which the force moves or through which the resistance is overcome. The product is always, then, given in foot-pounds, and represents the number of pounds which must be raised one foot or the number of feet through which one pound must be raised, in order that the work in either case may be equal to the work actually performed.

In comparing the work of different machines, it is very desirable to know the work performed by each per second. This is called "rate of work," and is evidently equal to the resistance overcome in pounds, into the distance passed through by the working point in one second—*i. e.*, to the product of the resistance and *velocity*. The rate of work may then be expressed in units of work (as foot-pounds) per second, per minute or per hour, as the case may be. There is a peculiar and entirely conventional unit of universal use, called a "HORSE-POWER," of frequent use in the estimation of the work performed by a machine. This unit is equivalent to 550 ft. lbs. per second, or 33,000 ft. lbs. per minute, or 1,980,000 ft. lbs. per hour. It serves as a convenient unit to measure the rate of work or the "power" of different machines. Thus to say that a machine has 10 horse-power is to say that it performs 5500 ft. lbs. of work every second—*i. e.*, is able in every second to raise 5500 lbs. through a distance of one foot, or one pound through 5500 ft.

**3.—Inertia—Momentum—Mass.***—"It is a fact familiar to every one, that a body cannot be moved from rest without the application of force, and also that a body already in motion cannot be stopped without the exertion of force. This property of bodies, by virtue of which they resist any change of state, either of rest or motion, is called *inertia.*"

The *mass* of a body depends upon the amount of matter it contains. We assume as the unit by which to measure the mass of a body, a certain quantity of matter, such that if originally at rest and acted upon by a force or pressure of one pound for one second, it will have imparted to it a velocity of one foot per second. The unit assumed for mass, then, is that quantity of matter which, acted upon by one unit of force during one unit of time, will have one unit of velocity. What the *weight* of this unit of mass is we shall presently see.

Now, it is clear that if a force equal to one pound pressure acting for one second upon a certain unit of mass, originally at rest, imparts to this unit of mass a velocity of one foot per second, then a force of two pounds pressure will impart the same velocity in the same time to two such units ; three pounds pressure will impart the same velocity in the same time to three such units, and so on. Also, it is clear that two pounds pressure (instead of one) will impart to the same unit of mass in the same time a double velocity ; three pounds pressure will then impart to the unit of mass in one second a velocity of three feet per second, and so on. If, now, a certain pressure acts upon a body having a mass of say five units and imparts in one second a velocity of one foot per second, we know that the intensity of that pressure must be five pounds, or five times that requisite to give in the same time the same velocity to one unit of mass. If the pressure, however, gave to the five units of mass in one second a velocity of six feet a second, it must be six times as great as that which in the same time gives to the same five units a velocity of one foot per second, or thirty pounds. In general, *the number of pounds constant pressure which can impart to $M$ units of mass in one second a velocity of $V$ feet per second, is equal to the product $M\ V$.*

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* The following views upon momentum are taken from a series of articles in Van Nostrand's Magazine for September and October, 1877, by J. J. Skinner, Instructor in Mathematics in the Sheffield Scientific School. The argument of Mr. Skinner is closely followed, sometimes in his own words, and the conclusions arrived at are substantially his. In Van Nostrand's Magazine for May, 1877, et seq., will be found interesting discussions upon this subject.

This product of the mass by the velocity is a measure of the *momentum* of the body. The product $M\ V$ gives then the number of pounds constant pressure which, acting on the body of $M$ units of mass, originally at rest, will give to it *in one second* a velocity $V$. If, then, any body having $M$ units of mass is moving with a velocity of $V$ feet per second, then, no matter how long it may have been moving, or how intense the force which originally set it in motion, we know that the same velocity would have been given to it *in one second* by a constant pressure of $M\ V$ pounds ; and, inversely, we know that a constant opposing pressure of $M\ V$ pounds will bring that body to rest in just one second. This, then, is what the product $M\ V$ or the "momentum" of a body really means.

*The product $M V$, or the measure of the momentum of a body of mass $M$, moving with a velocity $V$, gives then the number of pounds constant pressure which will bring the body to rest* IN ONE SECOND.

Momentum then is neither motion nor is it "quantity of motion," as defined in nearly every text-book. It is simply pressure exerted under certain conditions. Being a pressure, it will, if multiplied by the distance passed through in coming to rest in one second, give the *work* performed in thus coming to rest. It is, therefore, no measure in any sense of "motion," but if a number of bodies all pass through equal distances while gradually coming to rest in one second, then momentum is proportional to the rate of work in each case, and may then be considered as a measure of the relative "power" of the moving bodies, or their respective capabilities of work in equal times. It does not give the work in each case, but simply varies as the work. The same of course holds true for bodies starting from rest and attaining in one second certain velocities. Momentum may then, under certain arbitrary restrictions, be taken as a measure of "power," or rate of work, but never of motion. The force necessary to destroy or produce in certain bodies in a specified time certain velocities, does not measure and is not proportional to those velocities, unless the masses are equal, and in this case the measure of "quantity of motion" is *velocity*, and nothing more. The measure of "quantity of motion" can never properly be any thing else than feet per second. But the *power* or capability of performing work, of different bodies having different velocities, may be relatively compared by their momenta, provided these velocities are supposed generated from rest, or destroyed continuously, *in just one second.* Without this arbitrary limitation of time, momentum means nothing and measures nothing. For the actual force of which momentum is a true measure, see Art. 7.

**4.—Weight of the Unit of Mass—Acceleration of Gravity—Value of $g$.**—We have defined as our unit of mass that quantity of matter which, acted upon for one second by a constant pressure of one pound, will have a velocity of one foot per second—*i.e.*, *the amount of matter which, acted upon by the unit of force during the unit of time, will have the unit of velocity.* Evidently this quantity of matter must possess a certain weight. Let us see what this weight is.

If a quantity of matter which weighs one pound fall freely in a vacuum, we have a constant pressure of one pound acting upon a certain definite quantity of matter (for the same latitude), viz., the amount of matter which at the place of fall weighs one pound. The velocity attained at the end of one second is found by experiment to vary slightly with the latitude, is constant at the same latitudes, and is universally denoted by the letter $g$. For the latitude of London, the value of $g$ is 32·1912 ft. per second, or very nearly 32·2 ft. This value of $g$ is that ordinarily assumed in mechanical calculations. But, if a pressure of one pound acting for one second upon the quantity of matter which weighs a pound, will give it a velocity of $g$ ft. per second, then a pressure of one pound acting for one second upon a quantity of matter which weighs $g$ *pounds* would produce in *this* quantity of matter a velocity only $\frac{1}{g}$ part as much, or one foot per second. But we have already assumed, as stated above, this quantity of matter as the unit of mass. *The weight of one unit of mass is then $g$ pounds.* If now we have any body whose weight is $G$ pounds, we may reason as follows: The weight of *one* unit of mass is $g$ pounds; in a body weighing $G$ pounds then there will be as many units of mass as $g$ is contained in $G$, or $\frac{G}{g}$ units of mass. We have then generally $M = \frac{G}{g}$, or the mass of a body is equal to its weight divided by the acceleration of gravity. The acceleration of gravity, as already remarked, varies with the latitude, directly as the force of gravity or as the weight of the body itself varies. The quotient $\frac{G}{g}$, or the mass $M$, is then a constant for all latitudes. If a body which registers one pound upon a spring balance at the equator were taken to either pole, it would register upon the same balance somewhat more than a pound. But at the pole the acceleration of gravity $g$ would also be found increased by a corresponding amount, and thus $\frac{G}{g}$ would still be the same. The value of $g$ for any place is given by the formula

$$g = 32 \cdot 1724 \left[1 - 0 \cdot 00256 \cos. 2\phi\right] \quad \text{(Ganot's Physics)}$$

where $\phi$ is the latitude of the place in question. For the weight of a body, then, we have $G = Mg$, or

$$G = 32 \cdot 1724\, M \left[1 - 0 \cdot 00256 \cos. 2\phi\right].$$

**5.—"Vis Viva" or Living Force of a Body.**—"It is found by experiment that the velocity acquired by a body, when acted on by a constant force, is proportional to the time during which the force acts. Thus, since a pressure of one pound acting upon one unit of mass during one second gives it a velocity of one foot per second, the same pressure of one pound, if it acts for say six seconds, will impart to the unit of mass a velocity of six feet per second. Now, from the laws of uniformly accelerated motion, we know that if a body start from rest and move under the action of a constant force, the distance passed over in any time is half the distance which would have been passed over in the same time with a uniform velocity equal to the final velocity—*i. e.*, a mean between the initial and final velocities. In the case above, for instance, the time is six seconds, and the velocity at the end of that time is six feet per second. Then the distance actually passed over by the unit of mass, in acquiring the velocity of six feet per second from the action of a constant pressure of one pound acting during six seconds, is 18 ft. A pressure of one pound has then been exerted through a distance of 18 ft., or a work of 18 ft. lbs. has been performed. If the same velocity be imparted to the same unit of mass by a constant pressure of *two* pounds, the time requisite will only be one half as great, or three seconds. The distance passed through is then only nine feet, and the foot-pounds of work performed is 2 × 9, or 18, as before. If the pressure is three pounds, the time required is only two seconds, and the distance passed over is only six feet, but the work performed is as before, 3 × 6, or 18 ft. lbs. In general, if $V$ be the final velocity in feet per second at the end of $T$ seconds, then the distance passed over will be $\frac{TV}{2}$ ft. For one unit of mass, we see, from the above, that for one pound pressure the number of units in the time is equal to the number of units in the final velocity, or $T = V$, while for $P$ pounds pressure we have $T = \frac{V}{P}$. Substituting this value of $T$, we have the distance passed over by one unit of mass, while acquiring any velocity $V$, by the action of a constant pressure $P$, equal to $\frac{V^2}{2P}$. But the work is equal to the pressure into the distance through which it moves. Hence, if $W$ represent the number of units of work, we have $W = \frac{1}{2} V^2$.

"This is the work performed in giving the velocity $V$ to *one* unit of mass. To give the same velocity to $M$ units of mass would require a work of

$W = \frac{M V^2}{2}$. This product, $\frac{1}{2} M V^2$, or the product of half the mass into the square of the velocity, was named by Leibnitz, *vis viva* or *living force*. It represents the work which a body moving with the velocity $V$ must perform before it comes to rest. It is, therefore, the power of a moving body of performing work or giving motion to other bodies, which power is exhausted or may be said to *die* at the instant the body comes to rest." In order to find the number of units of work performed by any constant pressure while increasing the velocity of a mass $M$, already having the velocity $V$, from this velocity to the velocity $V'$, we have simply to observe that the work required to give it the velocity $V'$, starting from rest, is $\frac{1}{2} M V'^2$, and the work required to give it the velocity $V$, starting from rest, is $\frac{1}{2} M V^2$; hence, the work performed while increasing the velocity from $V$ to $V'$ is $\frac{1}{2} M [V'^2 - V^2]$.

"Moreover, the work performed in giving the mass $M$, starting from rest, any velocity $V$, will be equal to $\frac{1}{2} M V^2$, whether the impelling pressure be constant or variable. For if it be variable, we may suppose the time divided into very small periods, during each of which the pressure may be regarded as constant. Then if $V'$, $V''$, $V'''$, etc., be the velocities at the end of the first, second, and third instants, we shall have for the several works, $\frac{1}{2} M V'^2$; $\frac{1}{2} M [V''^2 - V'^2]$; $\frac{1}{2} M [V'''^2 - V''^2]$; and so on. Now, the sum of all these elementary quantities of work is the total work performed, or

$$W = \tfrac{1}{2} M V'^2 + \tfrac{1}{2} M [V''^2 - V'^2] + \tfrac{1}{2} M [V'''^2 - V''^2] + \ldots.$$

If $V$ be the final velocity, the sum of this series will evidently be $\frac{1}{2} M V^2$, since each of the other positive terms will be cancelled by an equal negative term."

We have, therefore, the following universal principle: *If any mass $M$ be moved from rest and acquire any velocity $V$, then, whether the impelling force be constant or variable, no matter what its intensity or what the direction through which it acts, or what the time employed—the number of units of work performed in giving to the mass that velocity is always the same, and is equal to $\frac{1}{2} M V^2$.*

Inversely, *$\frac{1}{2} M V^2$ is the work which a mass $M$ moving with a velocity $V$ will exert upon another body before coming to rest.* Unlike momentum, then (Art. 3), *vis viva*, or living force, measures the power of a moving body to perform work independently of any assumptions as to time or as to the nature of the impelling or resisting pressure, while momentum is not that work, but is merely proportional to it under certain arbitrary assumptions, both as to the nature of the force and its duration. Momentum may be used in comparison of rates of work. It is not a measure but a means of comparison. *Vis viva* gives the work and is an accurate measure.

**6.—Height Due to Velocity.**—We see now, from the preceding, that momentum ($M V$) is the constant pressure which, acting upon a moving body, will bring it to rest *in one second*, while *vis viva* ($\frac{1}{2} M V^2$) is the work performed by the body in thus coming to rest, *no matter what the time employed*.

or whether the opposing resistance be constant or variable. The unit of momentum is one pound, that of *vis viva* is one *foot-pound.* The unit of *mass* is such a quantity of matter as weighs $g$ pounds, $g$ being determined for the place where the body is. Accepting these views, the results at which we arrive are always consistent, and the confusion which too generally prevails upon this point is avoided.

Thus, if a body whose mass is 15 units (and weight accordingly 483 lbs.), is moving with a velocity of 10 ft. per second, and is brought to rest in one second by a constant force, we know that the space passed through will be five feet. The constant force which brings it in one second to rest is $M\,V = 10 \times 15 = 150$ lbs. pressure. The work which the mass performs while thus coming to rest is then $5 \times 150 = 750$ ft. lbs., which again is equal to the *vis viva*, or $\frac{1}{2}\,15 \times 100 = 750$ ft. lbs.

Now, we can put $\frac{1}{2} M\,V^2 = G\frac{V^2}{2g}$, since $G = Mg$ or $M = \frac{G}{g}$. Since now we know that $\frac{1}{2} M\,V^2$ is the work performed, if we suppose the constant moving force acting upon the mass $M$ to be that of gravity, $G$ will be the pressure, and since pressure into distance is work, $\frac{V^2}{2g}$ must be the corresponding distance. *The height, then, through which a body, starting from rest, must fall in order to acquire a velocity of $V$ is $\frac{V^2}{2g}$.* This quotient $\frac{V^2}{2g}$, or the square of the velocity divided by two times the acceleration of gravity, is called " the height due to the velocity."

If, then, in our example above, the body were to fall, starting from rest, through a distance of $\frac{100}{64 \cdot 4} = 1 \cdot 553$ ft., it would attain the velocity of 10 ft. The work performed upon it by the force of gravity, then, in giving it that velocity, is $483 \times 1 \cdot 553 = 750$ ft. lbs., as before.

If, then, a body whose weight is known is moving through space with a velocity $V$, we can at once find the work which that body is capable of doing, or the work which it will perform in coming to rest, by multiplying the weight by the height $\frac{V^2}{2g}$ through which the weight or the body would have to fall, starting from rest, in order to acquire the given velocity. If a body start from rest, then, and if the constant pressure acting upon it is its own weight, the distance through which it moves in acquiring the velocity $V$ is $\frac{V^2}{2g}$; and inversely, if a body be moving through space with a velocity $V$, then a constant opposing pressure equal to its own weight will bring the body to rest in the distance $\frac{V^2}{2g}$. The work which the body will perform in thus coming to rest is $G\frac{V^2}{2g} = \frac{1}{2} M\,V^2$, and this is the amount of work which must have been originally expended in giving to the body starting from rest the velocity $V$. If, however, the constant opposing pressure be $M\,V$ pounds instead of the weight or $G$ pounds, then the body will come to rest in *exactly* *one second*, and the distance through which it will pass in thus coming to rest will be $\frac{V}{2}$ feet. In either case, we have for the work performed $G\frac{V^2}{2g}$ or $MV\frac{V}{2}$, and since $M=\frac{G}{g}$, the work is in each case precisely the same—as should be.

**7.—Inertia—Work of Inertia.**—A force in general may be defined as that which changes or tends to change the state of a body or mass, whether in respect to rest or motion. There can be no action of a force without an equal and contrary reaction. This is a law of nature, and our knowledge of it comes from experience. If a force act upon a body retained by a fixed obstacle, the latter will oppose an equal and contrary resistance. If it act upon a free body, the latter will change its state, and in the act of doing so, its *inertia* will oppose an equal and contrary resistance. *Action and reaction are ever equal, contrary, and simultaneous.*

We can only measure a force by its effect—*i. e.*, by the pressure or tension, attraction or repulsion, which it causes. The measure of a force is then always the number of pounds' pressure or tension which it exerts. If a body is free to move, then the effect of the force, or the pressure or tension it exerts, will cause the body to move with a certain velocity; if the body be always the same, and the duration of action of the force the same, then the *velocity* in such case may be taken as a measure of the force, since it will vary directly as the pressure or tension itself. Thus, if we take the unit of mass or $g$ pounds as the body, and suppose the force to act *during one second* upon this unit of mass, the velocity, or number of feet moved in one second, will be equal, numerically, to the number of pounds' pressure exerted by the force which causes the motion. This velocity in such case then directly measures the force, the force necessary to give in one second one unit of velocity to one unit of mass, or *one pound*, being the unit of measurement in this case also. A force which will give to a mass $M$ this velocity in the same time will then be $M$ times as great, and if in the same time it gives to the mass $M$ a velocity $V$, it will be $MV$ times the unit.

The unit for the measurement of force is therefore always one pound. *Work* is simply force or pressure exerted through distance, without reference to time, and its unit of measurement is then one pound raised one foot, or one "foot-pound." Machines are compared by means of the work done by each in a unit of time—*i. e.*, by their *rate of work*. This is called the *power* of the machine. "Power," then, is work performed per second, and the unit for its measurement is one foot-pound per second. Thus, *time* is an essential element in the estimation of power—work is independent of time and dependent simply on force and distance, while force is always to be measured in pounds.

Since, now, action and reaction are equal, the resistance of the body $M$ to motion must be equal to that force which is necessary to produce the motion. The product $MV$, then, or the so-called "momentum," is really the *inertia* of the body. As the space passed through in acquiring the motion $V$ is $\frac{V}{2}$, the product $\frac{1}{2} M V^2$ *is the work of inertia.* This we have already seen to be equal to the work of the force producing motion—*i.e.*, the work of the force is equal to that of the resistance. We see, then, that inertia is a passive force, not originally existing, but called forth by the active or outer forces, and that it varies for the same mass directly as the velocity, or since $V = \frac{\text{space}}{\text{time}}$, directly as the space and inversely as the time. Thus the resistance of a body or its inertia is greater or less according as the velocity is imparted in a shorter or longer time.

We should do well, then, to discard the term "momentum," and replace it by *inertia*. The product $M V$ then really measures the inertia or resistance of a body to change of state. We then see at once the appropriateness of supposing the force constant and the time constant. The unit of inertia, then, or the standard of comparison, is the *resistance* in pounds offered by $g$ pounds of matter, when a velocity of one foot per second is imparted to it in one second. (See Art. 3.)

**8.—Remarks in Illustration of the Preceding.**—The ideas of the preceding articles are seldom stated in current text-books with proper exactness, and are sometimes even illustrated in a way calculated to convey an erroneous impression as to their significance.

Thus [Ganot's Physics, Art. 28] we read that "momentum or quantity of motion is a magnitude varying as the mass of a body and its velocity jointly, and therefore is expressed numerically by the product of the number of units of mass which it contains and the number of units of velocity in its motion. Thus a body containing 5 lbs. of matter, and moving at the rate of 12 ft. per second, has a momentum of 60."

The question at once arises : 60 what ? Is it 60 lbs. or 60 ft. ? What is the *unit* of which we are to take 60 times ? These very natural and obvious questions are nowhere answered, nor is any further mention made of the matter. Text-books generally define momentum as "quantity of motion," and give numerical illustrations as above, without stating what the numerical result obtained signifies. In no text-book known to the writer is it shown what momentum really is, and in what units it is measured. "Motion" is measured, and only properly measured, in feet per second. "Momentum" is not "quantity of motion" at all, but a force, and is measured in pounds.

In Olmsted's "Natural Philosophy," Art. 15, we read : "A ship weighing 336,000 lbs. is dashed against the rocks in a storm, with a velocity of 16 miles per hour : with what momentum did she strike ? *Ans.* 7,884,800 lbs."

This answer is incorrect, because momentum is here taken as $W V$ or weight into velocity, instead of $M V$. It therefore measures nothing whatever. However, this mention of "lbs." would indicate that momentum is pressure, yet nowhere do we find it so stated, nor is any thing anywhere said as to the *limitation of time involved*, without which the "answer" is manifestly worthless. According to the elasticity of the rock and ship, the time of coming to rest will vary, and the pressure exerted by supposing proper degrees of compressibility and elasticity may have any value whatever from zero to infinity. Out of all these possible values, the above "answer" holds good (or would hold good if the mass of the ship had been taken in place of its weight) only when the compressibility of ship and rock is so taken and so proportioned to the given values of mass and velocity that the ship is brought to rest *in exactly one second*—a series of assumptions, it need hardly be said, little likely to be realized in "practice," and rendering the practical value of the illustration exceedingly small. Again we read:

"On the supposition that Goliath of Gath presented an obstacle of 350 lbs., and that the stone hurled by David's sling weighed two ounces: with what velocity must it have been thrown to have prostrated the giant?"

If we suppose here that "prostration" means penetration of the cranium, we may, by the principle of "momentum," find the velocity required, provided it took the stone *just one second to get* into said cranium, but not otherwise.

The true answer to the first question above, is, that the *inertia* of the ship is $\frac{7884800}{g}$ times the unit of inertia, or the resistance offered by a body whose weight is $g$ pounds when given in one second a velocity of one foot per second. This resistance is (Arts. 3 and 4), as we have seen, just one pound; hence, the answer is, 7244840 lbs. would bring the ship to rest *in one second.* The "force with which she strikes" is unknown, and must be unknown unless the time is specified in which the mass is brought to rest.

**9.—Differential and Integral Formulæ for Motion.**—The velocity of a body for uniform motion is equal to the space divided by the time, or $V = \frac{S}{T}$. From this we obtain at once $S = T V$ and $T = \frac{S}{V}$. For the uniformly accelerated motion of gravity, we have also found (Art. 6) for the space passed through $S = \frac{V^2}{2g}$, hence for uniformly accelerated or retarded motion $V = \sqrt{2gS}$. Also since the distance is equal to half that passed through in the same time with the final velocity (Art. 5): $\frac{TV}{2} = S = \frac{V^2}{2g}$ or $V = g\,T$, $T = \frac{V}{g}$, $g = \frac{V}{T}$. For variable motion in general, we must take $v = \frac{ds}{dt}$ or $ds = v\,dt$, whence $s = \int v\,dt$ and $t = \int \frac{ds}{v}$. Also $p = \frac{dv}{dt}$ or $dv = p\,dt$, whence $v = \int p\,dt$ and $t = \int \frac{dv}{p}$ where $p$ is the acceleration. Also $v\,dv = p\,ds$ or $s = \int \frac{v\,dv}{p}$ and $\frac{v^2}{2} = \int v\,dv = \int p\,ds$; where $p$ is the acceleration. The preceding comprise the fundamental formulæ of motion.

## 10.—Examples Illustrative of the Preceding.

1. A body fell from the top of a tower which was 150 ft. high. How long was it in falling, and what velocity had it acquired when it got to the bottom?

*Ans.* $V = \sqrt{2 g S} = 98 \cdot 2$ ft. $T = \frac{V}{g} = 3 \cdot 05$ seconds.

2. A body has been falling 6 seconds. What space has it fallen through in that time, and what velocity has it acquired?

*Ans.* $V = g T = 193 \cdot 2$ ft. per second. $S = \frac{T V}{2} = 579 \cdot 6$ ft.

3. A cannon-ball, fired perpendicularly upwards, was gone 10 seconds before it returned to the ground. How high did it go, and what was the velocity of projection?

*Ans.* Time of descent was 5 seconds, hence $V = g T = 161$ ft. per second.

$$S = \frac{T V}{2} = \frac{v^2}{2 g} = 402 \cdot 5 \text{ ft.}$$

Of course, in all the above examples the resistance of air is neglected.

4. What is the greatest possible velocity a body can acquire in falling under the action of gravity, disregarding the resistance of the air?

Let $r =$ the radius of the earth, $v =$ the velocity, at any point distance $s$ from the surface, $g =$ the acceleration at the surface, and $g' =$ the acceleration at the point whose distance from the surface is $s$.

Then, since the force of gravity is inversely as the square of the distance, and since the acceleration is directly as the force,

$$g' : g :: r^2 : (r + s)^2, \qquad \text{or} \qquad g' = \frac{r^2 g}{(r + s)^2}.$$

But

$$v\, d v = g'\, d s, \qquad \text{or} \qquad v\, d v = \frac{r^2 g\, d s}{(r + s)^2}.$$

Integrating, we have

$$\frac{v^2}{2} = - \frac{r^2 g}{r + s} + C.$$

For $s = 0$, $v = 0$, and $C = r g$, hence

$$v^2 = 2 r g - \frac{2 r^2 g}{r + s}$$

is the general value for the velocity at any point distant $s$. As $s$ increases, the last term becomes smaller and smaller. The limiting value for $v$ is then

$$v^2 = 2 r g. \qquad \text{or} \qquad v = \sqrt{2 r g}.$$

If $r$ is 20900000 ft., and $g = 32 \cdot 2$ ft., then $v = 36687$ ft., or $6 \cdot 95$ miles per second.

5. Find the time and velocity of expansion of a coiled spring which is fixed at one end, and whose other end is in contact with a body whose mass is $M$ units.

Take the position of the end of the spring, before expansion, as the origin. Then upon the supposition that the force of expansion diminishes directly as the expansion increases, we have $P : P' :: s : s - x$, where $P$ is the initial pressure, and $P'$ the pressure at any point distant $x$ from the origin. Hence

$$P = \frac{P' s}{s - x}.$$

But if the acceleration at the distance $x$ is $p$, then $P^1 = M p$, and hence

$$P = \frac{M p s}{s - x}, \quad \text{or} \quad p = \frac{P (s - x)}{M s}.$$

Now, since $v\,d\,v = p\,d\,x$, $v\,d\,v = \frac{P (s - x)\,d\,x}{M s}$. Integrating, we have, since for $x = 0$ $v = 0$, and therefore the constant of integration is also zero,

$$\frac{v^2}{2} = \frac{2 P s x - P x^2}{2 M s}, \quad \text{or} \quad v = \sqrt{\frac{P}{M s}} \sqrt{s^2 - (s - x)^2}.$$

For $x = s$, we have the final velocity

$$v = \sqrt{\frac{P s}{M}}.$$

For the time of expansion, we have, since $d\,t = \frac{d\,x}{v}$,

$$d\,t = \frac{d\,x}{\sqrt{\frac{P}{M s}} \sqrt{s^2 - (s - x)^2}}.$$

Let $s - x = -z$, then $d\,x = d\,z$ and

$$d\,t = \sqrt{\frac{M s}{P}}\ \frac{d\,z}{\sqrt{s^2 - z^2}}.$$

Integrating, we obtain

$$t = \sqrt{\frac{M s}{P}} \cdot \cos.^{-1} \frac{z}{s} + C.$$

For $z = s$, the constant is zero, hence

$$t = \sqrt{\frac{M s}{P}}\ \cos.^{-1} \frac{z}{s}$$

For $z = 0$, we have

$$t = \frac{\pi}{2} \sqrt{\frac{M s}{P}}$$

for the time of expansion required, and

$$t = \pi \sqrt{\frac{M s}{P}}$$

for the time of a complete oscillation. If the pressure $P$ is equal to the weight of the body moved,

$$P = Mg \qquad \text{and} \qquad t = \pi \sqrt{\frac{s}{g}}\,;$$

that is, the same as the time of vibration of a simple pendulum whose length is $s$.

6. What influence has the recoil of a gun upon its range?*

The state of the case may be approximately represented by a coiled spring compressed between two bodies free to move under the action of the spring. The pressure must, of course, be the same at both ends of the spring; action and reaction being equal and opposite. But since the pressures are the same, the velocities imparted to the two bodies will be inversely as the masses, or $V : V' :: M' : M$. If, now, $W$ represent the work of the spring in expanding, and $\frac{W}{x}$ be the portion performed in moving the mass $M$, and $W - \frac{W}{x}$ that expended in moving $M'$, we have

$$\frac{W}{x} = \tfrac{1}{2}\, M\, V^2 \text{ and } W\frac{(x-1)}{x} = \tfrac{1}{2}\, M'\, V'^2.$$

But from the above proportion, we have

$$V' = \frac{M\, V}{M'}\,;$$

hence

$$W\frac{(x-1)}{x} = \tfrac{1}{2}\,\frac{M^2\, V^2}{M'}.$$

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* Solution by Mr. Skinner.

We have also the proportion between the works

$$\frac{W}{x} : \frac{W(x-1)}{x} :: \tfrac{1}{2} M V^2 : \tfrac{1}{2} \frac{M^2 V^2}{M'}, \qquad \text{or} \qquad 1 : x - 1 :: 1 : \frac{M}{M'};$$

hence

$$x = \frac{M' + M}{M'}.$$

Substituting this value of $x$ in the equation

$$\frac{W}{x} = \tfrac{1}{2} M V^2,$$

we have

$$\frac{W M'}{M' + M} = \tfrac{1}{2} M V^2, \qquad \text{or} \qquad V = \sqrt{\frac{2 W M'}{M M' + M^2}}.$$

We see now, from this value of $V$, that if $W$ and $M$ are constant, $V$ will increase with $M'$. Thus we may put the equation in the form

$$V = \sqrt{\frac{2 W}{M + \dfrac{M^2}{M'}}},$$

where we see at once, that as $M'$ increases the denominator diminishes, and hence the value of the fraction under the radical increases. If $M'$ be zero the denominator is infinity, and $V$ is zero. If, however, $M'$ be so large that $M^2$ may be neglected in comparison with $M M'$, we have

$$V = \sqrt{\frac{2 W}{M}},$$

which is the greatest possible value of $V$, and is the same as that which we should obtain if we had supposed the whole work $W$ to be employed in imparting velocity to $M$.

If $M$ is the mass of the bullet, say 1, and $M'$ that of the gun, say 100, and if $W = 505000$, we shall have

$$V = \sqrt{\frac{2 W M'}{M M' + M^2}} = 1000.$$

Now, if $M$ and $W$ remain the same, but $M'$ be doubled—*i. e.*, 200—we have $V = 1002{\cdot}48$. If $M'$ be again doubled, or $= 400$, we have $V = 1003{\cdot}73$. The difference is very slight. If $M'$ be so great that $M^2$, in comparison with $M M'$, may be neglected—in other words, if the gun be immovable and have *no recoil*, still we have for $V$ only 1004·9 ft. per second, or only 4·9 ft. more than in the first case supposed. In any practical case, the mass of the gun will be much more than 100 times that of the bullet, and hence the difference in velocity will be even less than 4·9 ft. This very small increase even between the widest extremes would hardly measurably increase the range. As the mass of the gun can never be infinite, or as there can never be absolutely no recoil, this will also further diminish the range of difference.

Theoretically, then, leaving out of account heat developed, etc., a gun shoots stronger the *less the recoil.* Practically, the difference in velocity is too small for observation. Experiment confirms our result. No measurable increase of range has ever been detected experimentally.

**11.—Impact—Work Lost in Impact.**—Since any body is capable of performing, before coming to rest, the same amount of work expended in setting it in motion, if a mass $M$, with velocity $V$, strikes another body $M'$ at rest, and the two move on with velocity $V'$, then, if *all* the work performed by $M$, by reason of the reduction of its velocity from $V$ to $V'$, were employed in giving the velocity $V'$ to $M'$, the work which both bodies could perform before coming to rest would be the same as $M$ could perform before impact. We have therefore

$$\tfrac{1}{2}(M + M')\, V'^2 = \tfrac{1}{2}\, M\, V,$$

or

$$V' = V \sqrt{\frac{M}{M + M'}}.$$

But, on the other hand, we know that the same pressure which destroys velocity in $M$ communicates velocity to $M'$, and the resulting velocities are therefore, for perfectly inelastic bodies, inversely as the masses. Hence

$$M : M' :: V' : V - V',$$

or

$$V' = V \cdot \frac{M}{M + M'}.$$

Comparing, now, these two values for $V'$, we see that they are not alike. The second is founded upon what we know to be a true principle. Experience, moreover, confirms this result. The first is founded upon an *assumption*—viz., that no work is lost. Experience contradicts this assumption. We know that a certain amount of work goes to the production of *heat*, or molecular motion, and, as *mass motion*, is lost. This is correctly indicated by our result, and from it we may determine the amount of work thus lost. Thus the work in the first case which the combined mass can perform is $\frac{1}{2} M V^2$, while that which in the second case can be performed is only

$$\tfrac{1}{2}\, M\, V^2 \frac{M}{M + M'}.$$

Since $M + M'$ is always greater than $M$, this work is less than the first, and the work lost is

$$\tfrac{1}{2} M V^2 \left(1 - \frac{M}{M + M'}\right).$$

If the two bodies are equal, *half* the whole work, potential in the striking body before impact, is lost. If the struck body $M'$ is very small, the work lost is small, and increases as $M'$ increases. *In impact, then* (*unless the bodies are perfectly elastic*), *work is always lost.*

The above illustrates strikingly the fallacy of supposing that momentum is "quantity of motion." Thus the ordinary method of proof is to assert that $M V$ represents "quantity of motion," and then to assert that motion before equals motion after impact; hence

$$M V = (M + M') V' \text{ or } V' = \frac{M V}{M + M'},$$

a result which, to be sure, is correct. But the very assumption that no motion is lost, we know to be untrue. Motion *is* lost, and disappears as heat; if, therefore, momentum really represented motion, or were in any sort a measure of motion before and after impact, we should have no right to equate the momenta—for thus equating them would be equivalent to asserting that which is known to be false. If momentum really represented quantity of motion, then we ought to obtain an incorrect result. But this is not found to be the case—the result *is* correct—and yet motion is also lost; we obtain a result which contradicts our assumption. If momentum is quantity of motion, the result ought not to be correct. But the result is correct. There is then a compensation of errors; first, in assuming that no motion is lost, and, second, in asserting that $M V$ represents motion. Our explanation of what momentum really is (Art. 3) shows, however, that our result must be correct, and avoids all confusion. $M V$ is the pressure which will bring $M$ to rest in one second. Suppose this is the time in which it really does come to rest; then, since while coming to rest the two masses (if perfectly inelastic) move together and come to rest together, this pressure must be equal to that which in the same time brings the mass $M + M'$ to rest, and if $V'$ is the velocity of the combined masses, this latter pressure is $(M + M') V'$. Hence $M V = (M + M') V'$. If it takes the bodies longer than one second, say $T$ seconds, then we have in precisely similar manner

$$\frac{M V}{T} = \frac{(M + M') V'}{T},$$

where $T$ cancels and gives the same as before. We recommend the student to consult the various text-books accessible, and take note of those in which the subject of "momentum" is *correctly* presented and illustrated. The ordinary method of reasoning by which the equation $M V = (M + M') V'$ is obtained is a striking illustration of how, in the words of Berkeley, "error may bring forth truth, though it cannot bring forth science."

**12.—Angular Velocity—Moment of Inertia—Centre and Radius of Gyration.**—If a body revolves about an axis, different points of it will have different velocities, depending upon the distance of each point from the axis of revolution. If we take a certain point at a distance of one unit from the axis, *the space passed through by that point in one second, or the velocity of that point, is the angular velocity.* Calling the angular velocity then $\omega$, the velocity at any point distant from the axis $r$ units is $\omega r$. We have also $\omega = \frac{2\pi}{t}$, where $2\pi$ is the whole circumference in which the point at one unit distance moves, and $t$ the time in making one entire revolution.

Since, now, the velocity at any point is $\omega r$, the inertia of that point is, from Article 7, $\omega m r$, where $m$ is the mass of the point in question. The *moment* of this force of inertia will be then $\omega m r \times r = \omega m r^2$. If the angular velocity is unity—*i. e.*, if the point at one unit distance has one unit of velocity—then the moment of the inertia of the point will be simply $m r^2$. This product is called the "*moment of inertia*" of the point whose mass is $m$ and distance from the axis $r$. *The moment of inertia, then, is the product of the mass into the square of the distance from the axis.*

The moment of inertia of a body is the sum of the moments of inertia of all the particles contained in it. Thus, if, for instance, a rectangle revolve about its centre line, we have for the moment of inertia of any very small strip whose width is $dx$, and distance from the axis $x$, the breadth being $b$, $b\,dx \times x^2$, since $b\,dx$, or the area of the strip, is the same as its mass.

$$\int_{x=-\frac{1}{2}h}^{x=\frac{1}{2}h} b x^2 \, dx$$

will then give the sum of the moments of inertia of all strips between the top and bottom—viz., $\frac{1}{12} b h^3$. In similar manner, the moment of inertia for any given form of cross-section with reference to any given axis may be found. Its value for various forms of cross-section, in terms of the given dimensions, is given in every text-book of mechanics.

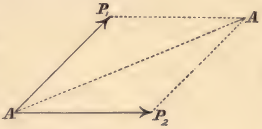
The moment of inertia of a body may also evidently be given by the product of the whole mass, or sum of all the elementary areas, multiplied by the square of a certain radius, this radius to be so chosen that the product shall be the same as above. The point thus given, at which, if the entire mass were concentrated, the moment of inertia would remain unchanged, is called the *centre of gyration*, and the distance from this point to the axis is called the *radius of gyration.* The moment of inertia is then also equal to the mass into the square of the radius of gyration. In the case of the rectangle, the mass is equal to the entire cross-section or $b h$. The radius of gyration is then

$$\sqrt{\tfrac{1}{12} h^2} = h \frac{1}{2\sqrt{3}}.$$

If the whole mass were concentrated at a single point at this distance from the axis, the moment of inertia would be the same as for the actual case.

**13.—Centrifugal Force, Etc.**—A body, by reason of its inertia, if once set in motion in a certain direction by the action of a force, will continue to move in that direction unless pulled aside by another force. If such another force acts, then, by reason of its inertia, it opposes a resistance to the motion caused by that force equal and opposite to the force itself. If a body is acted upon simultaneously by two forces not acting in the same line, it must then take up a position intermediate between the positions due to each force alone. It obeys each force as though the other were not acting. Thus, if, in Fig. 1, two forces, as $P_1$ and $P_2$, given in direction and intensity by the lines $A\ P_1$ and $A\ P_1$, act upon the point $A$, and if the force $P_1$, acting alone, would take this point to $P_1$ in one second, and $P_2$, acting alone, would move it to $P_2$ in the same time, then if both act together, the point $A$, at the expiration of one second, will be found at $A'$, found by completing the parallelogram by drawing $P_2\ A'$ parallel to $A\ P_1$, and $P_1\ A'$ parallel to $A\ P_2$. Thus the point $A$ passes in one second to $A'$ along the line $A\ A'$, and in so doing passes over the same distances parallel to each force that it would have described under the influence of each separately. This is known as the principle of the "parallelogram of velocities." The path of the body is always the resultant or diagonal of the parallelogram constructed upon the two velocities. If, now, a body be acted upon by a single force which gives it motion in a certain direction, and the force then cease to act, the body, unless arrested by some other force, will continue its direction forever at a constant velocity—viz., that which it had when the force ceased to act. If, now, while thus moving, it be acted upon by another force, it will, in say the first second, move in the resultant of its first direction and that of the second force. In the next second, its now new direction and the force will determine another resultant inclined to the first. If the second applied force, instead of acting at intervals, acts continuously, the broken path will become a curve. If this second applied force continually change its direction of action so as to always be perpendicular to the direction of motion at every instant, this curve will be a circle; if not perpendicular, the motion will be in some other curve. The body, by reason of its inertia, opposes a resistance to this second force, which is, by our principles, equal to its mass into the velocity caused by this force (Art. 7). This resistance is called the *centrifugal force*, and is therefore nothing more than the inertia of the body due to the deflecting force. The deflecting force itself is called the *centripetal force*. The one is a passive force or resistance, called into action by the other actual and active force, and always equal to it and opposite in direction. In the case of motion in a circle, these terms have a special significance, because then the centripetal force is always directed towards the centre; and if at any instant it should suddenly cease, the equal and now unbalanced centrifugal force would cause the body to fly out radially away from the centre.

FIG. 1.



Now, if $\omega$ is the angular velocity (see Art. 12), then the velocity at the circumference is $\omega r$. In passing from one position on the circle to another, lower down, for instance, the body in its passage passes through a certain distance along the radius to the circle at the first position, equal to the projection of its path upon this radius. If $c$ is the velocity along this radius, and $\omega r$ is the velocity on the circumference, and if the points are "consecutive," or so near together that there is no third point between them, then we have, since for very small angles the arc is nearly equal to its sine, $c : \omega r :: \omega r : r$, or $c = \omega^2 r$. We have then $\omega^2 r$ for the velocity along the radius. The mass into this velocity is then the inertia or centrifugal force, or $m \omega^2 r$. Since $\omega r = v$ or $\omega = \frac{v}{r}$ we can also put $m \omega^2 r = \frac{m v^2}{r}$ = the centrifugal force at any point. If the original impelling force is continuous, so that the velocity $v$ is constantly accelerated, we see that the centrifugal force increases as the square of the velocity. In a rapidly revolving body, it may even reach thus such an intensity as to overcome the cohesion of the particles of the body. Observe the analogy between the expression for centrifugal force and that for *work* (Art. 5). For work or *vis viva*, we have $\frac{1}{2} M V^2$; we may put

$$\frac{M V^2}{r} = \frac{2 M V^2}{d}$$

where $d$ is the diameter of the circle. For a circle whose diameter then is unity, the centrifugal force is equal to twice the *vis viva* at the circumference. In general, *the inertia of a body revolving in a circle whose radius is r, is equal in pounds to* $\frac{1}{r}$ *part of twice the work due at any moment to the velocity at the circumference.* If $C$ is the centrifugal force, then we have

$$C = m \omega^2 r = \frac{m v^2}{r}.$$

For a body whose mass is unity, we have, therefore,

$$C : v :: v : r,$$

or the velocity is a mean between the centrifugal force and the radius. Also putting

$$m = \frac{G}{g}, \quad \text{we have} \quad C = \frac{2 G v^2}{2 r g} = \frac{v^2}{2g} \cdot 2 \frac{G}{r}, \quad \text{or} \quad C : G :: 2 \cdot \frac{v^2}{2g} : r$$

That is, the centrifugal force is to the weight as twice the height due to the velocity is to the radius.

**14.—Parabolic Motion.***—In order thoroughly to understand the motion produced by the combination of velocity and acceleration, we must be able to give for any time ($t$) the *direction*, *velocity*, and the *space described*. Let the velocity parallel to $A\,Y$, Fig. 2, be constant and $= c$, and that parallel to $A\,X$ be variable and equal to $p\,t$. If we construct with these velocities $O\,Q = c$ and $O\,P = p\,t$ the parallelogram $O\,P\,R\,Q$, we have for the resultant velocity, $O\,R$, which is the velocity $v$ with which the body at $O$ moves. This velocity is then

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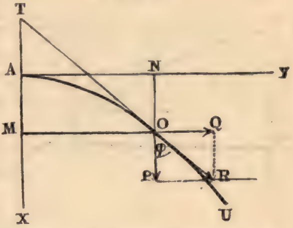
* The following articles are in large part selected from Weisbach, Vol. I. As they are frequently referred to in the present volume, we have thought it well to insert them here for the greater convenience of the reader.

$$v = \sqrt{c^2 + (p\,t)^2}.$$

FIG. 2.



$O\,R$ gives also the tangent to the curve, or the direction in which the body moves for an instant. Therefore, for the angle

$$P\,O\,R = X\,T\,O = \phi,$$

which the direction of motion at any instant makes with the direction $X\,A$ of the second motion, we have

$$tan.\ \phi = \frac{O\,Q}{O\,P} = \frac{c}{p\,t}.$$

If we put

$$tan.\ \phi = \frac{d\,y}{d\,x}$$

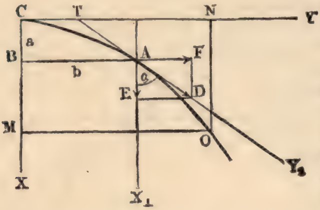
where $y$ is represented by $B\,N$ and $x$ by $N\,O$, we have

$$\frac{d\,y}{d\,x} = \frac{c}{y},$$

and integrating, $y^2 = 2\,c\,x$, which is the equation of a parabola. Now, if a body be projected horizontally by an impulsive force which acts but for a specified time, it will have a uniform velocity $c$. If at the same time it is acted upon by a constant force, then we see that the resultant moment is a parabola. A body projected horizontally, then, and acted upon by the force of gravity, describes a parabola, resistance of the air being neglected. To obtain the space described, or the arc $A\,O = S$, we have only to find by the calculus the length of the arc, or, the time being given, we can put $\sigma = v\,\tau$, where $\sigma$ is the space described in a very small part of the time $t$, and thus calculate the small portions considered as elements.

**15.**—We have previously supposed that the primitive directions of motion were at right angles. Let us now suppose that the direction of the impelling force makes any angle with that of the acceleration. Thus, if the impulsive force imparts to the body at $A$, Fig. 3, a velocity $c$ at an angle $\alpha$ with the direction of the force of acceleration $A E = p$, $A$ is no longer the vertex of the parabola as before, and $A X_1$ is no longer the axis, but is simply parallel to the axis $C X$. The vertex of the parabola is situated now at a point whose co-ordinates with respect to $A$ are $C B = a$ and $A B = b$, of which the first lies in the axis and the second is at right angles to it. The velocity $A D = c$ is decomposed into the two components $A F = c \sin. \alpha$ and $A E = c \cos. \alpha$. The first of these must be constant, while the latter is variable, and always equal to the variable velocity $p t$, if the body passes in the time $t$ from $C$ to $A$.

FIG. 3.



Hence we have

$$c \cos. \alpha = p t, \quad \text{whence} \quad t = \frac{c \cos. \alpha}{p},$$

and therefore

$$C B = a = \frac{p t^2}{2} = \frac{c^2 \cos.^2 \alpha}{2 p}$$

and

$$B A = b = c \sin. \alpha \cdot t = \frac{c^2 \sin. \alpha \cos. \alpha}{p} = \frac{c^2 \sin. 2 \alpha}{p}$$

By means of these distances, the vertex $c$ of the parabola is determined, and then, starting from this point, we can, for any given time, determine the position $O$ of the body. If, in the general equation for the parabola, $y^2 = 2 P x$, we put the above value of $B A$ for $y$ and of $C B$ for $x$, we find the parameter

$$P = \frac{c^2 \sin.^2 \alpha}{p},$$

and hence

$$y = c \sin. \alpha \sqrt{\frac{2 x}{p}}$$

and

$$x = \frac{p y^2}{2 c^2 \sin.^2 \alpha}$$

are the co-ordinates for any point of the curve.

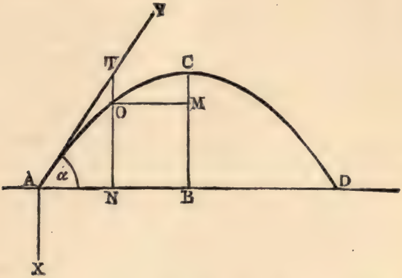
**16.—Motion of Projectiles.**—A body projected in the direction $A y$ at an angle of elevation $y A D = \alpha$, ascends to a certain height $B C$, which is called the height of projection, and it reaches the horizontal plane from which it started at $A$, at a distance $A D$, which is called the *range*.

From the velocity $c$, the acceleration $g$, and the angle $\alpha$, we obtain, according to the preceding article, when we replace $p$ by $g$ and $\alpha$ by $90 - \alpha$, or $cos.\ \alpha$ by $sin.\ \alpha$, etc.,

$$CB = a = \frac{c^2\, sin.^2\, \alpha}{2\, g} \quad \text{and} \quad AB = b = \frac{c^2\, sin.\ 2\, \alpha}{2\, g}.$$

From this last formula, we see that the range is a maximum for $sin.\ 2\, \alpha = 1$, or $2\, \alpha = 90°$, or $\alpha = 45°$. A body projected at an angle of 45° attains the greatest range.

FIG. 4.



We have also

$$a = \frac{g\, b^2}{2\, c^2\, cos.^2\, \alpha},$$

and for a point $O$ in the path of the body, for which $CM = x$ and $MO = y$,

$$x = \frac{g\, y^2}{2\, c^2\, cos.^2\, \alpha},$$

or, when its position is given by the co-ordinates $AN = x_1$ and $NO = y_1$,

$$x = CM = BC - NO = a - y_1$$
$$y = MO = AB - AN = b - x_1,$$

we have

$$a - y_1 = \frac{g\, (b - x_1)^2}{2\, c^2\, cos.^2\, \alpha}, \quad \text{whence} \quad y_1 = a - \frac{g\, (b - x_1)^2}{2\, c^2\, cos.^2\, \alpha},$$

or since

$$a - \frac{g\, b^2}{2\, c^2\, cos.^2\, \alpha} = 0, \quad y_1 = x_1\ tan.\ \alpha - \frac{g\, x_1^2}{2\, c^2\, cos.^2\, \alpha}.$$

Substituting in this equation for $\frac{1}{cos.^2\ \alpha}$ the value $1 + tan.^2\, \alpha$, and then solving for *tan.* $\alpha$, we have for the angle of elevation necessary to hit a point whose co-ordinates are $x_1$ and $y_1$,

$$tan.\ \alpha = \frac{c^2}{g\,x_1} \pm \sqrt{\left(\frac{c^2}{g\,x_1}\right)^2 - \left(1 + \frac{2\,c^2\,y_1}{g\,x_1^2}\right)}.$$

When

$$\left(\frac{c^2}{g\,x_1}\right)^2 = 1 + \frac{2\,c^2\,y_1}{g\,x_1^2}, \quad \text{or} \quad c^4 - 2\,g\,y_1\,c^2 = g^2\,x_1^2,$$

we have

$$c = \sqrt{g\,(y_1 + \sqrt{x_1^2 + g_1^2})}, \quad \text{and hence} \quad tan.\ \alpha = \frac{c^2}{g\,x_1}.$$

Smaller values of $c$ make *tan.* $\alpha$ imaginary, and larger values of $c$ give two values for *tan.* $\alpha$; in the first case, the point will not be reached, and in the second it will be attained either in the rise or fall of the body.

EXAMPLE 1.—A jet of water rises with a velocity of 20 ft. at an angle of 66°. The height due to the velocity is $h = 0{\cdot}0155,\ 20^2 = 6{\cdot}2$ ft., and the jet ascends to a height $a = h\ sin.^2\ \alpha = 6{\cdot}2\ (sin.\ 66^\circ)^2 = 5{\cdot}17$ ft.; the range of the jet is $2\,b = 2{\cdot}6,\ 2\ sin.\ 132^\circ$, or $2{\cdot}6,\ 2\ sin.\ 48 = 9{\cdot}21$ ft. The time which each particle of water requires to describe the entire arc $A\,C\,D$ of the parabola is

$$t = \frac{2\,c\ sin.\ \alpha}{g} = \frac{2 \cdot 20\ sin.\ 66^\circ}{32{\cdot}2} = 1{\cdot}14 \text{ seconds.}$$

The height corresponding to the horizontal distance $A\,N_1 = 3$ ft. $= x_1$ is

$$y_1 = 3\ tan.\ 66^\circ - \frac{32{\cdot}2 \cdot 9}{2 \cdot 400 \cdot cos.\ (66^\circ)^2} = 6{,}738 - \frac{0{\cdot}36225}{0{\cdot}16543} = 6{\cdot}738 - 2{\cdot}189 = 4{\cdot}549 \text{ ft.}$$

EXAMPLE 2.—A jet of water discharged from a horizontal tube has for a height 1¾ ft. and a range of 5¼ ft.; how great is its velocity?

From the formula

$$x = \frac{g\,y^2}{2\,c^2} = \frac{y^2}{4\,h}, \quad \text{we deduce} \quad h = \frac{y^2}{4\,x},$$

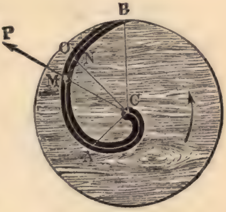
in which we must substitute $x = 1{\cdot}75$ and $y = 5{\cdot}25$, and thus obtain

$$h = \frac{5{\cdot}25^2}{4 \times 1{\cdot}75} = 3{\cdot}937 \text{ ft.},$$

and the corresponding velocity $c = 15{\cdot}92$ ft.

**17.—Mechanical Effect of Centrifugal Force.**—If the path $C A B$, Fig. 5, in which a body moves is not at rest, but turning about an axis $C$, it imparts to the body a centrifugal force $P$, by reason of which the body either performs work or work is performed upon it, according as it departs from or approaches the axis of rotation $C$. Let $M$ be the mass of the body, $\omega$ the constant angular velocity with which the path turns about $C$, and let $r$ denote the variable distance $C M$ of the body which is moving in the path $C A B$. We have the centrifugal force of the body (Art. 13) $P = \omega^2 M r$, and for the work done by this force while the body describes an element $M O$ of its path, and hence, while the radius $C M = r$ is increased by an amount

FIG. 5.



$$N O = d r,\ d \omega = \omega^2 M r\, d r.$$

Hence, for the whole work,

$$W = \frac{\omega^2 M r^2}{2}.$$

Now the velocity of rotation at the distance $r$ is $v = \omega r$, and therefore

$$W = \frac{M v^2}{2} = \frac{v^2}{2 g} G,$$

when we substitute, instead of the mass $M$, the weight $G = M g$.

If the body begins its motion, not at $C$, but at any other point, $A$, at a distance $r_1$ from $C$, where the velocity of rotation is $v_1 = \omega r_1$, we have for the work done while passing from $A$ to $M$,

$$W = \tfrac{1}{2} \omega^2 M r^2 - \tfrac{1}{2} \omega^2 M r_1^2 = \tfrac{1}{2} \omega^2 M (r^2 - r_1^2) = \tfrac{1}{2} M (v^2 - v_1^2) = \frac{(v^2 - v_1^2) G}{2 g}.$$

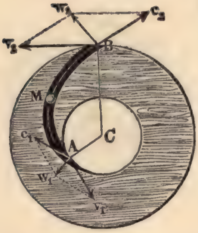
If, then, a body moves in a rigid path or groove, which revolves about a fixed axis, the *vis viva* of the body is increased or diminished by the product of the mass ($M$) and the difference of the squares of the velocities ($v^2$ and $v_1^2$) at the two ends ($A$ and $M$) of the path, or by the product of the weight ($G$) and the heights

$$\left(\frac{v^2}{2 g} \text{ and } \frac{v_1^2}{2 g}\right)$$

due to these velocities. The increase takes place when the motion is from within outward, and the decrease when the motion is from without inward.

**18.**—If a body begins its path *A M B* at *A*, Fig. 6, with a relative velocity $c_1$, and leaves at *B* with the relative velocity $c_2$, and if the velocity of rotation of the path itself at *A* and *B* is $v_1$ and $v_2$, the work which the body is capable of performing before coming to rest is

FIG. 6.



$$W = \frac{c_2^2 - c_1^2}{2g} G = \frac{v_2^2 - v_1^2}{2g} G,$$

and therefore

$$c_2^2 - c_1^2 = v_2^2 - v_1^2, \text{ or } c_2^2 = c_1^2 + v_2^2 - v_1^2,$$

and consequently the velocity of exit is

$$c_2 = \sqrt{c_1^2 + v_2^2 - v_1^2} = \sqrt{c_1^2 + \omega^2 (r_2^2 - r_1^2)},$$

$\omega$ denoting the angular velocity, and $r_2$, $r_1$ the distances *C A* and *C B* of the points *A* and *B* of entrance and exit from the axis of rotation *C*.

The relative velocity of exit $c_1$ is determined, in like manner, when the body enters at *B* with the relative velocity $c_2$, and moves along the path from without inwards. It is then

$$c_1 = \sqrt{c_2^2 - (v_2^2 - v_1^2)} = \sqrt{c_2^2 - \omega^2 (r_2^2 - r_1^2)}.$$

Since the body, in describing the path *A M B*, has besides its relative velocity $(c)$ in the path, also the velocity $v$ of the path itself, it must be introduced at *A* with an absolute velocity of $w_1$ or *A* $w_1$, which is determined in direction and intensity by the diagonal of the parallelogram constructed with $c_1$ and $v_1$, and the body leaves at *B* with an absolute velocity $B w_2 = w_2$, determined by the diagonal of the parallelogram $B\, c_2\, w_2\, v_2$, constructed with the relative velocities $c_2$ and $v_2$.

The work performed by or upon the body, in describing the path *A M B*, and which has therefore been gained or lost by the revolving path, is

$$W = \pm \left(\frac{w_2^2 - w_1^2}{2g}\right) G.$$

If a body should impart all its energy $\frac{w_1^2}{2g} G$ to the path while passing from *A* to *B*, the absolute velocity of exit must be $w_2 = 0$, and $c_2$ must be not only equal to $v_2$, but also exactly opposite to it. Hence in this case the path must be tangent to the circumference at *B*.

EXAMPLE.—If the interior radius is $C A = r_1 = 1$ ft., and the exterior one $C B = r_2 = 1\frac{1}{2}$ ft., and if the whole path revolves about *C* one hundred times a minute, then the angular velocity is

$$\frac{2 \pi u}{60} = \frac{\pi u}{30},$$

where $u$ is the number of revolutions. Hence

$$\omega = 3{\cdot}1416 \cdot \frac{10}{3} = 10{\cdot}472 \text{ ft.}$$

The velocity, then, at the interior circumference is

$$v_1 = \omega\, r_1 = 10{\cdot}472 \text{ ft.},$$

and at the exterior one

$$v_2 = \omega\, r_2 = 10{\cdot}472 \times 1{\cdot}5 = 15{\cdot}708 \text{ ft.}$$

Now, if we cause a body whose velocity is $w_1 = 25$ to enter at $A$ in such a direction that the angle $\omega_1 A\, v_1$ formed by this absolute velocity with the direction of revolution is $\alpha = 30°$, we have for the relative velocity $c_1$ with which the body begins its motion in the path,

$$c_1^2 = v_1^2 + w_1^2 - 2\, v_1\, w_1 \cos. \alpha = 109{\cdot}66 - 453{\cdot}45 + 625, \text{ or } 281{\cdot}21,$$

hence

$$c_1 = 16{\cdot}77 \text{ ft.}$$

If the body is to enter without impact, we must have for the angle $v_1\, A\, C_1 = \beta$ of the path with the inner circumference,

$$\frac{w_1}{c_1} = \frac{\sin. \beta}{\sin. \alpha}, \text{ or } \sin. \beta = \frac{25 \sin. 30°}{16{\cdot}77}, \text{ whence } \beta = 48°\ 12\tfrac{1}{2}'.$$

For the relative velocity of exit $c_2$ we have

$$c_2^2 = c_1^2 + v_2^2 - v_1^2, \text{ or } c_2^2 = 281{\cdot}21 + 109{\cdot}66\, [(\tfrac{3}{2})^2 - 1^2] = 418{\cdot}28,$$

and hence

$$c_2 = 20{\cdot}45 \text{ ft.}$$

On the other hand, for the absolute velocity of exit $w_2$, when the path or groove $A\, M\, B$ forms with the exterior circumference an angle $\delta = 20°$, or $v_2\, B\, C_2 = 160°$, we have

$$w_2^2 = c_2^2 + v_2^2 - 2\, c_2\, v_2 \cos. \delta = 418{\cdot}28 + 246{\cdot}74 - 603{\cdot}72 = 61{\cdot}30,$$

and hence

$$w_2 = 7{\cdot}80 \text{ ft.}$$

Finally, the heights due to the velocities are

$$\frac{w_1^2}{2\,g} = 0{\cdot}0155 \times 625 = 9{\cdot}69 \text{ ft.} \quad \text{and} \quad \frac{w_2^2}{2\,g} = 0{\cdot}0155 \times 61{\cdot}31 = 0{\cdot}95 \text{ ft.},$$

and the amount of work imparted to the body in which the groove or path exists, which turns about $c$ by a body whose weight is $G$, while passing from $A$ to $B$, is

$$W = \left(\frac{w_1^2 - w_2^2}{2\,g}\right) G = (9{\cdot}69 - 0{\cdot}95)\, G = 8{\cdot}74\, G.$$

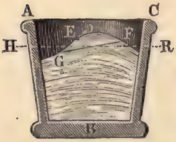
If $G = 10$ lbs.,

$$w = 8{\cdot}74 \times 10 = 87{\cdot}4 \text{ ft. lbs.}$$

The foregoing theory of the motion of a body within another revolving body finds direct application to turbine-wheels.

**19.—Surface of Water.**—In consequence of the action of gravity upon water, all the elements of it tend to descend, and if not prevented really do descend. In order to keep a quantity of water together, it is necessary, then, to confine it in a vessel. The water in a vessel, *A B C*, Fig. 7, can only be in equilibrium when the free surface *H R* is at right angles to the direction of gravity, or horizontal; for so long as this surface is curved or inclined to the horizon, there will be elements of the water, such as *E*, which, being situated above the others, will, in consequence of their weight and mobility, or freedom of motion, slide down those below them as upon an inclined plane. Since, when the surface is very great, the direction of gravity cannot be considered as parallel at every point, the free surface in a very large vessel—*i. e.*, in a large sea—will not form a plane surface, but a portion of the surface of a sphere.

FIG. 7.



If another force acts, in addition to gravity, upon the elements of the water, then, when equilibrium exists, the free surface of the water must be at every point at right angles to the resultant of this force and of the force of gravity.

If a vessel *A B C*, Fig. 8, is moved forward with the *constant acceleration* $p$, the free surface of the water forms an inclined plane *D F*; for in this case every element *E* of this surface is drawn vertically downwards by its own weight *G*, and at the same time in a horizontal direction by its inertia $P = \frac{G}{g} p$. These two forces give rise to a resultant *R*, whose direction, since $p$ and *G* are constant, forms with that of gravity a constant angle $R E G = \alpha$. This angle is at the same time the angle *D F H* formed by the surface of the water (which must be at right angles to the resultant) with the horizon. It is given by the equation $tan. \alpha = \frac{p}{g}$.

FIG. 8.

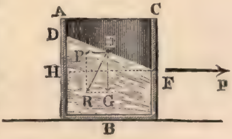
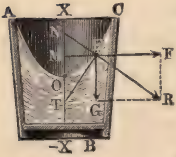


FIG. 9.



If, on the contrary, a vessel *A B C*, Fig. 9, is caused to *revolve uniformly* about its vertical axis *X X*, the surface of the revolving water forms a hollow *A O C*, whose cross-section through the axis is a *parabola*. If $\omega$ is the angular velocity of the vessel and water in it, *G* the weight of an element *E* of the water, and $y$ its distance *M E* from the vertical axis, we have for the centrifugal force of this element

$$F = \frac{M V^2}{r} = \frac{G}{g} \omega^2 y,$$

and therefore for the angle *R E G* or *T E M* = $\phi$ formed by the resultant with the vertical, or by the tangent $E\,T$ to the profile of the water at any point $E$ with the horizon,

$$tan.\ \phi = \frac{F}{G} = \frac{\omega^2 y}{g}.$$

At the point $E$ we have

$$\frac{dx}{dy} = tan.\ \phi = \frac{\omega^2 y}{g},$$

and integrating, we have for the equation of the curve $A\,O\,C$,

$$y^2 = 2\,\frac{g}{\omega^2}\,x,$$

which is the equation of a parabola whose parameter is $\frac{g}{\omega^2}$.

If we cause a vessel $A\,B\,H$, Fig. 10, to move uniformly in a vertical circle around a horizontal axis $C$, the surface of the water will assume a cylindrical form, with a circular cross-section $D\,E\,H$. If we prolong the direction of the resultant $R$ of the weight $G$ and of the centrifugal force $F$ of any element $E$ until it cuts the vertical line $C\,K$, passing through the centre of rotation, we obtain two similar triangles $E\,C\,O$ and $E\,F\,R$, for which we have

$$\frac{C\,O}{E\,C} = \frac{F\,R}{E\,F} = \frac{G}{F}.$$

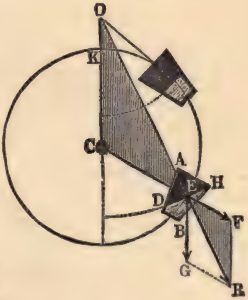
But if the radius of rotation $E\,C = y$, we have

$F = \frac{\omega^2\,G\,y}{g}$, whence it follows that

$$C\,O = \frac{G\,y}{F} = \frac{g}{\omega^2} = g\left(\frac{30}{\pi\,u}\right) = \frac{2936}{u^2}\ \text{ft.}$$

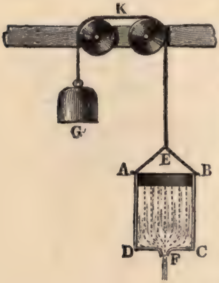
where $u$ denotes the number of revolutions per minute. Since this value of $C\,O$ depends only upon $u$, it is the same for all the elements of the water, and hence the resultant at every point of the cross-section $D\,E\,H$ is directed towards $O$, and since the cross-section at every point is at right angles to this direction, it forms the arc of a circle described from $O$. Hence the surfaces of the water in the buckets of an overshot water-wheel are always cylindrical described from the same horizontal axis at $O$.

FIG. 10.



**20.—Efflux from a Vessel in Motion.**—The *velocity* of efflux changes when a vessel originally at rest or moving uniformly is set in motion, or when a change in its condition of motion takes place, since in this case every molecule of the water acts upon those surrounding it not only by its weight, but also by its inertia.

FIG. 11.



If the vessel $A\,C$, Fig. 11, is moved with an *accelerated motion vertically upwards*, while the water flows through an opening $F$ in the bottom, the velocity of efflux is augmented, and if it descends with an accelerated motion, the velocity is diminished. If the acceleration is $p$, every molecule $M$ of the water presses not only with its weight $M g$, but also with its momentum or inertia $M p$, and in the first case we must put the force of each molecule equal to $(g + p)\,M$, and in the second case equal to $(g - p)\,M$, or generally $(g \pm p)\,M$ for both cases. Since, now, work is pressure through distance, we have, if the distance passed through is $h$,

$$\tfrac{1}{2} M v^2 = (g \pm p)\ M h,$$

$$\frac{v^2}{2} = (g \pm p)\,h,$$

and hence the *velocity of efflux* is

$$v = \sqrt{2 (g \pm p)\,h}.$$

If the vessel rises with the velocity $g$, we have $p = g$, and hence

$$v = \sqrt{2 \cdot 2 g h} = 2 \sqrt{g h},$$

and therefore the velocity is in this case 1·414 times as great as it would be if the vessel stood still. If the vessel *falls* by its own weight, $p = g$ and $v = \sqrt{0} = 0$, and no water runs out at all. If the vessel moves uniformly upwards or downwards, there is no acceleration, and $p = 0$, and $v$ remains $= \sqrt{2 g h}$, but if its rise is retarded, we have

$$v = \sqrt{2 (g - p)\,h},$$

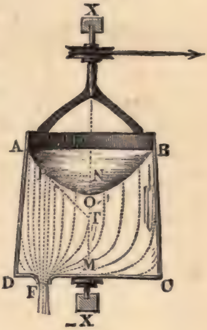
and if its fall is retarded

$$v = \sqrt{2 (g + p)\,h}.$$

If the vessel from which the water flows is moved horizontally or at an acute angle to the horizon, the surface (see preceding article) becomes oblique to the horizon, and a variation of the velocity of efflux is the result.

If a vessel $A\,C$, Fig. 12, is caused to *revolve about its vertical axis* $X\,X$, its surface will assume, as we have just seen, the shape of a parabolic funnel $A\,O\,B$, and at the centre $M$ of the bottom the head of water $M\,O$ is least, and increases as we go towards either side. The water will then flow more slowly through an orifice at the centre than through any other equally large aperture in the bottom. If $h$ denotes the head of water $M\,O$ at the centre $M$, the velocity of efflux at that point will be $= \sqrt{2gh}$, but if $y$ denotes the distance $M\,F = N\,P$ of an aperture $F$ from the axis $X\,X$ and $\omega$ the angular velocity, we have, since for a parabola the subtangent $T\,N$ for any point $P$ is equal to twice the ordinate $O\,N$, for the corresponding elevation of the water above $F$,

FIG. 12



$$h + O\,N = h + \tfrac{1}{2} T\,N = h + \tfrac{1}{2} P\,N \; tan.\, N\,P\,T.$$

But

$$tan.\, N\,P\,T = tan.\, \phi = \frac{\omega^2\, y}{g}$$

(see preceding article), hence if the angular velocity $\omega y$ of $F$ is $w$, we have

$$O\,N = x = \tfrac{1}{2} y \frac{\omega^2\, y}{g} = \frac{\omega^2\, y^2}{2\,g} = \frac{w^2}{2g}.$$

Therefore the velocity of efflux through the orifice $F$ is

$$v = \sqrt{2g\left(h + \frac{w^2}{2g}\right)} = \sqrt{2gh + w^2}.$$

This formula holds good for a vessel of any shape, even when closed on top and full to the top, so that the funnel $A\,O\,B$ cannot be completely formed. Here also $h$ is the depth $M\,O$ of the orifice below the vertex $o$ of the funnel, and $v$ the velocity of efflux. It will be employed repeatedly in the discussion of reaction-wheels and turbines in the following pages.

EXAMPLE 1.—If a vessel which, when filled with water, weighs 350 lbs., is drawn upwards by a weight of 450 lbs. by means of a cord and pulley, it rises with an acceleration

$$p = \frac{450 - 350}{450 + 350} \cdot g = \frac{100}{800} \cdot g = \tfrac{1}{8}\, g,$$

because the accelerations are as the pressures producing them, and the acceleration of the combined weights is $g$, hence

$$450 + 350 : 450 - 350 :: g : p.$$

For the velocity of efflux, then,

$$v = \sqrt{2\,(g + p)\,h} = \sqrt{2 \cdot \tfrac{9}{8}\, g\,h} = \sqrt{\tfrac{9}{4}\, g\,h}.$$

Now, if the head of water were four feet, the velocity of efflux would be

$$v = \sqrt{9g} = 3\sqrt{32 \cdot 2} = 17 \cdot 02 \text{ ft.}$$

2. If a vessel filled with water makes 100 revolutions per minute, and if the orifice $F$ is two feet below the level of the water at the centre, and at a distance from the axis of three feet, the velocity of efflux is

$$v = \sqrt{2gh + w^2} = \sqrt{64\cdot4 \times 2 + \left(\frac{3\,.\,\pi\,.\,100}{30}\right)^2} = \sqrt{128\cdot8 + 100\,\pi^2}$$

$$= \sqrt{128\cdot8 + 987} = \sqrt{1115\cdot8} = 33\cdot4 \text{ ft.}$$

If the vessel stands still, we have $v = \sqrt{128\cdot8} = 11\cdot35$ ft.

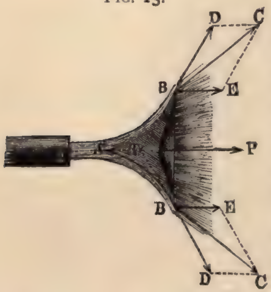
**21.—Impulse and Resistance of Water.**—Water or any other fluid, when it impinges upon a solid body, imparts a force or impulse to it, and thus produces a change in its state of motion. We may distinguish from one another, first the impact of an isolated stream; secondly, the impact of a limited or bounded stream; and, third, the impact of an unlimited stream. Impact of the first sort takes place when a stream discharged from a vessel encounters a body, as, for instance, the bucket of an overshot water-wheel. Impact of the second sort occurs when the water in a canal or trough strikes against a body which entirely fills the cross-section of the trough. Finally, impact of the third kind occurs when running water strikes upon a body immersed in it, and the cross-section of the body is but a small part of that of the stream, as the float of a wheel in an open current.

We may distinguish also impact against bodies at rest and bodies in motion, against curved and plane surfaces; the latter may be either direct or oblique.

We shall now consider the general case of an isolated stream striking against a *surface of revolution*, moving in the direction of the stream, which latter coincides with the axis of the surface.

**22.—Impact of an Isolated Stream.**—Let $B\,A\,B$, Fig. 13, be a surface of revolution, $A\,B$ its axis, and $F\,A$ a stream of water moving in the direction of the axis, and impinging against it. Let us put the velocity of the water $= c$, that of the surface $= v$, and the angle $B\,T\,P$, which the tangent $D\,T$ at $B$, or which each fibre $B\,D$ of the stream which leaves the surface, makes with the axis $= \alpha$. We also assume that the water experiences no friction while passing over the surface, and hence no *vis viva* is lost from this cause. The water, now, impinges upon the surface with a velocity $c - v$, passes over the surface with that velocity, and leaves it in a tangential direction $T\,B$ with this velocity. From the velocity $B\,D = c - v$ and $B\,E = v$, we obtain the absolute velocity $B\,C = c_1$ of the water after it has impinged upon the surface, by the formula

FIG. 13.



$$c_1 = \sqrt{(c - v)^2 + v^2 + 2\,(c - v)\,v\,cos.\,\alpha.}$$

Now, a discharge $Q$ can produce by its *vis viva* a mechanical effect $\frac{c^2}{2g} Q \gamma$, if $\gamma =$ the weight of a unit of volume, and in producing this effect loses its velocity entirely. The energy remaining in the water is then $\frac{c_1^2}{2g} Q \gamma$, and that imparted to the surface is

$$P v = \frac{c^2}{2g} Q \gamma - \frac{c_1^2}{2g} Q \gamma = \frac{c^2 - c_1^2}{2g} Q \gamma$$

or

$$P v = \frac{[c^2 - (c - v)^2 - 2 (c - v) v \cos. \alpha - v^2]}{2g} Q \gamma$$

$$= \frac{2 c v - 2 v^2 - 2 (c - v) v \cos. \alpha}{2g} Q \gamma = (1 - \cos. \alpha) \frac{(c - v) v}{g} Q \gamma.$$

Therefore, the force or impulse in the direction of the axis is

$$P = (1 - \cos. \alpha) \frac{(c - v)}{g} Q \gamma.$$

If the surface moves with a velocity $v$ *opposite* to that of the water, we have

$$P = (1 - \cos. \alpha) \frac{(c + v)}{g} Q \gamma,$$

and if the surface does not move, or $v = 0$, the pressure of the stream in the direction of the axis is

$$P = (1 - \cos. \alpha) \frac{c}{g} \cdot Q \gamma.$$

Hence, we see that, other things being the same, *the impulse of a certain mass of water is proportional to the relative velocity* $c \mp v$ *of the water.*

If the area of the cross-section of the stream is $F$, the volume of the impinging water is $F (c \mp v)$; hence

$$P = (1 - \cos. \alpha) \frac{(c \mp v)^2}{g} F \gamma,$$

or for $v = 0$,

$$P = (1 - \cos. \alpha) \frac{c^2}{g} F \gamma.$$

Therefore, *if the cross-section of the stream remains the same, the pressure upon a surface at rest increases as the square of the velocity of the stream.*

**23.—Impact against a Plane Surface.**—The impulse of a stream of water depends, according to the above, upon the angle $\alpha$ at which the water moves off from the axis after impact; it is zero when this angle is $= 0$, and, on the contrary, a maximum when and $= 2 \frac{c \mp v}{g} Q \gamma$ when $\alpha$ is 180°, or when its cosine is $-1$, in which case the water quits the surface in the opposite direction to that in which it struck. In general, then, the impact is greater against concave than plane or convex surfaces, for in the first case the angle $\alpha$ is obtuse, and $1 - \cos. \alpha$ becomes $1 + \cos. \alpha$.

If the surface is plane, and $\alpha = 90°$, or $\cos. \alpha = 0$, the impulse is

$$P = \frac{c \mp v}{g} Q \gamma.$$

When the surface is at rest we have

$$P = \frac{c}{g} Q \gamma = \frac{c^2}{g} F \gamma = 2 \frac{c^2}{2g} F \gamma = 2 F h \gamma.$$

Therefore, *the normal impulse of water against a plane surface is equal to the weight of a column of water the cross-section of whose base is equal to that of the stream, and whose height is twice that due to the velocity* $\left(2h = 2\frac{c^2}{2g}\right)$.

**24.—Maximum Work Due to Impact.**—The work or mechanical effect

$$P v = (1 - \cos. \alpha) \frac{(c - v) v}{g} Q \gamma$$

depends upon the velocity $v$ of the surface upon which the water strikes. It is zero not only for $v = c$, but also for $v = 0$, hence there must be some intermediate velocity for which the work done is the greatest possible. It is evident that this velocity is that which makes $(c - v) v$ a maximum. If we put $y = (c - v) v = c v - v^2$, differentiate and put the first differential equation equal to zero, we have $\frac{dy}{dv} = c - 2v = 0$; hence, $c = 2v$ or $v = \frac{c}{2}$. The water produces its maximum mechanical effect, then, when $v = \frac{c}{2}$, or when the surface struck moves in the direction of the stream with half the velocity of the stream. The maximum work is then

$$P v = (1 - \cos. \alpha) \tfrac{1}{2} \cdot \frac{c^2}{2g} \cdot Q \gamma = (1 - \cos. \alpha) \tfrac{1}{2} Q h \gamma.$$

Now, if $\alpha = 180°$—*i. e.*, if the motion of the water is reversed by the impact, we have for the work done, $2 \cdot \frac{1}{2} Q h \gamma = Q h \gamma$; but if $\alpha = 90°$—*i. e.*, if the stream strikes against a plane surface—the work done is only $\frac{1}{2} Q h \gamma$, and the water transmits to the surface only one half of its actual energy, or but one half of the mechanical effect corresponding to its *vis viva*.

EXAMPLE 1.—If a stream of water, the area of whose cross-section is 40 square inches, delivers five cubic feet of water per second, and strikes normally against a plane surface, which moves away with a velocity of 12 ft., the impulse is

$$P = \frac{(c - v)}{g} Q \gamma = \left(\frac{5 \cdot 144}{40} - 12\right) 0 \cdot 031 \times 5 \times 62 \cdot 5$$

$$= 6 \times 0 \cdot 031 \times 312 \cdot 5 = 58 \cdot 125 \text{ lbs.},$$

and the work performed is $P v = 58 \cdot 125 \times 12 = 697 \cdot 5$ ft. lbs. The maximum effect is when $v = \frac{c}{2} = \frac{1}{2} \frac{5 \times 144}{40} = 9$ ft., and is equal to

$$\frac{1}{2} \frac{c^2}{2 g} Q \gamma = \frac{1}{2} \times 18^2 \times 0 \cdot 0155 \times 5 \times 62 \cdot 5$$

$$= 81 \times 0 \cdot 155 \times 62 \cdot 5 = 784 \cdot 6875 \text{ ft. lbs.}$$

The corresponding impulse or hydraulic pressure is $\frac{784 \cdot 6875}{9} = 87 \cdot 19$ lbs.

2. If a stream, the area of whose cross-section is 64 square inches, impinges with a velocity of 40 feet upon an immovable cone whose angle of convergence is 100°, the hydraulic pressure in the direction of the stream is

$$P = (1 - cos.\ \alpha) \frac{c}{g} Q \gamma,$$

or $$= (1 - cos.\ 50^\circ) \times 40 \times 0 \cdot 031 \times \frac{64}{144} \times 40 \times 62 \cdot 5$$

$$= (1 - 0 \cdot 64279) \times 1 \cdot 24 \times \frac{10000}{9} = 0 \cdot 35721 \times 1377 \cdot 8 = 492 \cdot 16 \text{ lbs.}$$

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END OF PART FIRST.

# SECTION II.

## APPLICATION OF MECHANICS TO MACHINES.

### INTRODUCTION.

§ **108.—Machines.**—All artificial arrangements by which forces are enabled to perform work may be called *machines* (Fr. maschines; Ger. Maschinen). They differ, then, from *structures* (Fr. constructions; Ger. Bauwerke), in that these latter are designed merely to hold certain forces in equilibrium. *Tools* or *instruments* (Fr. instruments; Ger. Werkzeuge) are, then, essentially machines; they serve in general merely for the performance of a small amount of work by hand-power.

In every machine, we must distinguish between the *power* and the *resistance.* The *power* (Fr. force; Ger. Kraft) is the cause of the motion, and the *resistance* (Fr. résistance; Ger. Widerstand) is that which opposes the motion, to overcome which is the object of the machine. Those powers which are made use of to produce motion of the machine are called *motors* (Fr. moteurs; Ger. Beweger). These motive powers or motors are the various forces of nature, such as the physical force of men and animals, inertia, elasticity, gravity, expansive force of heat, etc. The resistance is to be overcome while changing the form of a body, by compression or division, etc., or while moving the body from one place to another.

In every machine, we may distinguish three principal parts: the point at which the motive power is applied, or the *receiving point* (Fr. récepteur; Ger. Umtriebsmaschine); the point at which the resistance is directly overcome, or the work performed, or the *working point* (Fr. opérateur; Ger. Arbeitsmaschine); the part intermediate between these which transmits or communicates the motion of the receiving point to the working point, with such change of direction or velocity, or both, as may be desired to accomplish the end in view. We may call this then, the *transmitter* (Fr. communicateur; Ger. Zwischenmaschine). In the ordinary flour mill, for instance, considered as a machine, the water-wheel receives the motive power, the millstones do the work, and between the two a train of gearing transmits the motion of the wheel to the stones with the required changes of direction and velocity.

REMARK.—These three divisions may not always be evident. Thus the transmitter may be wanting, because the receiving point may itself have precisely the motion which is necessary for the performance of the work desired. In the ordinary wheelbarrow, these three parts are united; the handles are the receiving parts, as also the transmitters, and the body the working part, yet all three constitute only a single body.

[We may then define a machine as a contrivance or collection of bodies, intended to receive at certain points certain forces, and to exert at other points other forces, which generally differ from the first in their intensity, direction, and points of application. We may thus regard a machine as a transmitter of power. It would, perhaps, be better to regard a machine as simply a *transmitter of motion*, instead of power. A machine which performs work or transmits power, we may call an *engine*, while to communicators of motion generally we may apply the term, *mechanism*. (See Willis's "Principles of Mechanism.")]

§ **109.—Mechanical Effect.**—The *mechanical effect* or *delivery* (Fr. effet; Ger. Wirkung, Leistung) of a machine is measured by the *work* performed by it in a given time, as one minute or one second (see Vol. I., Art. 71), or by the product of the resistance at the working point, into the velocity or distance passed over in one unit of time by that point. Thus, if $P$ is the force, and $s$ the distance passed through in one second, we have as the measure of the mechanical effect or delivery of the machine: $L = P\ s$ ft. lbs., if $P$ is measured in pounds and $s$ in feet. [*Work* is simply pressure or force into distance, without regard to time. The work performed is then the same, whether the time occupied be one second, one minute, or one hour, provided that in each case the force and distance passed over are equal. The mechanical effect or delivery, however, is *work performed in a unit of time.* The total work performed by two machines may then be the same; but if one takes only half the time to do it in, its mechanical effect or delivery is twice as great—*i. e.*, at the same rate, *in the same time*, it would perform twice the work of the other. In comparing machines, we must compare the *rate of work*, or the work of each, in equal times.]

It is usual to take a greater unit than the foot-pound per second, as a more convenient unit of measurement for machines. This unit is entirely arbitrary, and is called a *horse-power* (Fr. cheval-vapeur ; Ger. Pferdekraft). It is taken as 550 ft. lbs. per second, English measure, or 33,000 ft. lbs. per minute. The horse-power is, in French measure, 75 *kilogrammes* raised 1 *metre* in one second, or 75 *kilogrammetres* per second, or 4500 *kilogrammetres* per minute. [The metre may be taken as 3·281 ft., the kilogramme as 2·2 lbs., approximately. The French horse-power is then only 542½ ft. lbs., English measure.]

It is also necessary to distinguish the *useful effect*, the *lost effect*, and the *total effect*, or delivery. The *useful effect* (Fr. effet utile ; Ger. Nutzleistung) is the work actually done. The *lost effect* (Fr. effet perdu ; Ger. Nebenleistung) is the work consumed by the resistances inherent in the machine, such as friction, shocks, etc. The total effect (Fr. effet total ; Ger. Totalleistung) is the sum of the preceding ; the work inherent in the moving power, or the work of the motor, which takes effect at the receiving point. The more perfect a machine, the less the ratio of the lost effect to the useful or total effect, or the greater the useful effect as compared with the total—*i. e.*, the less the work consumed by the resistances of friction, etc. We may, therefore, take the ratio of the useful to the total effect as a measure of the perfection of a machine. This ratio we call the *efficiency* (Fr. rendement; Ger. Wirkungsgrad). If $L$ is the total effect, $L_1$ the useful, and $L_2$ the lost effect, then the efficiency is

$$\eta = \frac{L_1}{L} = \frac{L - L_2}{L}.$$

A machine is, therefore, more perfect the nearer this ratio approaches to unity. Since the resistances inherent in the machine, such as friction, resistance of the air, etc., or the so-called prejudicial resistances, can never be reduced to zero, $L_2$ must always have a certain value, and it is impossible for the efficiency to reach unity. The efficiency of a machine is, therefore, always less than unity, and approaches nearer to unity the more perfect the machine.

Example.—A stamp-mill consists of 20 stampers, each weighing 250 lbs., and each is raised 40 times per minute to a height of 1 foot. The machine which performs this work is a water-wheel, using 260 cubic feet of water per minute, with a fall of 20 feet. Required the efficiency of the wheel.

The work actually performed, or the useful effect per second, is—

$$L_1 = \frac{20 \times 250 \times 40 \times 1}{60} = 3333\tfrac{1}{3} \text{ ft. lbs.} = 6 \cdot 06 \text{ horse-power.}$$

Since, however, in each second $\frac{260}{60}$ cubic feet $= \frac{260 \times 62 \cdot 5}{60} = 270 \cdot 8$ lbs. of water fall through 20 feet, we have for the total effect, $L = 270 \cdot 8 \times 20 = 5416$ ft. lbs. per second, or 9·85 horse-power. The lost effect is, therefore,

$$L_2 = 5416 - 3333\tfrac{1}{3} = 2082\tfrac{2}{3} \text{ ft. lbs.} = 3 \cdot 78 \text{ horse-power.}$$

The efficiency of the machine, then, is,

$$\eta = \frac{L_1}{L} = \frac{3333\frac{1}{3}}{5416} = 0 \cdot 615.$$

Remark.—For "horse-power," see a treatise by Reuleaux, "Civilingenieur," Band III.

§ **110.—Useful and prejudicial Resistance.**—We may likewise divide the resistance overcome by the machine, into useful and prejudicial; since, however, the power, useful resistance, and prejudicial resistance act in general at different points, we cannot directly place the power equal to the sum of the other two, without a preliminary reduction. This reduction may be made either by means of the spaces simultaneously passed through at the different points, or by means of the lever arms of the resistances.

If the force $P$ acts through the space $s$, while the useful resistance $P_1$ acts through the space $s_1$ and the prejudicial resistance $P_2$ through the space $s_2$, we have

$$P s = P_1 s_1 + P_2 s_2 \text{, hence } P = \frac{s_1}{s} P_1 + \frac{s_2}{s} P_2.$$

The point at which $P$ acts or can be conceived as applied is the receiving point, and the point at which the resistance $(P_1 + P_2)$ acts is the working point. We have in

$$\frac{s_1}{s} P_1,$$

then, the useful resistance reduced to the receiving point, and in

$$\frac{s_2}{s} P_2$$

the prejudicial resistance reduced to the receiving point. *The power is, therefore, equal to the sum of the useful and prejudicial resistances, reduced to the point of application of the power.* We have also

$$P_1 = \frac{s}{s_1} P - \frac{s_2}{s_1} P_2,$$

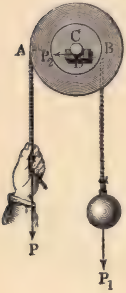
or, *the useful resistance is the difference of the power and prejudicial resistances reduced to the point of application of the useful resistance.*

We can therefore put for the efficiency of a machine,

$$\eta = \frac{P_1 s_1}{P s} = \frac{s_1}{s} P_1 \div P = P_1 \div \frac{s}{s_1} P,$$

*i. e.*, the efficiency is equal to the quotient of the useful resistance reduced to the power point divided by the power, or to the quotient of the useful resistance divided by the power reduced to the point of application of the useful resistance.

FIG. 211.



Most machines are combinations of the wheel and axle (Vol. I., Art. 165), and hence the reduction may be made by means of the lever arms. In the wheel and axle $A B C$, Fig. 211, let the radius $C A = a$, radius of drum $C B = b$; then for the moment of the power $P$, we have $P a$, and for that of the useful resistance, $P_1 b$. Hence the useful resistance reduced to the power point $A$ is

$$\frac{P_1 b}{a} = \frac{b}{a} P_1,$$

and the power reduced to the working point $B$ is

$$\frac{P a}{b} = \frac{a}{b} P.$$

If the prejudicial resistance $P_2$ consists only of the axle friction $\phi (P + P_1 + G)$, and if $r$ is the radius $C D$ of the axle, we have for the moment of $P_2$, $P_2 r$, and hence the prejudicial resistance reduced to the power point is

$$\frac{P_2 r}{a} = \frac{\phi r}{a} (P + P_1 + G),$$

while the prejudicial resistance reduced to the working point is

$$\frac{P_2 r}{b} = \frac{\phi r}{b} (P + P_1 + G).$$

We have then for the power,

$$P = \frac{b}{a} P_1 + \frac{\phi r}{a} (P + P_1 + G),$$

and for the useful resistance,

$$P_1 = \frac{a}{b} P - \frac{\phi r}{b} (P + P_1 + G).$$

Finally, we have for the efficiency,

$$\eta = \frac{b}{a} P_1 \div P = P_1 \div \frac{a}{b} P = \frac{P_1 b}{P a}.$$

EXAMPLE.—A wheel and drum weigh 250 lbs. The radius of the wheel is 30 inches, that of the drum 6 inches, and that of the axle ½ an inch. The useful resistance is 500 lbs.; the coefficient of friction is $\frac{1}{10}$.

For the useful resistance reduced to power point, we have then

$$\frac{b}{a} P_1 = \frac{6}{30} 500 = 100 \text{ lbs.}$$

For the prejudicial resistance reduced to power point

$$\frac{\phi r}{a} (P + P_1 + G) = \frac{1}{10} \times \frac{1}{2 \times 30} (750 + P) = \frac{5}{4} + \frac{P}{600}.$$

Hence,

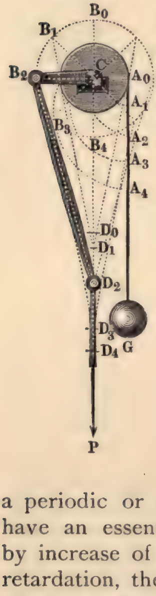
$$P = 100 + \frac{5}{4} + \frac{P}{600} \text{ or } P = 101 \cdot 25 \times \frac{600}{599} = 101 \cdot 42 \text{ lbs.},$$

and the efficiency is

$$\eta = \frac{100}{101 \cdot 42} = 0 \cdot 986.$$

§ **III.—Working Condition.**—After a machine is set in motion, it settles down very soon to its *working condition*—*i. e.*, there is a regular and periodic recurrence of the same state at equal intervals. In this condition we generally assume machines to be, but this working condition may be either uniform or variable. If all the parts of a machine move uniformly, the working condition is *uniform ;* if, however, within any period these move not uniformly, the working condition is a variable one. The causes of the latter condition are: variation of the power, of the resistance, or of the mass of the machine, as also the variations in the spaces described by the power and resistance in equal times, due to the combination of moving parts. In a steam-engine the power is variable when the steam acts expansively—*i. e.*, when the steam is cut off during the action of the piston. In the forge hammer, power and mass are variable, because the hammer, when falling, is *out of gear*, or disconnected from the machine. Both machines have then a variable or periodic working condition. If, now, both machines are united, and the hammer is raised by the steam-engine, the condition is variable for three reasons. When a weight $G$, Fig. 212, is raised by means of a wheel, $C A_0$, and a crank, $C B_2$, operated by a steam-engine with *uniform* steam pressure, the machine has still a variable condition, because equal distances, $A_0 A_1$, $A_1 A_2$, $A_2 A_3$, $A_3 A_4$, of the resistance correspond to very unequal distances, $D_0 D_1$, $D_1 D_2$, $D_2 D_3$, $D_3 D_4$, described by the power, and hence the ratio of speed of power to resistance during a half revolution is variable. The action is repeated at equal intervals, and the condition is periodic.

FIG. 212.



In uniform working condition, the inertia of the parts of the machine is without influence upon its action, because only at first, while the machine is still accelerating its motion, does the inertia absorb work. Later, when uniform motion is established, there is neither loss nor gain of mechanical effect (Vol. I., Art. 55). If, on the other hand, a machine has a periodic or variable motion, the inertia of the parts have an essential influence upon the motion, because by increase of speed they absorb work, which, for equal retardation, they give out again. If $M$ is the sum of all the masses reduced to the point of application of the power or resistance, $v_1$ and $v_2$ the minimum and maximum velocities of one of these points, we have for the work which the inertia of the masses required during the transition from $v_1$ to $v_2$, and which they perform while passing from $v_2$ to $v_1$,

$$\frac{v_2^2 - v_1^2}{2} M.$$

Accordingly, by reason of inertia, the resistance is alternately increased and diminished by the above amount during each change, and hence the total effect for the entire period, or the mean effect, is the same as if these masses were not there. Therefore the formula

$$P s = P_1 s_1 + P_2 s_2$$

holds good generally, if, for variable working condition, we understand by $s$, $s_1$, $s_2$, the spaces described during a complete period, or if for $P$, $P_1$, $P_2$, we substitute the mean values of power, useful and prejudicial resistances. For accelerating motion,

$$P s = P_1 s_1, + P_2 s_2, + \left(\frac{v_2^2 - v_1^2}{2}\right) M,$$

hence,

$$v_2 - v_1 = \frac{P s - (P_1 s_1 + P_2 s_2)}{\left(\frac{v_2 + v_1}{2}\right) M.}$$

This formula shows that the variations of velocity of a machine are not only less, the less the difference between the work of the power and the sum of the works of the resistances, but also the greater the mass and the velocities of the various parts of the machine.

Remark.—It does not follow because the mass of the parts affects only the motion but not the efficiency, that therefore it is a matter of indifference whether the parts of a machine have more or less mass. Changes in velocity, as also greater weight of parts, increase often the prejudicial resistances, such as friction, etc., cause disturbing vibrations, and often cause shocks. Many machines also furnish a poorer product by reason of variable motion, etc. It is thus often necessary to apply means to increase the uniformity of motion of the machine. When a machine or part of a machine must pass from rest into motion, and from motion to rest, we must seek to attain not a uniform motion, but such a variable one that the velocity may increase gradually from zero up to a certain maximum, and gradually decrease again to zero; because sudden changes of velocity will cause vibrations and shocks accompanied by loss of work (Vol. I., Art. 335).

# PART FIRST.

## THE MOTIVE POWER OF ANIMALS, WATER, AND WIND, AS ALSO MACHINES FOR THEIR UTILIZATION.

## CHAPTER I.

### THE MEASUREMENT OF MOTIVE POWERS AND THEIR EFFECTS.

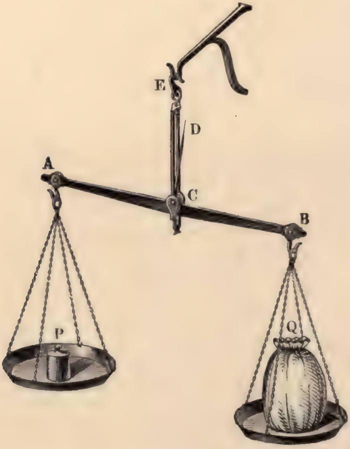
§ **112.—Dynamometer.**—In order to estimate the mechanical effect of powers and machines, three elements are necessary: *the magnitude of the power, the distance passed over by it*, and its corresponding *time of action.* Dynamometers serve to measure the power, chains and measuring lines and rods the distance, and the pendulum or chronometer the time. If $P$ is the magnitude of the power as given by the dynamometer, $s$ the distance passed over by it in the time $t$, we have for the work performed during this time, $Ps$, and hence the mechanical effect or work per second is

$$L = \frac{Ps}{t}.$$

We shall treat here only of dynamometers, of which there are three kinds: *weight*, *spring*, and *friction* dynamometers. Weight and spring dynamometers are not essentially different from weight and spring balances. While the last serve more especially for weighing or measuring the weight of bodies, the first are applied to the measure of power generally. The friction brake or dynamometer finds its application only in determining the mechanical effect of a revolving axle.

Weight dynamometers or balances are simple or compound levers, by which the force or weight to be measured is held in equilibrium by known weights. Spring dynamometers are steel springs which measure the magnitude of the force acting upon them by their change of form as indicated by a suitably attached index. Balances have either equal or unequal arms, and these last are simple or compound according as they consist of one or more unequal armed levers.

FIG. 213.



§ **113.—Common Balance.**—The common balance is essentially a lever of equal arms *A B*, Fig. 213, by which the weight $Q$ to be measured is held in equilibrium by the equal known weight $P$. We may distinguish in it the *balance beam* $A\ B$ (Fr. fléau; Ger. Wagebalken), the *index* $C\ D$ (Fr. aiguille; Ger. Zunge), the *fork* $C\ E$ (Fr. support; Ger. Scheere), the *axis* $C$ (Fr. and Ger. axe), formed of a three-sided prism or knife edge, and the *scales* (Fr. bassins; Ger. Wagschalen), suspended from the balance beam by chains, hooks, etc.

In such a balance the requisites are: that it shall take a horizontal position, and the index a vertical position and coincide with the fork, when the weights in the scales are equal, and only then; secondly, it should possess *sensitiveness* and *stability*—that is, it must incline promptly for a very slight difference of weight in either scale, and must quickly and readily recover its horizontal position when the weights are made equal. In order that a balance with equal weights in the scales may be in adjustment, the arms must be precisely of equal length. If $a$ is the length of one and $b$ of the other, $P$ the weight in one pan and $Q$ that in the other, then for a horizontal position we have $P\,a = Q\,b$. If, however, we exchange the weights, then for a horizontal position of the beam we must also have $P\,b = Q\,a$. From both equations we have $P^2\,a\,b = Q^2\,a\,b$, or $P = Q$, and hence also $a = b$.

If, therefore, by transposing the weights, equilibrium is not disturbed, it is a proof of the truth of the balance. This test may also be made as follows: If we bring, one after the other, two weights, $P$, $P$, in equilibrium with a third, $Q$, in the other pan, these two weights must be equal each to each, though not necessarily to $Q$, unless the balance is true. If, then, removing $Q$, we lay these two weights in opposite scales, we shall have, in case equilibrium is undisturbed, $P\,a = P\,b$, and hence $a = b$. Thus the horizontality of the beam under equal weights is a direct proof of the accuracy of the balance. Small inaccuracies can be corrected by adjusting screws $K$, $L$, as shown in Fig. 214.

If a balance gives for the same body the weights $P$ and

$Q$, according as it is placed in one or the other pan, then we have for the true weight $X$:

$$X a = P b \quad \text{and} \quad X b = Q a,$$

hence

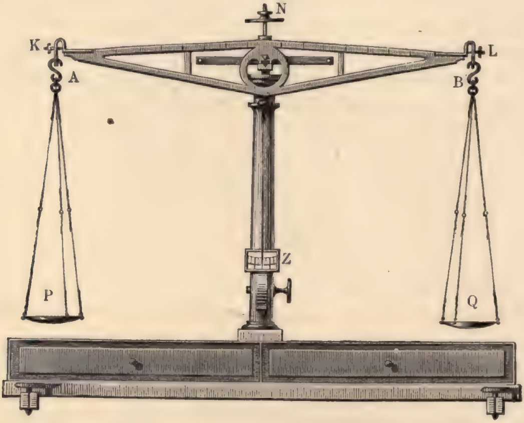
$$X^2 a b = P Q a b,$$

or

$$X^2 = P Q \quad \text{and} \quad X = \sqrt{P Q}.$$

*The true weight is therefore a geometrical mean between the two observed values.*

FIG. 214.



We can also put

$$X = \sqrt{P(P + Q - P)} = P\sqrt{1 + \frac{Q - P}{P}},$$

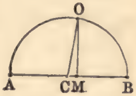
or approximately, if, as is usually the case, the difference $Q - P$ is small compared with $P$,

$$X = P\left(1 + \frac{Q - P}{2P}\right) = \frac{P + Q}{2}.$$

We may therefore take the true weight *as the arithmetical mean of the two observed weights.*

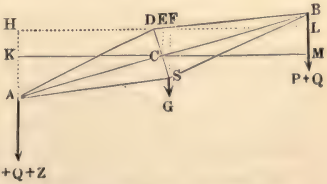
If upon the sum $\overline{AB}$ of $\overline{AM} = P$ and $\overline{BM} = Q$, we describe a semicircle $A O B$, Fig. 215, the radius $\overline{CA} = CB = CO$ represents the approximate value $\frac{P + Q}{2}$, and the ordinate $\overline{MO}$ the exact value $\sqrt{PQ}$ of the weight $X$. If also we cause equilibrium by placing sand, shot, etc., in one scale, the body being in the other, then remove the body and replace it by known weights till equilibrium is restored, these weights will give the true weight of the body.

FIG. 215.



§ **114.—Sensitiveness of the Balance.**—In order that the balance may move as freely as possible, and especially not be impeded by the axle friction, the fulcrum consists of a three-sided prism of steel or *knife edge*, which rests upon hardened steel plates, or upon agate. In order, further, that the direction of the resultant of the loaded or unloaded scale may pass through the point of suspension without deviation by friction, and therefore that the lever arm may be constant, it is necessary to hang the scales also upon knife edges. When, then, such a balance is loaded, we may assume that the weights act directly at the points of suspension, and that hence the point of application of the resultant of the weights is in the line joining their points of suspension. Since, according to Vol. I., Art. 131, a suspended body is in stable equilibrium only when the centre of gravity is below the centre of suspension, it is evident that the fulcrum $D$, Fig. 216, of the balance should always be above the centre of gravity $S$ of the unloaded beam, and also not below the line $A B$ through the points of suspension. For the sake of generality, we shall, therefore, assume in what follows that the fulcrum $D$ is above, and the centre of gravity $S$ below the line $A B$.

FIG. 216.



The deviation of the beam from horizontality for a given load in one pan, is the measure of the sensitiveness of the balance, and we have then to investigate the dependence of this upon the difference of the weights in the two scales. Thus we represent the length of the arms $C A$ and $C B$ by $l$, the distance $C D$ of the fulcrum from the line $A B$ through the points of suspension of the scales by $a$, the distance $S D$ of the centre of gravity from the fulcrum by $s$, the angle of deviation from the horizontal by $\phi$, the weight of the unloaded beam by $G$, the weight upon one end $P$, and upon the other $P + Z$, $Z$ being thus the difference of the weights applied, and finally the weight of one scale, together with its suspending chains, hooks, etc., by $Q$. We have then for the statical moment upon one side of the beam,

$$(P + Q + Z)\,\overline{D H} = (P + Q + Z)\,(\overline{C K} - \overline{D E})$$
$$= (P + Q + Z)\,(l \cos. \phi - a \sin. \phi),$$

and upon the other side,

$$(P + Q)\,\overline{D L} + G \times \overline{D F} = (P + Q)\,(C M + D E) + G \times \overline{D F}$$
$$= (P + Q)\,(l \cos. \phi + a \sin. \phi) + G\, s \sin. \phi\,;$$

therefore, for equilibrium,

$$(P + Q + Z)\,(l \cos. \phi - a \sin. \phi) = (P + Q)\,(l \cos. \phi + a \sin. \phi) + G\, s \sin. \phi,$$

or introducing *tan.* $\phi$ and transforming

$$(\,[2\,(P + Q) + Z]\,a + G\,s)\tan. \phi = Z\,l,$$

therefore,

$$\tan. \phi = \frac{Z\,l}{[2\,(P + Q) + Z]\,a + G\,s}.$$

This expression shows that the angle $\phi$, and therefore the sensitiveness of the balance, increases directly as the length of the arms and the difference of the weights, and diminishes as the weights $P$, $Q$, $G$, $Z$, and the distances $a$ and $s$ increase. A heavy balance is then less sensitive than a light one, other things being the same, and its sensitiveness diminishes the greater the applied weights. In order, then, to increase the sensitiveness of a balance, we must bring the line $A\,B$ and the centre of gravity $S$ near the fulcrum $D$. If $a$ and $s$ were zero, or $D$ and $S$ in the line $A\,B$, we would have

$$tan.\,\phi = \frac{Z\,l}{0} = \infty, \qquad \text{or} \qquad \phi = 90^\circ.$$

In such a case, the least difference would cause a deviation of 90°. For $Z = 0$ we should also have

$$tan.\,\phi = \frac{0}{0}$$

that is, under equal applied weights the balance would remain at rest in any position, or is in indifferent equilibrium, and therefore practically worthless. If $a$ alone is zero, or if the fulcrum $D$ is in the line $A\,B$, we have

$$tan.\,\phi = \frac{Z\,l}{G\,s};$$

*i. e.*, the sensitiveness in this case, friction being neglected. is independent of the amount weighed, and the balance is especially useful. By means of a counter-weight $N$, as shown in Fig. 214, we can regulate the sensitiveness.

§ **115.—Stability and Motion of the Balance.**—The stability or the statical moment $S$ which tends to cause an equally loaded balance to return to equilibrium when it has been inclined an angle $\phi$ is given by

$$S = 2\,(P + Q)\,\overline{D\,E} + G \times \overline{D\,F} = [2\,(P + Q)\,a + G\,s]\,sin.\,\phi.$$

The measure of the stability $[2\,(P + Q)\,a + G\,s]$ increases then with the weights $P$, $Q$, and $G$, and with the distances $a$ and $s$, but is independent of the length of the beam.

A vibrating balance may be compared to a *pendulum*, and the time of vibration computed according to the theory of the latter. Thus, the statical moment with reference to $D$ is $2\,(P+Q)\,a$, and the moment of inertia of the loaded scales is $2\,(P+Q)\,\overline{A\,D}^2 = 2\,(P+Q)\,(l^2+a^2)$; also $G\,s$ is the statical moment of the unloaded beam. If we put the moment of inertia of the unloaded beam $= G\,k^2$, we have for the length of the pendulum isochronous with the balance (Vol. I., Art. 327),

$$r = \frac{2\,(P+Q)\,(l^2+a^2)+G\,k^2}{2\,(P+Q)\,a+G\,s},$$

and hence the time of vibration is

$$t = \pi\sqrt{\frac{2\,(P+Q)\,(l^2+a^2)+G\,k^2}{g\,[2\,(P+Q)\,a+G\,s]}},$$

or, when $a$ is very small or equal to zero,

$$t = \pi\sqrt{\frac{2\,(P+Q)\,l^2+G\,k^2}{g\,G\,s}}.$$

We see, therefore, that the time of vibration increases with $P$, $Q$, and $l$, and as $a$ and $s$ decrease. For equal weights, then, a balance vibrates more slowly the more sensitive it is. Weighing with such a balance is thus a slower process than with one less sensitive. For this reason, it is often useful to furnish sensitive balances with a scale and index. In order to interpret the indication of such a scale, let $Z = 0$ in the denominator of the formula,

$$tan.\ \phi = \frac{Z\,l}{[2\,(P+Q)+Z]\,a+G\,s},$$

and put $\phi$ in place of $tang.\ \phi$. We have then

$$\phi = \frac{Z\,l}{2\,(P+Q)\,a+G\,s}.$$

If, now, we put $Z_1$ and $\phi_1$ for $Z$ and $\phi$, we have

$$\phi_1 = \frac{Z_1\,l}{2\,(P+Q)\,a+G\,s},$$

hence,

$$\phi : \phi_1 = Z : Z_1.$$

Therefore, *for small difference of the weights, the angle of deviation is proportional to that difference.* Accordingly

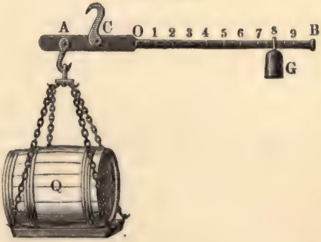
$$\phi : \phi_1 - \phi = Z : Z_1 - Z \qquad \text{or} \qquad Z = \frac{\phi}{\phi_1 - \phi}(Z_1 - Z).$$

We can then find the difference of weights corresponding to a deviation $\phi$ by trying how much the deviation is increased when the difference of the weights is increased by a given small quantity, and then multiplying this increase $Z_1 - Z$ by the ratio of the first deviation to the increase of the same.

REMARK.—Equal armed balances occur of all dimensions and of various degrees of perfection. The most ordinary balance used in trade is shown in Fig. 213. The finest are assay balances and such as are used for chemical or physical purposes, as shown in Fig. 214. Such balances will bear but little more than a pound, but will turn and indicate $\frac{1}{50}$ of a grain or $\frac{1}{350000}$ of a pound. The finest balances show even the millionth of the load, but are used only for very light weights and extremely delicate work. If the beam is graduated and carries a fine wire or *rider*, then, by changing its position, we can determine small differences without very small weights. Large balances which will weigh tons may also be constructed with a high degree of sensitiveness, especially if made light, the beam of wood, etc. (See Lardner's and Kater's "Mechanics.")

§ **116.—Unequal armed Balances.**—Of these there are three kinds, the *steelyard* with sliding counter-weight, steelyard with proportional weight, and steelyard with fixed weight. The steelyard with sliding weight (Fr. balance romaine; Ger. Schnellwage mit Laufgewicht), Fig. 217, consists of a lever, *A B*, from whose shorter arm, *C A*, a scale pan is suspended, and furnished with a movable counterpoise *G* upon the longer graduated arm *C B*, which holds in equilibrium the body to be weighed at *Q*. If $l_0$ is the lever arm *C O* of the weight *G* when the unloaded balance is horizontal, we have for the statical moment of the empty scale $X_0 = G\,l_0$.

FIG. 217.



If, on the other hand, $l_n$ is the lever arm $C\,G$, when the sliding weight $G$ balances the loaded scale, we have for the statical moment of the scale in this case,

$$X_n = G\,l_n;$$

and hence, by subtraction, we have for the moment of the weight $Q$,

$$X_n - X_0 = G\,(l_n - l_0) = G \times \overline{O\,G}.$$

If, now, we denote by $a$ the lever arm $C\,A$ of the load, and by $x$ the distance $\overline{O\,G}$ of the counterpoise from $O$, we have,

$$Q \times a = G \times x,$$

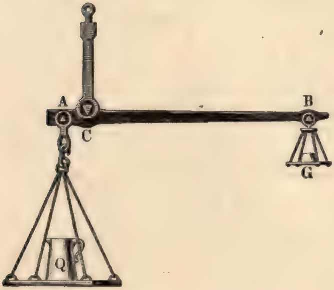
hence for the load itself

$$Q = \frac{G}{a}\,x.$$

The load, therefore, is proportional to the distance $x$ of the counterpoise from the point $O$, at which point the counterpoise balances the empty scale. If $x$ is doubled, $Q$ must be doubled; if $x$ is three times as great, $Q$ is three times as great, etc. The graduated scale upon $O\,B$ is therefore one of equal divisions with zero at the point $O$. The unit of the scale is to be found by observing what weight $Q_n$ will be balanced by the sliding weight at the extremity $B$. $Q_n$ is then the number of divisions, and $\frac{O\,B}{Q_n}$ is the unit of division for the scale along $O\,B$. Thus, for example, if when the counterpoise is at $B$ the load $Q = 100$ pounds, we have to divide $O\,B$ into 100 parts, and hence the distance for one pound or the unit of the scale is $\frac{O\,B}{100}$. If, now, for another load the counterpoise must be placed at a distance $x = 80$ by scale, then $Q = 80$ lbs.; if the counterpoise is at 53, the load $Q = 53$ lbs., and so on.

In the steelyard with proportional weights, Fig. 218, the body to be weighed is suspended from the extremity of the shorter arm $C A = a$, and the counter-weights from the extremity of the longer arm $C B = b$. The ratio $\frac{C B}{C A} = \frac{b}{a}$ is usually a simple one, as 10 to 1, for example, in which case the balance is a *decimal* one. The balance being first brought to horizontality by a certain counterpoise at $B$, then for any load $Q$ we have

FIG. 218.



$$Q a = G b$$

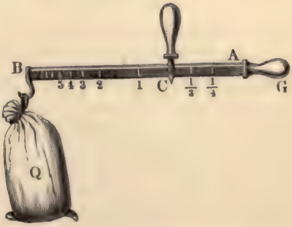
where $G$ is the weight added to restore equilibrium, or

$$Q = \frac{b}{a} G.$$

The weight of the body is found, therefore, by multiplying the known weight $G$ by the known number ratio $\frac{b}{a}$, which for the decimal balance is 10—*i. e.*, the body is 10 times the applied counter-weight.

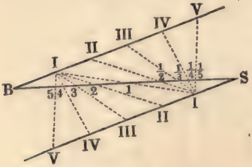
The steelyard with fixed weight, called the Danish balance, Fig. 219, has a movable fulcrum, $C$, furnished with a handle, and the beam itself is slid over this fulcrum till equilibrium exists between its weight and that of the body to be weighed. The division of the beam is an unequal one, as will appear in the following remark.

FIG. 219.



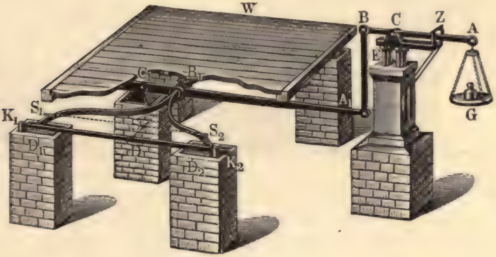
REMARK.—In order to determine the division of the beam for the Danish balance, Fig. 220, draw through its centre of gravity $S$ and the point of suspension $B$ two parallel lines, and set off on these from $S$ and $B$ equal divisions, and unite the first point from $B$ or I with each of the points I, II, III, etc., upon the line through $S$, also join the first point from $S$ or I with each of the points I, II, etc., upon the line through $B$. The intersections of these last lines with the axis $B\ S$ give the required points of division. The point of intersection 1 of the line II lies half way between $B$ and $S$. When the fulcrum is at that point, therefore, and equilibrium exists, the weight $Q$ is equal to the weight $G$ of the steelyard. The point of intersection 2 of the line I, II, is as far again from $S$ as from $B$. When there is equilibrium and the fulcrum is at 2, then we have $Q = 2\ G$. In similar manner for fulcrum at 3 we have $Q = 3\ G$, and so on. It is also easily seen that for fulcrum at the points $\frac{1}{2}$, $\frac{1}{3}$, etc., we have, when equilibrium subsists, $Q$, $\frac{1}{2}\ G$, $\frac{1}{3}\ G$, etc. The points of division for large loads are, therefore, close together, and for small ones farther apart, and the sensitiveness of the balance is hence very variable.

FIG. 220.



§ **117.—Platform Scales.**—Compound balances consist of two, three, or more levers or balance beams. To these belong weighing tables, platform scales, toll scales, etc. They serve for the weighing of great weights, and are therefore usually *proportional* balances. The scale pan for the load is here replaced by a large platform, which must be so supported and connected with the levers that the body to be weighed may be received and removed as conveniently as possible, and that the indications of the scale may be independent of the position of the body on the platform. An excellent platform scale (Fr. balance à bascule; Ger. Brückenwage) is shown in Fig. 221, as made by Schwilgue in Strasbourg. It consists of a two-armed lever, $A\ C\ B$, of a simple one-armed lever, $A_1\ B_1\ C_1$, and of two fork-shaped single-armed levers, $B_1\ S_1\ D\ S_2$, etc. These levers have their fulcrums at $C$, $C_1$, $D_1$ and $D_2$. Only a part of the platform $W$ is shown, and only one of the fork-shaped levers is visible. The platform rests ordinarily on the four bolts $K_1$, $K_2$, etc., but during the operation of weighing is supported upon the four knife edges, $S_1$, $S_2$, etc., which are attached to the fork-shaped levers. In order to effect this, the supports $E$ of the beam $A\ B$ can be moved up and down by a rack and pinion (not shown in the figure). When the load is brought on to the platform, the support $C$ is raised, weights are placed in the pan $G$, and after equilibrium is produced the platform is lowered to its bearings $K_1$, $K_2$ again, and the load removed.

FIG. **221.**

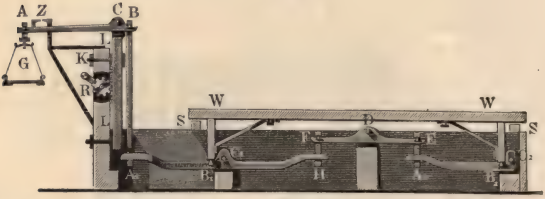


Ordinarily, the ratio of the lever arms $\frac{C\ A}{C\ B} = 2$, the ratio $\frac{C_1\ A_1}{C_1\ B_1} = 5$, and the ratio $\frac{D\ B_1}{D\ S} = 10$. For a weight in the scale pan, therefore, the force at $B$ or $A_1$ is $2\ G$, the force at $B_1 = 5$ times that at $A_1$, or $= 10\ G$, and finally the force at $S = 10$ times the force at $B_1$, or $= 100\ G$. When in equilibrium then, the load is 100 times as great as the counterweight $G$, and the balance is a *centesimal* or 100-fold proportional balance.

Another platform scale made by W. Becker, in Strasbourg, is shown in Fig. 222. The platform $W$ of this balance rests by means of four columns upon supports at $B_1$ $B_2$, etc., on the fork-shaped single-armed levers, $A_1\ B_1\ C_1$, $A_2\ B_2\ C_2$, the last of which is united by an equal-armed lever, $D\ E\ F$, with a prolongation $C_1\ H$ of the first. Before the operation of weighing, the platform rests upon the supports $S$, $S$, but when the load is brought on, the support

*L L* of the fulcrum *C* is raised, together with the entire system of levers, by means of the rack and pinion at *K R*, and then weights are placed in the scale pan *G* till equilibrium exists. Wherever the load $Q$ may be placed on the platform, the sum of the forces at $B_1$, $B_2$, etc., is equal to the load. But the ratio $\frac{C_2\,A_2}{C_2\,B_2}$ of the lever arms is equal to the ratio $\frac{C_1\,A_1}{C_1\,B_1} = \frac{a_1}{b_1}$; also $D\,E = D\,F$, as also $C_1\,H = C_1\,A_1$. It is therefore a matter of indifference whether a portion of the load $Q$ is borne by $B_2$ or directly by $B_1$; or the conditions of equilibrium of the lever $C_1\,B_1\,A_1$ are the same, whether the entire load $Q$ is directly supported at $B_1$, or only a portion at $B_1$, and the rest, at $B_2$, is transmitted by the levers $C_2\,B_2\,A_2$, $E\,D\,F$ and $C_1\,H$ to $C_1\,B_1\,A_1$. If, now, $\frac{a}{b}$ is the ratio of $\frac{C\,A}{C\,B}$, we have for the force in the rod $B\,A_1$,

FIG. 222.



$$Z = \frac{a}{b}\,G,$$

and hence the load on the previously adjusted platform is

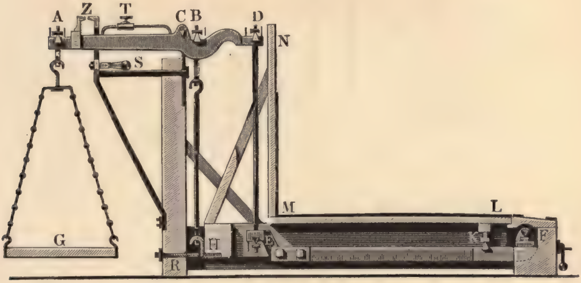
$$Q = \frac{a_1}{b_1}\,Z = \frac{a_1}{b_1} \times \frac{a}{b}\,G.$$

Ordinarily $\frac{a}{b} = \frac{a_1}{b_1} = \frac{10}{1}$; hence $Q = 100\,G$, and the balance is a *centesimal* one.

REMARK.—Street or toll scales require only small platforms if first the fore-wheels and then the hind-wheels of the wagon are brought on and weighed. The weight of the entire wagon is then the sum of the two weighings, whatever be the distribution of the load upon the two axles.

§ **118.—Portable Platform Scales.**—In factories, warehouses, etc., we find platform scales of various forms and sizes such as designed by Quintenz. Such a scale is shown in Fig. 223, and consists of three levers, *A C D*, *E F*, and *H K*. Upon the first hangs the scale pan *G* and two rods *D E* and *B H*. The rod *D E* is attached to the lever *E K F* which turns about *F*, and *B H* carries the lever *H K* whose fulcrum *K* rests upon the lever *E F*. In order to give the two last levers a safe position, they are made fork-shaped, and the fulcrum at *K* and at *F* is a knife edge. Upon the lever *H K* rests the platform, which is provided with a back-board *M N* in order to protect the more delicate parts of the balance. Before and after the process of weighing, the platform rests upon three supports, of which only one, *R*, is shown in the figure, but the beam *A D* is supported by a catch and handle *S*. When the load is placed upon the platform, the catch is released and weights placed in the pan *G* till equilibrium is produced. Then the catch is raised so that the platform again rests upon the three fixed supports, and the load can then be removed without injuring the scale. The horizontal position of *A D* is indicated

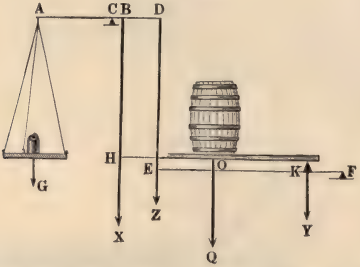
FIG. 223.



by the index $Z$, and the empty scale is adjusted by a sliding weight $T$, or by a weight in $G$.

In this as in all platform scales, it is necessary that its indications should be independent of the position of the load upon the platform. But in order to satisfy this condition, it is necessary that the ratio $\frac{EF}{KF}$ of the lever arms of the lever $EKF$, Fig. 224, shall be equal to the ratio $\frac{CD}{CB}$ of the lever arms of the beam $AD$.

FIG. 224.



A portion $X$ of the load $Q$ upon the platform acts through the rod $BH$ upon the beam $AD$ and has the moment $\overline{CB} \times X$. Another portion $Y$ passes from $K$ to the lever $EF$ and causes at $E$ a force

$$Z = \frac{KF}{EF} Y.$$

This latter force takes effect through the rod $DE$ at $D$ and has the moment

$$\overline{CD} \times \frac{KF}{EF} Y,$$

which is the same as a force

$$\frac{CD}{CB} \times \frac{KF}{EF} Y,$$

acting at the point $B$.

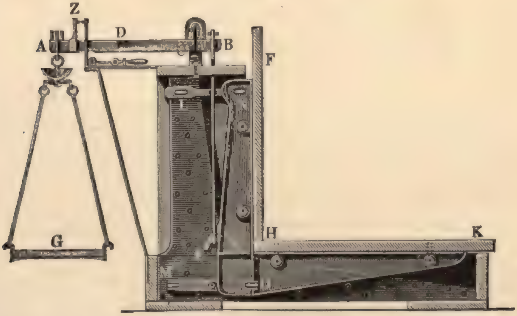
In order now that the equilibrium may not depend upon $X$ or $Y$ alone, but upon their sum $Q = X + Y$, it is necessary that the force $Y$ should have the same effect at $B$ as if it were directly applied at that point, therefore

$$\frac{CD}{CB} \times \frac{KF}{EF} Y = Y, \text{ or } \frac{CD}{CB} \times \frac{KF}{EF} = 1, \text{ or } \frac{CD}{CB} = \frac{EF}{KF}.$$

If we denote the lever arms $CA$ and $CB$ by $a$ and $b$, we have, as for the simple balance, $G\,a = (X + Y)\,b = Q\,b$, and hence $Q = \frac{a}{b} G$. That is, $Q = 10\,G$ when $CA$ is 10 times $CB$. We may test such a balance by observing whether a weight placed at various points of the platform holds in equilibrium always a weight of $\frac{1}{10}$ as much in the scale.

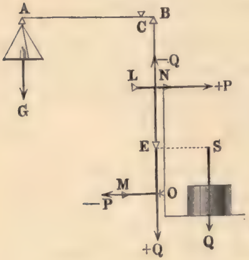
§ **119.**—Another peculiar platform balance is the *balance-bascule*, by George of Paris ("Bulletin de la Société d'Encouragement," Avril 1844; or Dingler's "Polyt. Journal," Bd. 93). The essential construction of such a balance is as follows: $A\,C\,B$, Fig. 225, is a decimal balance with scale pan

FIG. **225.**



$G$ and index Z, which to the right of $D$ divides into two arms, each of which rests upon a knife edge at $C$, while the platform *F H K* rests upon a rod *B E*, which is suspended from a knife edge at *B*. In order that the platform may not turn about *E*, it is provided with two pair of horizontal knife edges at *L*, *M* and *N*, *O*, each pair connected by couplings *L N* and *M O*, so that the resultant couple due to eccentric loading of the platform takes effect as tension in *L N* and compression in *M O*.

FIG. 226.



If we conceive at the point of suspension *E* of the platform *H K*, Fig. 226, two equal vertical forces $+Q$, $-Q$, the first $-Q$ forms with the load $Q$ a couple which is balanced by the couple formed by the forces in *L N* and *M O*, while the other force $+Q$ is transmitted by the rod *B E* to the beam *A C B*.

If $d$ is the distance *E S* of the point of suspension *E* from the load $Q$, and $e$ the distance *N O* between the knife edges *N* and *O*, or *L* and *M*, we have (Vol. I., Art. 93) for the forces $+P$, $-P$,

$$P e = Q d,$$

or

$$P = \frac{d}{e} Q.$$

If, further, $a$ and $b$ are the lever arms *C A* and *C B* of the balance beam, and if $G$ is the counter-weight, we have for equilibrium

$$G a = Q b,$$

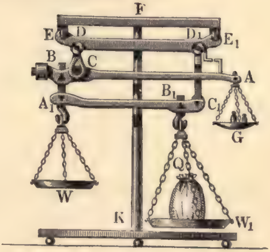
or

$$G = \frac{b}{a} Q.$$

Therefore, only the horizontal force $\pm P$ depends upon the distance $e$ or upon the position of the load, while the weight $G$ is independent of it.

§ **120.—Ship Balance.**—To the simpler balances with proportional weights belong the so-called Swedish *ship balances*. These consist essentially of two unequal-armed beams so connected that the force of the under one acts as a load upon the upper one. If, therefore, in both beams, the lever arm of the load is $\frac{1}{10}$ of that of the force, the counterpoise $G$ in the scale at the end of the longer arm of the first lever will balance a weight $Q$ 100 times as great at the end of the shorter arm of the second.

FIG. 227.



The decimal and centesimal balance of Joseph Beranger ("Polyt. Centralblatt," 1850) is constructed upon the same principle. It consists of two levers $A\ C\ B$ and $A_1\ C_1\ B_1$, Fig. 227, with the lever arm ratio

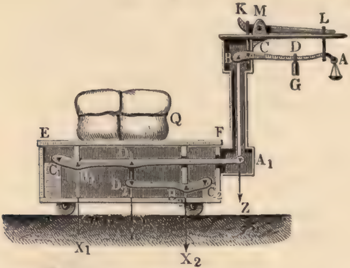
$$\frac{C\ A}{C\ B} = \frac{C_1\ A_1}{C_1\ B_1} = 10.$$

The knife edges $C\ D$, $C_1\ D_1$ are connected with a third beam $D\ D_1$, which, by means of two hooks at $E$ and $E_1$, is hung to the pedestal $K\ F$. The upper balance beam carries the scale pan $G$, while upon the under one we have two pans, $W$ and $W_1$, for the load. According as the body to be weighed is placed in one or the other of these, the counterpoise at $G$ is $\frac{1}{10}$ or $\frac{1}{100}$ of the load.

§ **121.—English Platform Balance.**—An English balance resting upon wheels is shown in Fig. 228. The platform $E\ F$ forms the top of a chest which contains the mechanism. The platform rests upon four supports $B_1$, $B_2$, etc., and the mechanism consists of two levers $C_1\ B_1\ D_1$, $C_2\ B_2\ D_2$, turning about knife edges at $C_1$ and $C_2$ and connected with each other at $D_1\ D_2$, and with the balance beam $A\ B\ C$ by the rod $B\ A_1$. The pivot $C\ K$ of the balance beam hangs from a lever $K\ L$ turning about $M$, whose end $L$ being pressed down raises $C$ and $E\ F$ and brings the balance into play.

If that part of the load borne by the double knife edges $B_1$ is $X_1$, that part borne by the double knife edges $B_2$, $X_2$,

FIG. 228.



and if the lever arms $C_1 A_1 = a_1$, $C_1 B_1 = C_2 B_2 = b_1$, and $C_1 D_1 = C_2 D_2 = d_1$, we have for the tension in the connection $D_1 D_2$:

$$Y = \frac{b_1 X_2}{d_1},$$

and for the tension in $B A_1$,

$$Z = \frac{b_1 X_1}{a_1} + \frac{d_1 Y}{a_1} = \frac{b_1 X_1 + b_1 X_2}{a_1} = \frac{b_1 (X_1 + X_2)}{a_1} = \frac{b_1 Q}{a_1}.$$

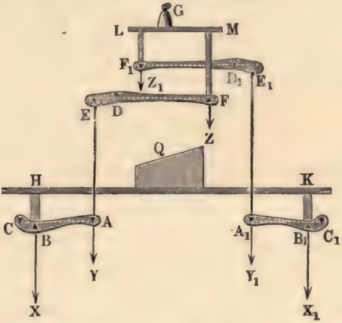
If, finally, $a$ is the variable lever arm $C D$ of the sliding weight $G$, and $b$ the lever arm $C B$ of the force $Z$, we have, under the supposition that the empty balance is first equilibrated by a special weight,

$$G a = Z b = \frac{b_1 b Q}{a_1}, \qquad \text{and hence,} \qquad Q = \frac{a a_1}{b b_1} G.$$

The construction of the platform scale of Kuppler is shown in Fig. 229. The load $Q$ is placed upon a platform $H K$ and the counterpoise upon $L M$. The first is supported upon the levers $A C B$ and $A_1 C_1 B_1$, and the second upon the levers $D E F$ and $D_1 E_1 F_1$, which are connected with the first by the rods $A E$ and $A_1 E_1$. If we denote the lever arms $C A = C_1 A_1$ by $a$, the lever arms $C B = C_1 B_1$ by $b$, $D F = D_1 F_1$ by $a_1$, and $D E = D_1 E_1$ by $b_1$, and represent the pressure upon $B$ and $B_1$ by $X$ and $X_1$; we have for the forces in the rods $A\,E$ and $A_1\,E_1$

$$Y = \frac{b}{a} X \qquad \text{and} \qquad Y_1 = \frac{b}{a} X_1.$$

FIG. 229.



The pressures due to $G$, upon the supports $F$ and $F_1$, are

$$Z = \frac{b_1}{a_1} Y = \frac{b\,b_1}{a\,a_1} X \qquad \text{and} \qquad Z_1 = \frac{b_1}{a_1} Y_1 = \frac{b\,b_1}{a\,a_1} X_1,$$

so that

$$G = Z + Z_1 = \frac{b\,b_1}{a\,a_1}(X + X_1) \doteq \frac{b\,b_1}{a\,a_1} Q,$$

or inversely,

$$Q = \frac{a\,a_1}{b\,b_1} G.$$

For example, for

$$\frac{a_1}{b_1} = \frac{a}{b} = 10,\ Q = 100\,G.$$

REMARK.—Platform balances will be found treated in detail in Hülse's "Allgemeiner Maschinenencyclopädie," Bd. II., also in Gerstner's "Mechanik," Bd. I. For Hofmann's platform balance, see Poggendorff's "Annalen," 1845, and Dingler's "Polyt. Journal," Bd. 97. Here also belong the balances of Kuppler and Baumann, which will be found treated in the "Baierischen Kunst- und Gewerbeblatt," Jahrgang, 1845, and in the above cited article in the "Allgemeinen Maschinenencylopädie." See also a description of platform scales for weighing loaded wagons by Däntzer and Schmidt in

Vol. 27 (1861) of the "Polyt. Centralblatt." A complete discussion of the balance of Burg will also be found in Prechtl's "Technologische Encyclopädie," Bd. 20. Rühlmann's "Allgemeine Maschinenlehre," Bd. I., 1862, is also to be recommended. A platform scale of peculiar construction has been described in a monograph by Prof. Schönemann, Wien, 1855.

§ **122.—Index or Bent Lever Balance.**—The bent lever balance (Fr., peson ordinaire; Ger., Zeigerwage) is a lever of unequal arms *A C B*, Fig. 230, which gives the weight *Q* of a body suspended at *B* by means of a pointer moving over a fixed graduated scale *D E*. The weight *G* upon the end of the pointer balances the weight *Q* of the body to be weighed. In order to develop the theory of this balance, let us consider first the simple case in which the axis of the pointer *C D* passes through the point of suspension *B* of the scale pan, Fig. 231. When the empty balance is in equilibrium, and therefore its centre of gravity $S_0$ directly under the pivot *C*, let the pointer stand at $C D_0$, and the point of application of the load is then at $B_0$. If, however, we apply a weight *Q*, $B_0$ comes to *B*, $D_0$ to *D*, and $S_0$ to *S*, and the load *Q* has the lever arm *C K*, and the weight *G* of the empty balance the lever arm *C H*. For the new position of equilibrium, then,

FIG. 230.

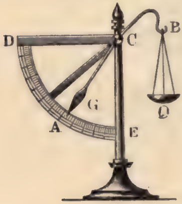
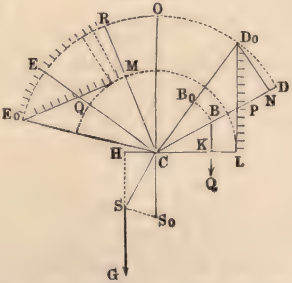


FIG. 231.



$$Q \times \overline{CK} = G \times \overline{CH}.$$

If we draw $D_0 N$ perpendicular to *C D*, the triangles $C D_0 N$ and *S C H* are similar, and hence

$$\frac{CH}{CS} = \frac{D_0 N}{C D_0}.$$

Since, now, the triangles $D_0 P N$ and $C B K$ are similar also, we have

$$\frac{C K}{C B} = \frac{D_0 N}{D_0 P}.$$

Therefore,

$$Q \frac{C B \times D_0 N}{D_0 P} = G \frac{C S \times D_0 N}{C D_0},$$

or

$$Q = \frac{C S}{C B} \times \frac{D_0 P}{C D_0} G;$$

or, putting

$C S = a$, $C B = b$, $C D_0 = C D = d$ and $D_0 P = x$,

$$Q = \frac{a}{b} \times \frac{x}{d} G.$$

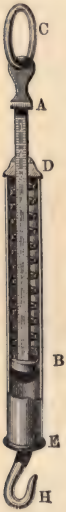
$Q$ therefore increases as the segment $D_0 P = x$ cut off by the pointer upon the vertical $D_0 L$, and we may then divide $D_0 L$ into equal parts. If, by applying a known load, we find the corresponding point $P$ upon this line, we can find the points corresponding to portions of this known load by dividing $D_0 P$ into the proper number of equal parts.

If the axis of the pointer does not pass through the point of suspension $B_0$, but has another direction $C E_0$, we find the corresponding scale of equal parts $E_0 M$ from the right-angled triangle $C E_0 M = C D_0 L$. In order finally to obtain the circular scale $E_0 R$, we draw through $C$ and the points of division of $E_0 M$ straight lines to intersection with the circle described by the end of the pointer.

REMARK.—There are other index balances, as, for example, that of Du Mont of Braby, etc. To this class also belongs Weber's chain balance, as also Steinhiel's platform balance with pointer, which is not supported upon knife edges, but hung by wires. In this balance the scale and weight form one whole, and a plummet borne by the scale pan serves as pointer. Index balances are used chiefly for weighing paper, letters, etc. See the article "Wage" in Vol. 20 of Prechtl's "Technologische Encyclopädie," as also Vol. 10 of Gehler's "Physicalisches Wörterbuch."

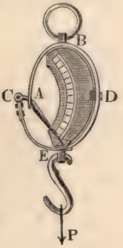
§ **123.—Spring Balance.**—Spring balances or dynamometers (Fr. pesons à ressort; Ger. Federwagen) consist of hardened steel springs upon which the force to be measured acts, and the change of form of the spring caused by this force is indicated by a pointer moving over a graduated scale. These springs must be perfectly elastic—*i.e.*, they must regain their original shape upon removal of the deflecting force. For this reason, we can load the spring balance only up to a certain proportion of its full strength. If the limit of elasticity is exceeded, they are no longer perfectly elastic, do not return exactly to their original shape, and are useless. The springs used in such balances are of various forms. Sometimes they are spiral and enclosed in a cylinder, so that the elongation or compression in the axis of this cylinder measures the applied force. Such a balance is shown in Fig. 232. The graduated rod *A B* has a ring *C* at top and a button *B* at bottom, and is surrounded by a spiral spring, which, together with the button *B*, is enclosed in a cylinder *D E*. Through the upper end of this cylinder the rod passes, and to the lower end is attached a hook *H* from which the body to be weighed is suspended. Since here the weight acts by means of the spring upon the button *B* of the rod *A B*, the spring will be compressed and the cylinder *D E* sink, and the portion *A D* of the rod will increase the greater the weight.

FIG. 232.



Another form of spring balance is the open steel ring, shown in Fig. 233. The pointer *C Z* is hinged at the end *C* and passes through a ring at the other end of the spring. If the ring is suspended by *B*, and a force *P* is applied to the hook *E P*, the ends of the spring, *A* and *C*, separate in the direction of the force, and the pointer *C Z* rises to a certain point upon the scale attached at *D*. If we have first divided the scale by applying known weights, then the position of the pointer upon this scale will indicate any unknown force which may afterwards be applied

FIG. 233.



In Fig. 234 we have a rear view of a French spring balance of this kind. The spring *A B C* is fastened at *A* to a circular graduated plate; the spring being furnished above with a ring *E* and below with a hook *H*. At this lower end *C* we have also a toothed arm *C K*, which engages a pinion *L* and turns a pointer *Z*. It is easy to see that the load at *H* draws the toothed arm *C K* down and thus sets the pointer *L Z* in motion, so that the position of this last gives the magnitude of the force.

Fig. 234.

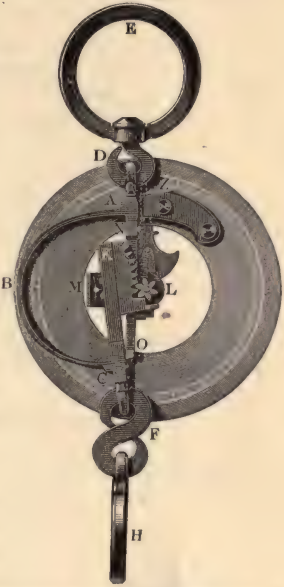
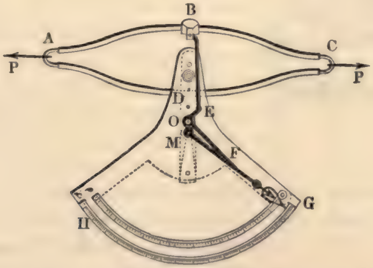


Fig. 235 represents the dynamometer of Regnier. *A B C D* is the steel spring forming a closed ring which by forces at *A* and *C* may either be drawn out or compressed. *D E G H* is a sector furnished with two graduated scales and fastened at *D* and *E* to the spring; while *M G* is a double pointer turn-

Fig. 235.



ing about *M*, and *E O F* is a bent lever which is turned about *O*, when, by the action of the forces the points *B* and *D* approach or recede. In order that the index may remain where the force may have put it for more convenient reading, the pointer has on its under side a friction washer.

FIG. 236.

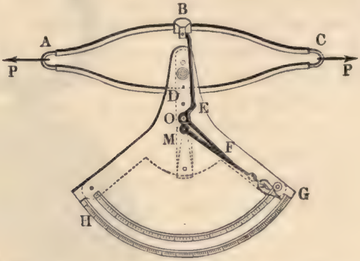
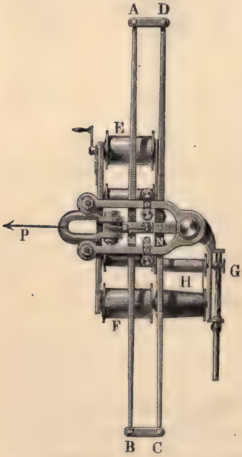


FIG. 237.



**§ 124.—Spring Dynamometer.**—The most perfect and useful spring dynamometer for mechanical purposes is that made use of by General Morin and described in his Treatise, "Description des appareils dynamométriques," Metz, 1838. It consists of two equal steel springs *A B* and *C D*, Fig. 237, from $\frac{1}{4}$ to $\frac{1}{2}$ metre in length (10 to 20 inches), and gives the magnitude of the force *P* applied at the centre *M* of one spring by the increase of the distance *M N* between the two. In order now to determine the magnitude of any force, as, for example, the traction of a horse on a wagon, the spring *C D* is fastened at its centre *N* to the wagon by a bolt, the horse is attached to $M$, and by means of a pointer at $M$ and a scale at $N$, the separation of the springs is indicated. If the springs are of uniform breadth and thickness, and we put the length $= l$, the breadth $b$, and the thickness $h$, we have from Vol. I., Art. 217, for the height of arc corresponding to the force $P$,

$$a = \frac{1}{48} \frac{P l^3}{W E} = \frac{1}{4} \frac{P l^3}{E b h^3}.$$

The deflection, therefore, increases directly as the force, and the scale is one of equal divisions. Since here we have the deflection of two springs, $s$, we have

$$s = \frac{1}{2} \frac{P l^3}{E b h^3}.$$

In order to save material, it is preferable to give the springs the *parabolic* shape of a body of equal strength, having a constant breadth, but thickness gradually decreasing towards the ends (Vol. I., Arts. 253–256). In this case the deflection is double that for a body of constant thickness $h$. Hence, for this case we have

$$s = \frac{P l^3}{E b h^3} = \frac{1}{E b} \left(\frac{l}{h}\right)^3 P = \nu P,$$

where $\nu$ is a number to be determined by experiment.

If, before the use of such an instrument, we apply a known weight and observe the deflection $s$, the number $\nu$ can be calculated and used in the construction of a scale. In the use of the best steel, it has been found that the deflection may reach $\frac{1}{10}$ of the length, before the relation between it and the force changes, and the limit of elasticity is exceeded.

**§ 125.—Recording and Registering Dynamometers.**—In general, the forces are not always uniform in their action, but act intermittently, and the main object is then to ascertain the mean action. But with the ordinary index apparatus the spring balance only gives the force at a single instant, or the maximum value. Hence for great variations in the force, as in the case of wagons, etc., the ordinary dynamometer is very uncertain in its results. For this reason, recording and registering instruments were first devised by Poncelet and applied by Morin. Both instruments give the work or product of the force into distance, and the mean value of the force can then be determined by dividing the work by the distance.

In the *recording dynamometer* (Fr. dynamomètre à style; Ger. Zeichnenapparate), Fig. 238, the work is indicated by a pencil *M S* fastened at *M*, which traces a line upon a continuous roll of paper which passes under it. The paper is unrolled from *E* and wound upon *F*, which is set in motion by bands or wheel work, as *G H*, etc., Fig. 237, by the motion of the wagon itself. When no force is applied and the spring not strained, simply a straight line is described upon the moving paper. But when by the action of force the spring bends and the pencil *M S* moves, we have a curve *O S*. The area between this curve and the line in the first case is a measure of the work performed, since its base is a line proportional to the distance, and its height at any point is proportional to the force at the corresponding moment.

FIG. 238.

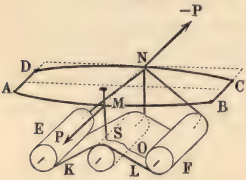
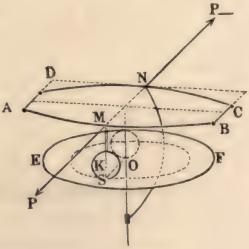


FIG. 239.



The *registering dynamometer* (Fr. dynamomètre à compteur; Ger. Zählapparat) consists essentially of a horizontal plate *E F*, Fig. 239, which is connected with the centre *N* of the rear spring *C D*, and set in rotation by the machine upon which the force to be determined acts; and of a vertical wheel *K S* connected with the centre $M$ of the forward spring $A\,B$, which presses lightly upon the plate $E\,F$, so that, by reason of friction upon the latter, it is made to revolve about its axis $K$.

If the plate makes $u$ rotations per minute, and if the distance $O\,S$ of the point of contact $S$ of the wheel from the centre of the plate is $z$, then the velocity of the point $S$ is

$$v = \frac{2\,\pi\,z\,u}{60} = \frac{\pi\,u\,z}{30}.$$

If, on the other hand, the wheel makes $u_1$ rotations per minute, and its radius $K\,S = r$, then its velocity of rotation is

$$v = \frac{\pi\,u_1\,r}{30},$$

so that we must have

$$u_1\,r = u\,z,$$

or

$$u_1 = \frac{u\,z}{r}.$$

If the wheel is at the centre $O$ of the plate when the spring is not strained, it will, upon the application of a force $P$, pass through a distance $z = O\,S$ proportional to that force, and since the number of revolutions of the plate $u$ depends upon the distance $s$ passed through, it follows that $u_1$ increases with the product of $P$ and $s$, or is proportional to the work.

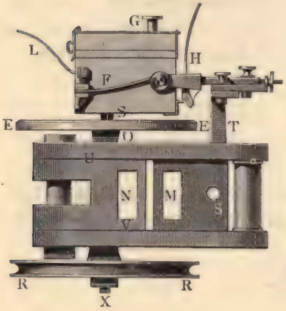
If we put $z = \alpha\,P$ and $u = \beta\,s$, we have then for the work $L$,

$$L = \frac{\alpha\,\beta\,P s}{r} = \mu\,P\,s$$

if we put $\frac{\alpha\,\beta}{r}$ = to a constant coefficient $\mu$.

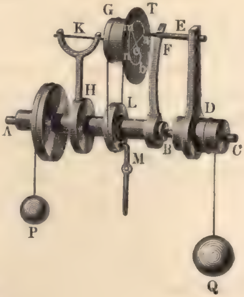
The special arrangement of the registering apparatus is shown in Fig. 240. *E E* is the plate which is made to revolve about the axis *O X* by means of the wheel *R R*, which is set in motion by a band, and *S* is the visible part of the small wheel which presses lightly upon the revolving plate *E E*. The remaining portion is enclosed in the box *F G*, which also contains a train of wheel-work, which sets in motion two graduated disks, which thus register the number of revolutions of the small wheel *S*. A pair of springs (*F*) press the box, together with the wheel *S*, gently upon the revolving plate during the observation, while at its close, by means of the hooks *L* and *H*, the box may be raised clear of the plate, and suspended above it. In order to mark the beginning and end of the observation, the button *G* is pressed, which by a simple mechanism dots with ink each of the two graduated wheels. The rear spring of the dynamometer is attached at *N* to the frame *U V*, which is connected with the wagon and carries the axis of the revolving plate, while the forward spring is fastened at *M* to the sliding support *S T* of the registering apparatus.

FIG. 240.



**§ 126.—Rotation Dynamometer.**—When it is required to determine the force of rotation of a revolving wheel, the above-described dynamometer requires modification. The essential arrangement of such a modified

FIG. 241.



dynamometer may be shown by the ideal construction represented in Fig. 241. Let the machine whose force of rotation and work we wish to determine be represented by the shaft $A B$ and force $P$ and the shaft $B C$ with the load $Q$, the two shafts being connected by a steel spring $B F$ attached to the shaft $A B$ and acting upon the arm $D E$, which revolves with the shaft $B C$. If, now, by means of a scale attached to the peg $E$, we measure the deflection of the spring $B F$, we have a measure of the force $R$ exerted by one shaft upon the other. If, now, the distance of the peg from the axis $A C$ is $a$, and if the number of revolutions $u$ is known, we can find at once the work of the force $P$ or $Q$ by the formula

$$L = \frac{\pi u a}{30} R.$$

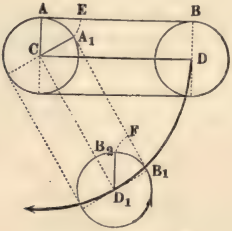
But we thus obtain always only a single value, and not the mean value of the force $R$. To obtain this last, we must replace the spring by such an apparatus as shall give a measure of the work of the force $R$. Such a contrivance consists first of a drum $G$, which turns about its own axis $K$, while at the same time it is carried around the axis $A B$ by means of the arm $H K$. In order to cause the drum to revolve about its own axis, it is connected by a band with the drum $L$, which rests upon the shaft $A B$, but does not revolve with it, being prevented by the bolt at $M$. Thus, while the axis $K$ revolves about $A B$, the drum $G$ is made to revolve about $K$. Let $A C$, in Fig. 242, be the fixed and $B D$ the revolving drum, connected with the first by the endless band, and of any desired size. If this drum revolves to $B_1 D_1$, so that its axis $D$ passes over the angle $D C D_1$, a portion $A E$ of the band equal to $A A_1$ is wound upon the drum $A C$, and a portion $B_1 B_2 = B_1 F$ is unwound from the drum $B_1 D_1$. Since $A_1 B_1 = A B$, we must have $B_1 B_2 = B_1 F = A E = A A_1$. If, now, the radius $C A = r_1$ and $D B = r_2$, and the angle $A C A_1 = D C D_1$ is $\phi_1$ and $B_1 D_1 B_2 = \phi_2$, we have,

$$A A_1 = r_1 \phi_1 \qquad \text{and} \qquad B_1 B_2 = r_2 \phi_2,$$

and hence the relation between the angular velocities about *D* and *C*,

$$\frac{\phi_2}{\phi_1} = \frac{r_1}{r_2}.$$

FIG 242.



If, for example, $r_2 = r_1$, then we have the ratio $\frac{\phi_2}{\phi_1} = 1$. In this case, then, the drum would turn exactly once about its axis *D*, while the axis itself revolves once about *C*. If

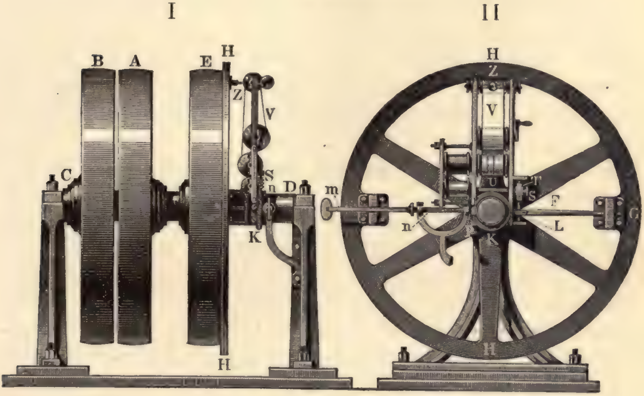
$$r_2 = 2\, r_1, \qquad \text{then} \qquad \frac{\phi_2}{\phi_1} = \tfrac{1}{2},$$

and hence the drum *B D* would have to revolve twice about *C* in order to turn once about its axis.

The simplest way to effect our purpose now, is to attach a paper-covered plate *T* to the drum *G*, Fig. 241, so that a pencil *a* fixed to the peg *E* may describe upon it a curve *a m n b*. The mean value of the force *R* with which the spring *F* presses upon the peg *E* during its passage through the distance corresponding to the angle of rotation *a c b*, is then the mean of the distances *c a*, *c m*, *c n*, *c b* . . . . of the various portions of the curve *a m n b* from the centre of the plate *c*.

§ **127.**—In order to obtain the work of a machine for a greater time or distance, we may replace the plate *T*, Fig. 241, by a pair of drums with an endless band of paper, so that the point *a* of the peg *E* may describe upon the band as it passes beneath it a curve, the quadrature of which will give a measure of the work performed by the machine, while the paper is passing through a certain distance. The arrangement of such a rotation dynamometer as given by Morin is shown in Fig. 243, I and II, and is essentially as follows: Upon the horizontal shaft *C D* is keyed a belt wheel *A* and two loose running wheels *B* and *E*. By the first the power of the motor is transferred to the shaft *C D* from which by means of *E* it is transmitted to the machine whose work we wish to determine. As long as the belt runs upon *B* and *E* is not fastened on to the shaft, there is, of course, neither motion of the shaft nor of the machine. In order to cause motion of the shaft, we have to shift the belt from *B* to *A*. The wheel *E* is fastened to the shaft by means of two dynamometer springs *F G* similar to those de-

FIG. 243.

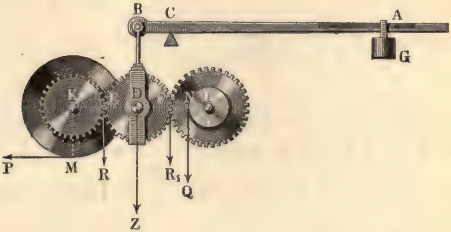


scribed above, which are fastened to the shaft, project radially from it, and are fastened at the end *G* to the ring *H G H*, which is firmly attached to the wheel *E*. By the motion of the axle, the wheel *E* is thus set in motion, and the springs *F G* bend more or less according to the resistance encountered, just like the spring *B F* in Fig. 241. Upon the shaft *C D* we have also a toothed wheel *K*, furnished with a projection *p* upon its side, so that, by means of a sliding bar *m n*, it may be prevented from revolving with the shaft during the experiment. This wheel engages with a smaller toothed wheel *L*, which turns a screw *S*, and thus sets in motion another wheel *T*, which latter, by means of the axle *U*, sets in motion a system of drums, by means of which a band of paper is unrolled from one roller passed over a second at *Z* and wound upon a third. During this motion of the paper, a pencil *Z* fixed in one of the arms of the ring *G H* presses against it, and we have thus a curve, which, as above seen, gives a measure of the work of the machine.

When the paper cannot be made to move by direct connection with the shaft, we may set it in motion by clock-work. The instrument, however, does not then give the product of the force and distance, but of the force and time. In order, therefore, to find the mean force, we must divide this product by the time, and then, to find the work of the machine, multiply the quotient by the distance.

§ **128.—Pillow-Block Dynamometer.**—By another kind of dynamometer, the pressure of the axle of a revolving shaft may be determined, and thus the magnitude of the force of rotation determined. The simplest dynamometer of this kind is the *steelyard dynamometer* of Hachette. This consists of an ordinary steelyard *A C B*, Fig. 244, which has a toothed wheel *D E F* in place of the scale pan. This wheel engages with two others *K E* and *L F*, whose rotating force is to be determined. If *P* is the rotating force of one shaft with the lever arm $K M = a$ and *Q* that of the other shaft with the lever arm $LN = b$, while $r$ is the radius $KE$ of one wheel and $r_1$ the radius $LF$ of the other,

FIG. 244.



we have for the forces with which these two wheels press upon the centre wheel at $E$ and $F$,

$$R = \frac{Pa}{r} \quad \text{and} \quad R_1 = \frac{Qb}{r_1}.$$

Since these two pressures have equal lever arms $DE$ and $EF$, we have $R = R_1$, and hence the load of the balance $ACB$ at $B$ is

$$Z = R + R_1 = 2\,R,$$

or, inversely, the pressure $R$ between the toothed wheels is $R = \frac{Z}{2}$. If, by moving the sliding weight $G$, we cause equilibrium with the force $Z = 2\,R$, we determine at once $Z$ and $R$, as also

$$P = \frac{r}{a}\,R = \frac{r}{a} \cdot \frac{Z}{2},$$

and

$$Q = \frac{r_1}{b}\,R = \frac{r_1}{b} \cdot \frac{Z}{2}.$$

If, now, we know the number of revolutions per minute $u$ of the wheel to which the power is applied, or the number $u_1$ of that to which the resistance is applied, we can determine the work of the machine by the formula,

$$L = \frac{\pi\, u\, a}{30} P = \frac{\pi\, u\, r}{30} \cdot \frac{Z}{2}$$

and

$$L = \frac{\pi\, u_1\, b}{30} Q = \frac{\pi\, u_1\, r_1}{30} \cdot \frac{Z}{2}.$$

By reason of the friction of the axle *D* and between the teeth at *E* and *F*, $R_1$ is somewhat less than *R*, and *R* is therefore somewhat greater than $\frac{Z}{2}$. Hence the result given by the formula

$$L = \frac{\pi\, u\, r}{30} \cdot \frac{Z}{2}$$

is somewhat too small.

In general, we shall have

$$R = \frac{Z}{2}(1 + \mu)$$

and

$$R_1 = \frac{Z}{2}(1 - \mu),$$

where $\mu$ is a number to be determined from experiment. We have then

$$P = (1 + \mu)\frac{r}{a} \cdot \frac{Z}{2}$$

and

$$Q = (1 - \mu)\frac{r_1}{b} \cdot \frac{Z}{2},$$

and hence,

$$\frac{P}{Q} = \frac{1 + \mu}{1 - \mu} \cdot \frac{r}{r_1} \cdot \frac{b}{a},$$

or inversely,

$$\mu = \frac{P\, a\, r_1 - Q\, b\, r}{P\, a\, r_1 + Q\, b\, r}.$$

If by a preliminary trial, we determine the two forces $P$ and $Q$, which just hold each other in equilibrium, we can calculate from the above formulæ the value of $\mu$, and then, by the aid of this value, can find for other cases the force

$$P = (1 + \mu)\, \frac{r}{a} \cdot \frac{Z}{2},$$

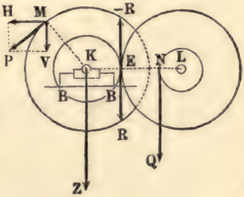
as also the work

$$L = (1 + \mu) \frac{\pi u r}{30} \cdot \frac{Z}{2} = (1 + \mu) \frac{\pi u r}{60} Z.$$

The dynamometer of Schinz (see "Poly. Centralblatt," 1848) is not essentially different from the above. The same is true of Rittinger's improved dynamometer (see "Oesterreich. Zeitschrift für Berg- und Hüttenwesen," 1855).

The *pillow-block dynamometer* (see Rittinger's Treatise in the "Oesterreich. Zeit. für Berg- und Hüttenwesen," 1856) is based upon the same principle as the steelyard dynamometer, except there is no intermediate wheel, the vertical axle pressure of one or the other wheel being directly determined, and from this the force of rotation calculated. To determine this axle pressure $Z$ for the shaft $MKE$, Fig. 245, we may employ a platform scale upon which the two pillow blocks $BB$ of the two axles $K$ rest. If the force acting upon the shaft has the lever arm $KM = a$ and makes an angle $\alpha$ with the horizon, while the radius $KE$ of a toothed wheel attached to the shaft is $r$, and the weight of the entire shaft with wheel is $G$, then we have

FIG. 245.



$$Z = G + P \sin.\alpha + \frac{a}{r} P = G + \left(\sin.\alpha + \frac{a}{r}\right) P.$$

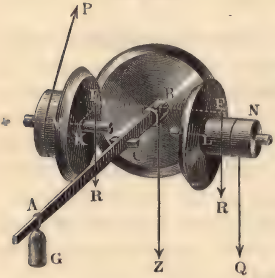
The force of rotation is therefore

$$P = \frac{Z - G}{sin.\ a + \frac{a}{r}}.$$

The determination of this force is, of course, more precise the less the weight $G$ of the shaft.

§ **129.—Differential Dynamometer.**—When the shafts $K$ and $L$, Fig. 244, whose rotating force is to be determined by the steelyard dynamometer, lie one behind the other so that their axes are in the same line, as shown in Fig. 246, the toothed wheels $K E$ and $L F$ must be conical, while every thing else, as, for instance, the steelyard $A C B$ which carries the centre wheel, remains the same. If, then, in this case, $Z$ is the axle pressure of the wheel $E F$ as given by the steelyard, the rotating pressure upon the teeth at $E$ is as before,

FIG. 246.



$$R = (1 + \mu) \frac{Z}{2},$$

and hence the rotating force $P$, whose lever arm is $a$, is

$$P = (1 + \mu) \frac{a}{r} \cdot \frac{Z}{2},$$

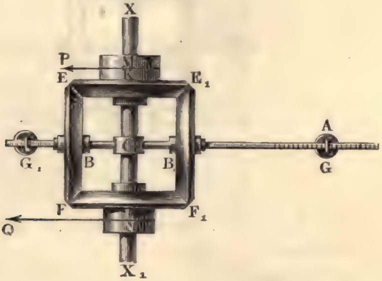
and the work of the shaft is

$$L = (1 + \mu) \frac{\pi u r}{60} Z,$$

where $r$ is the radius $K E$ of the wheel upon the shaft $K M$, and $\mu$ is the revolutions per minute of this shaft.

This dynamometer may be improved by furnishing the steelyard $A\,C\,B$ with two conical wheels so that the wheel $K\,E$ may act by means of both upon the wheel $L\,F$. The general arrangement of such a dynamometer is shown in Fig. 247 in plan. The power is applied to the drum $M$, which carries the conical wheel $E\,E_1$, and the resistance is applied to the drum $N$, which carries the conical wheel $F\,F_1$. Both wheels run loosely upon the fixed shaft $X\,X_1$, and are connected by the wheels $E\,F$ and $E_1\,F_1$. By the action of the force $P$ and resistance $Q$, the pressure of the wheels $E\,E_1$ and $F\,F_1$ upon $E\,F$ is downwards at $E$ and $F$, and upon $E_1\,F_1$, upwards at $E_1$ and $F_1$.

FIG. 247.



Such a mechanism is called a *differential gear*, and hence this dynamometer may be called a *differential dynamometer*.

If $R$ is the pressure between the teeth at each of these four places, then the wheels $E\,E_1$ and $F\,F_1$ exercise upon the lever $A\,C\,B$ an upward vertical pressure,

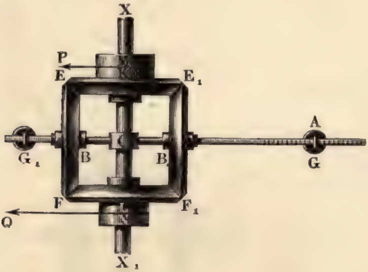
$$Z = 2\,R \text{ at the point } B,$$

and a downward vertical pressure,

$$Z = -\,2\,R \text{ at the point } B_1.$$

Both pressures form a couple which is held in equilibrium by the weight $G$ at $A$, and by the resistance $(-G)$

FIG. 248.



of the shaft $XX_1$ at $C$, about which at this point the lever turns by means of a collar.

If $a_1$ and $b_1$ are the lever arms $CA$ and $CB = CB_1$, we have

$$Ga_1 = Zb_1 + Zb_1 = 2\,Zb_1 = 4\,Rb_1.$$

If, as before, $a$ is the lever arm of the force $P$ and $r$ the radius of $EE_1$ and $FF_1$, we also have

$$Pa = Rr + Rr = 2\,Rr,$$

and hence

$$P = \frac{r}{a} \cdot 2\,R = \frac{a_1}{b_1} \cdot \frac{r}{a} \cdot \frac{G}{2},$$

in which of course there is no allowance for friction.

With reference to this, we have

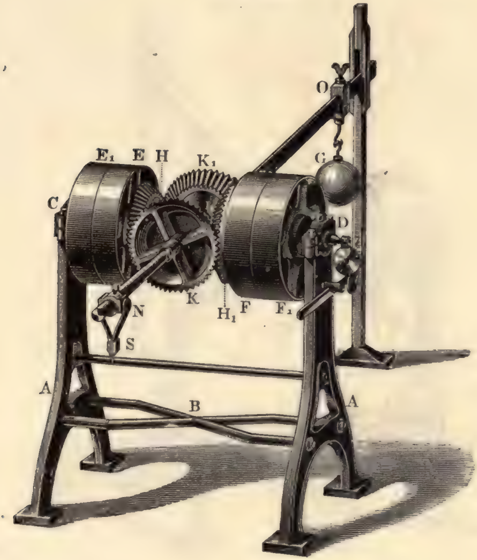
$$P = (1 + \mu)\,\frac{a_1}{b_1} \cdot \frac{r}{a} \cdot \frac{G}{2},$$

and for the mechanical work,

$$L = (1 + \mu)\,\frac{a_1}{b_1} \cdot \frac{\pi u r}{60}\,G.$$

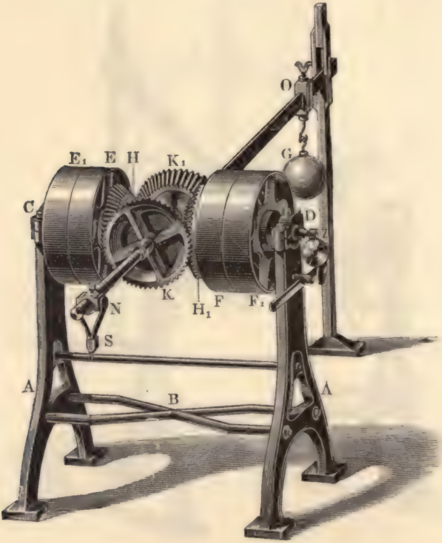
The dynamometer of Batchelder (see Dingler's "Polytech. Journal," 1844) is constructed upon the same principle as shown in Fig. 249. Two standards *A A* of cast-iron, united at bottom by the wrought-iron rods *B*, support the pillow blocks *C*, *D*, of the horizontal shaft *C D*, which carries two pair of equal band wheels $E, E_1$, and $F, F_1$, as well as the two conical toothed wheels $H, H_1$. The wheel *H* is fastened to *E* and the wheel $H_1$ to *F*. The wheel *E* is firmly attached to the shaft *C D*, while $E_1, F$ and $F_1$ revolve loosely upon it. Two other conical wheels $K, K_1$, which engage with the first two, revolve upon the axis *L M*, whose prolongation *L O* forms a balance beam with the sliding weight *G*. In the centre, between the wheels *K* and $K_1$, the axis *L M* enlarges into a collar which embraces the shaft *C D*, and

FIG. 249.



upon its other end *N* is a hook, to which may be hung the weight necessary to balance the apparatus before applying the weight *G*. Finally at *Z* we have an apparatus for registering the number of revolutions, which is set in motion by the screw *D* attached to the end of the shaft *C D*. Before the experiment, the band which connects the motor runs upon the loose wheel $E_1$, and that which connects with the working point runs upon the loose wheel $F_1$. When the observation commences, these bands are shifted to *E* and *F*, which are connected by the toothed wheels, so that the motion of the motor is transmitted to the receiver. If, then, by moving the weight *G*, the arm *L O* is made horizontal, we obtain the necessary element for the determination of the force of the machine.

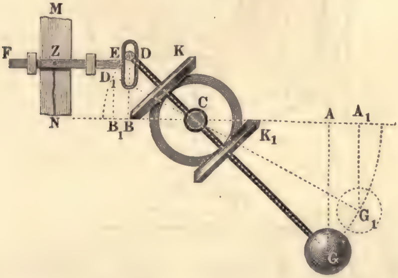
FIG. 250.



If by means of this instrument we wish to determine directly the work of the machine, we may, instead of the weight at *G*, attach a spring dynamometer, as shown in Fig. 237, at *N*, and cause the pencil to describe a curve upon a paper strip set in motion at *Z*.

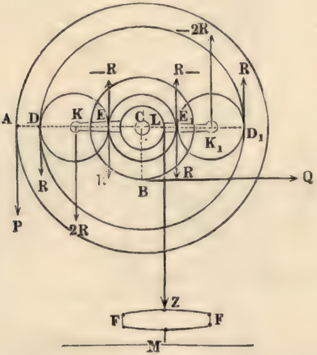
To obtain such a diagram, a spring dynamometer is not essential. We can cause the pencil to be set in motion by the weight *G* itself. Such a dynamometer has been patented by J. Wagner, as early as 1837. The essential construction of such an instrument is shown in Fig. 251. The prolongation of the axis of the conical wheels which forms the balance beam is movable about *C*, and has an inclined position. Upon its other end is a friction roller *D*, which travels in the stirrup *D*, and thus moves the bar *E F*, to which the pencil *Z* is fixed, to and fro. The paper *M N* is moved up or down by the machine or by a chronometric apparatus, and thus we obtain the diagram of work for the machines connected by the apparatus. As the force changes, the arm *C G* takes another position, and the lever arm *C A* becomes *C* $A_1$, and thus changes by a distance *A* $A_1$, which is proportional not only to the change of the force, but also to the projection *B* $B_1$ of the path *D* $D_1$ of the end *D* in the direction *C A*. Thus the motion of the bar *E F* with the pencil *Z* corresponds with the change in the force.

FIG. 251.



**§ 130.**—An apparently very suitable dynamometer for determining the work of machines, furnished with a recording and registering apparatus, is described by E. Hartig in the "Polytech. Centralblatt," 1857, No. 1. The principle of the instrument is as follows: To the interior of the wheel *C A*, Fig. 252, upon which the rotating force acts, is fixed a toothed wheel $D\,C\,D_1$, which engages at $D$ and $D_1$ with the two equal cog-wheels $D\,E$ and $D_1\,E_1$, both of which act upon a third cog-wheel $E\,E_1$. This last wheel revolves free upon the axle $C$ of the wheel $D\,D_1$ and is firmly attached to the drum $B\,C$ upon which the resistance $Q$ acts, while the other two wheels $D\,E$, $D_1\,E_1$, have their axes supported by a lever $K\,C\,K_1$ which revolves freely about $C$. To this lever a band roller $C\,L$ is attached, which carries a band, one end of which is fastened to the spring dynamometer $F\,F$, which latter is bolted down at $M$. We see at once that here the rotating force $P$ is held in equilibrium by two forces, $R, -R$, that out of these last arises a couple $-R, R$, which holds the resistance $Q$ in equilibrium, and that, therefore, the forces $2\,R$ and $-2\,R$ act at $K$ and $K_1$ and stretch the spring dynamometer with a certain force $Z$.

FIG. 252.



Let $a =$ the lever arm $C\,A$ of the force,
$b$ the lever arm $C\,B$ of the resistance,
$r$ the radius $C\,D = C\,D_1$ of the large cog-wheel,
$r_1$ the radius $C\,E = C\,E_1$ of the centre cog-wheel, and hence,
$\frac{r - r_1}{2}$ the radius $K\,D = K_1\,D_1$ of the two intermediate wheels,
$c$ the lever arm $C\,L$ of the force $Z$, then we have,

$$P\,a = 2\,R\,r,\ Q\,b = 2\,R\,r_1$$

and

$$Z\,c = 2\,R\,(r + r_1);$$

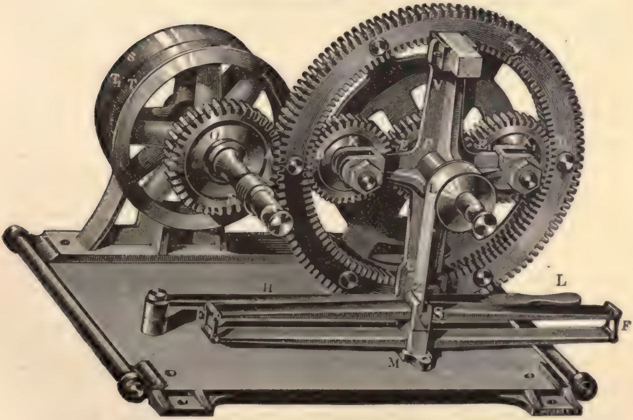
hence

$$\frac{P}{Q} = \frac{r}{r_1} \cdot \frac{b}{a}$$

and

$$\frac{P}{Z} = \frac{r}{r + r_1} \cdot \frac{c}{a}.$$

FIG. 253.

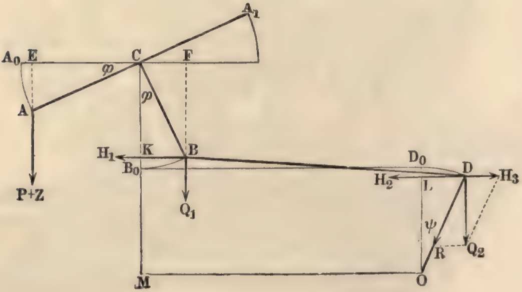


In the representation of this instrument given in Fig. 253, we see also at $T$ and $T_1$ the fast and loose power pulleys, and at $O$ the cog-wheel which transmits the power to the wheel $A\ D\ D_1$. We also see at $N$ the screw by which the recording or registering apparatus (not shown in the figure) is set in motion. The arms $K\ C$ and $K_1\ C$, which carry the cog-wheels $D\ K$ and $D_1\ K$, form, with the two other arms $U$ and $V$, and the drum $C\ L$ which revolves loosely upon the axis of the wheel $E\ E_1$, one whole. The drum is joined by the band $L\ Z$ to the dynamometer springs $F\ F$, of which one is furnished with a pencil at $S$, which describes a curve upon the moving paper. By the lever $H\ L$ which catches the end $U$, the instrument may be set in action or stopped. In order to prevent the overstretching of the springs, the end of the arm $C\ V$ is provided with a strong wooden projection, which for a certain position of the cross $K\ U\ K_1\ V$ brings up against some firm obstruction.

REMARK.—Bentall's dynamometer with spiral springs is described by M. Eyth in Dingler's Journal, Bd. 167 (1863).

§ **131.**—**Horizontal Dynamometer.**—For the measurement of horizontal forces of moderate magnitude the *horizontal dynamometer* of Prof. Schönemann may be advantageously used. It is essentially as follows: $A\ C\ A_1$ (Fig. 254)

FIG. 254.



is an ordinary balance beam with fulcrum at $C$, and $B\,D$ is the platform or scale pan for the reception of the force to be measured, one end $B$ being connected with the balance beam by an arm $C\,B$, and the other end $D$ resting upon an arm $O\,D$ which turns about $O$. Of course the points of support $A$, $B$, $C$, $D$ and $O$ must rest upon knife edges. By the equilibrium of the balance, the platform $B\,D$ has the horizontal position $B_0\,D_0$, and the arms $C\,B$ and $O\,D$ take the vertical positions $C\,B_0$ and $O\,D_0$. In this position of the balance, the vertical forces are transmitted direct by the arms $B_0\,C$ and $D_0\,O$ to the points of support $C$ and $O$, while the horizontal force of the table $B\,D$ acts by the lever $C\,B$ upon the balance beam $A\,C\,A_1$ and turns it about $C$. If, now, $H$ is the magnitude of this horizontal force, $P$ the weight at $A_0$ which holds this force in equilibrium, and if $b$ and $a$ are the lever arms $C\,B_0$ and $C\,A_0$, we have $P\,a = H\,b$, and hence for the horizontal force of the platform $B_0\,D_0$,

$$H = \frac{a}{b}\,P.$$

The addition of $Z$ to $P$ causes a deviation $A_0\,C\,A = \phi$ of the beam, which, upon the supposition that it is but small, can be determined as follows: The combined forces and weights of the platform $B\,D$ we can decompose into two vertical forces $Q_1$ and $Q_2$, and two horizontal forces $H_1$ and $H_2$, acting at $B$ and $D$. Further, the horizontal deviation $L\,D$ of the point $D$ is equal to that of the point $B$, viz., $K\,B$. Denote the length of the arm $O\,D = O\,D_1$ by $r$ and the angle of displacement $D_0\,O\,D$, which corresponds to the deviation $B_0\,C\,B = A_0\,C\,A = \phi$, by $\psi$. Then we have

$$r\,sin.\,\psi = b\,sin.\,\phi,$$

hence

$$sin.\,\psi = \frac{b}{r}\,sin.\,\phi, \qquad \text{or approximately} \qquad \psi = \frac{b}{r}\,\phi.$$

Since by deviation of the balance, $B_0$ rises by the distance $B_0\,K = b\,(1 - cos.\,\phi) = 2\,b\left(sin.\,\frac{\phi}{2}\right)^2 = \frac{b\,\phi^2}{2}$, and

$D_0$ falls by $D_0\,L = r\,(1 - \cos.\psi) = \frac{r\,\psi^2}{2} = \frac{b^2\,\phi^2}{2\,r}$, then, if the length $B\,D$ of the platform is $l$, we have for its angle of inclination $\mu$,

$$\sin.\mu = \frac{B_0\,K + D_0\,L}{B\,D} = \frac{b\,r\,\phi^2 + b^2\,\phi^2}{2\,r\,l} = \frac{(b+r)\,b}{2\,r\,l}\,\phi^2.$$

On account of the factor $\phi^2$ we have approximately, therefore, $\mu = 0$; *i.e.*, for a small deviation $\phi$, the platform remains nearly horizontal. The vertical force $Q_2$ at the point $D$ gives upon the point of support $O$ the component $R = \frac{Q_2}{\cos.\psi}$, while the horizontal component $H_3 = Q_2\,\tan.\psi$ unites with the horizontal force $H_2$, so that the entire horizontal force at $D$ is

$$H_2 - H_3 = H_2 - Q_2\,\tan.\psi = H_2 - \frac{Q_2\,b\,\phi}{r} \text{ approximately.}$$

Since now $B\,D$ is nearly horizontal, we can assume that this force is taken up by $B\,D$ and transmitted to $B$. We have, therefore, at $B$ the total horizontal force

$$H_1 + H_2 - H_3 = H_1 + H_2 - \frac{Q_2\,b\,\sin.\phi}{r}$$

acting with the lever arm $C\,K = C\,B\,\cos.\,B_0\,C\,B = b\,\cos.\phi$, and the vertical force $Q_1$ with the lever arm $C\,F = b\,\sin.\phi$, both acting opposed to the force applied to the balance beam $A\,C\,A_1$ with the lever arm $C\,E = a\,\cos.\phi$. We can therefore put

$$(P + Z)\,a\,\cos.\phi = \left(H_1 + H_2 - Q_2\frac{b\,\sin.\phi}{r}\right)\,b\,\cos.\phi + Q_1\,b\,\sin.\phi,$$

or

$$(P + Z)\,a = (H_1 + H_2)\,b + Q_1\,b\,\tan.\phi - \frac{Q_2\,b^2}{r}\,\sin.\phi,$$

or approximately,

$$(P + Z)\, a = (H_1 + H_2)\, b + \left(Q_1 - \frac{b}{r}\, Q_2\right) b\, \phi.$$

But, for $\phi = 0$,

$$P\, a = (H_1 + H_2)\, b = H\, b,$$

therefore we have

$$Z\, a = \left(Q_1 - \frac{b}{r}\, Q_2\right) b\, \phi,$$

and the deviation desired is hence

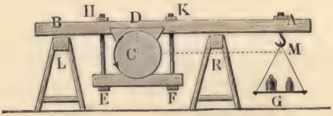
$$\phi = \frac{Z\, a}{\left(Q_1 - \frac{b}{r}\, Q_2\right) b}.$$

In this case, therefore, as in the ordinary balance, the deviation increases directly as the applied force, as the lever arm $a$, etc.

REMARK.—The monograph, "Das Horizontal-Dynamometer und seine Anwendung auf die Mechanik," by Th. Schönemann, Berlin, 1864, gives a complete discussion of the theory, with a description of this instrument, and treats also of various applications of it. The above is only a short outline of the theory.

§ **132.**—**Friction Brake.**—The friction brake (Fr. frein dynamométrique; Ger. Bremsdynamometer) is used to determine the force applied to and mechanical effect produced by a revolving shaft or rotating machine generally. In its simplest form, it consists of a beam $A\,B$, Fig. 255,

FIG. 255.



with a scale pan $A\,G$; and of two wooden circle segments $D$ and $E\,F$, which, by means of screw bolts $E\,H$ and $F\,K$, can be made to compress powerfully the revolving shaft $C$. When the power of the shaft $C$ for a given angular velocity or number of revolutions is to be determined by this instrument, we place weights in the scale pan $A\ G$ and screw up the bolts until the shaft not only makes the desired number of revolutions, but also the beam $A\ B$ maintains a horizontal position without coming in contact with the blocks $L$ or $R$. In this condition, the entire mechanical effect is expended in producing friction between the shaft and the wooden segments, and this mechanical effect is equal to the work or useful effect of the shaft when running at the same speed in the performance of its duty. Since the lever hangs freely, it is only the friction $F$, acting in the direction of revolution, which holds the weight at $G$ in equilibrium, and this friction may be easily deduced from the known weight. If we represent by $a$ the lever arm $C\ M$ of the weight $G$, then the statical moment of this weight, and therefore the moment of the friction, or the friction itself, if we consider the radius of the shaft as unity, is equal to $G\ a$. If, now, $\varepsilon$ is the angular velocity of the shaft, its mechanical effect per second is

$$L = P v = G a \cdot \varepsilon = \varepsilon a G.$$

If $u$ is the number of revolutions of the shaft per minute, we have

$$\varepsilon = \frac{2 \pi u}{60} = \frac{\pi u}{30},$$

and hence the work required is

$$L = \frac{\pi u a}{30} G.$$

Under the weight $G$ we must include not only the weights in the scale pan, but also the weight of the apparatus reduced to the point of suspension of the scale pan. In order to determine this last, we may set the apparatus upon a knife edge at $D$, and support $A$ from the end of a balance by a cord.

In order that a friction brake may possess stability, we should, just as in the ordinary balance, hang the weight $G$

from a knife edge at *A*, and not place this latter, as in Fig. 256, above, but, as in Fig. 257, below the axis *C* of the shaft.

FIG. 256.

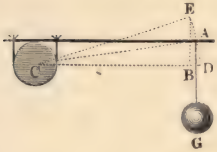
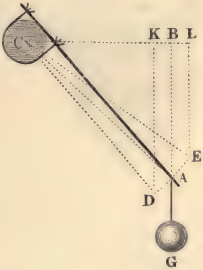


FIG. 257.

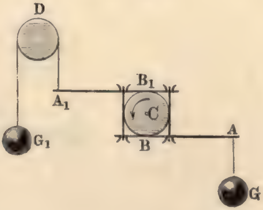


When in the latter case the weight *G* falls or rises, and hence the point of suspension is at *D* or *E*, the lever arm *C B* diminishes or increases, and the lever *C A* comes back to equilibrium.

In the first case (Fig. 256), on the other hand, with an increase or decrease of *G* we have also an increase or decrease of the lever arm $C B = a$, and the lever *C A* cannot of itself come into equilibrium.

In order not to increase the axle friction, it is advantageous to use two brakes $A\ B$, $A_1\ B_1$, Fig. 258, or to support

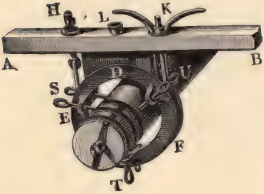
FIG. 258.



the single brake by a force $G_1 = G$ applied at $B_1$.

A better form of instrument is the friction brake shown in Fig. 259, with a cast-iron friction ring *D E F*, which by the three pairs of screws *S*, *T*, *U*, can be applied to any shaft not too large for the ring. In this apparatus the lower wooden segment or brake is replaced by an iron band which embraces half the circumference of the friction ring, which is grooved for this purpose. This band is attached to two bolts passing through the beam *A B*, and may be tightened at pleasure by two nuts at *K* and *H*.

FIG. 259.



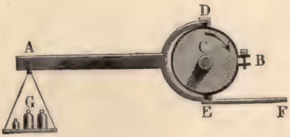
In order to prevent charring of the wood or too great heating of the iron, oil or water is supplied to the rubbing surfaces through the hole *L*. This apparatus is known in Germany under the name of "Egen's Friction Brake."

EXAMPLE.—To determine the mechanical effect of a water-wheel, a friction brake was applied to the axle, and the water being perfectly regulated for $u = 6$ revolutions per minute, the weight $G$, including the reduced weight of the instrument, was found to be 530 lbs., the lever arm for this weight $a = 10 \cdot 5$ ft. Hence the effect of the wheel was

$$L = \frac{\pi \times 6 \times 10 \cdot 5}{30} \times 530 = 3497 \text{ ft. lbs.} = 6 \cdot 3 \text{ horse-power.}$$

§ **133.**—In recent times, many forms of friction brake, more or less perfect, and some very complicated, have been adopted. We notice here only the simplest. Fig. 260 represents the Armstrong brake. This consists of an iron ring, which can be tightened upon the shaft by the screw *B*, and of a lever *A D E*, which bears upon one end the scale pan for the weight *G*, and at the other end spreads into a fork which grasps two projections *D*, *E*, upon the ring. One prong of this fork is prolonged as at *E F* for convenience in handling and in application. The

FIG. 260.



method of conducting and computing the experiment with this instrument is the same as above.

A small dynamometer composed of iron bars of 2¾ in. broad and 1 in. thick, Fig. 261, has been applied by Tauberth for the determination of the mechanical effect of a steam-engine of five horse-power. This dynamometer was applied to the band wheel *B D*, which revolved upon the 4⅛ in. shaft *C*, and the pressure of the brakes *E*, *F*, was caused by the screw *S*. The force was measured by a spring balance similar to Fig. 232, the lever arm *C A* measuring 118¾ inches. (See "Civilingenieur," Band III., 1856.)

FIG. 261.

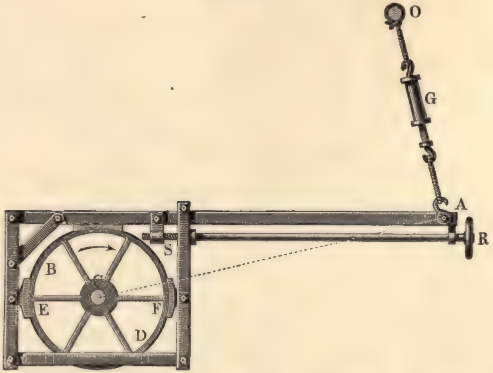
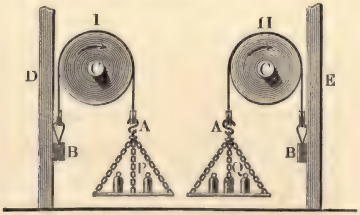


FIG. 262.



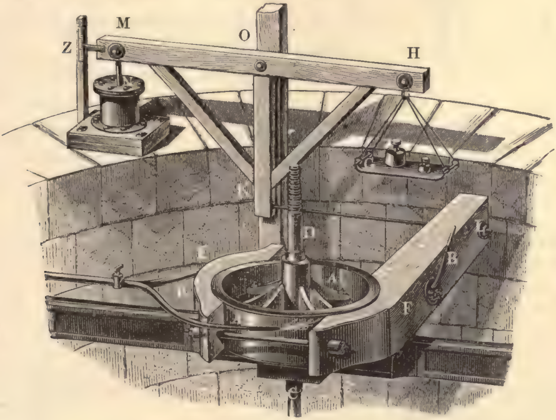
When a spring dynamometer is used, we can easily, by means of a tracing or registering apparatus, sum up or obtain directly the effect of the machine by the friction brake. Navier proposed to determine the force of a revolving shaft also by running an iron band around it, and attaching one end to a spring balance and the other to a weight sufficient to create friction enough to cause the shaft to revolve at the desired speed. The difference between this weight $Q$ and the force $P$ indicated by the balance gives the friction between the band and shaft. If, now, the circumference of the shaft is $p$, and if it makes during the experiment $u$ revolutions per minute, we have for the mechanical effect

$$L = F \frac{up}{60} = \frac{up}{60} (Q - P).$$

In default of a spring balance, a simple band, as in Fig. 262, is sufficient, if the experiment is made twice, the end $B$ being fastened first on one and then on the other side of the shaft, as shown in the figure (the arrows in the figure should be reversed). We thus obtain by one experiment $Q = P + F$, and in the other $Q = P$; because in one case the friction $F$ acting in the direction of revolution of the shaft counteracts the weight $P$ in the scale at $A$, and in the other case the weight acts with the friction. The method of determination of the effect by this apparatus, first used by the author, is the same as above. It can only be used for small effects, as the power has only a small leverage. In order to find the effect of more powerful machines, the author has replaced the weight at $A$ by a balance of unequal arms, and thus determined the tension of the band. In order that the axle pressure may not be increased, and also that the instrument may be used for greater forces, we may pass the band entirely round the shaft, and fasten one end above and the other below.

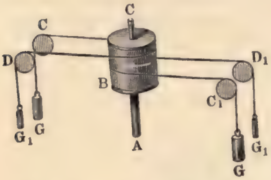
§ **134.**—If we wish to determine the force of rotation of a vertical shaft—for instance, that of a turbine—by the friction brake, we can, of course, no longer suspend the weight directly from the lever, but must insert a guide roller or bent lever, so that the vertical direction of the weight may be converted into a horizontal force. In Fig. 263, we have a representation of a friction brake for a vertical shaft. This dynamometer was used by Francis in his hydraulic experiments (Lowell hydraulic experiments) for the determination of the effect of a turbine of 75 horse-power. *A A* is the cast-iron friction wheel of $5\frac{1}{2}$ ft. diameter and $2\frac{1}{4}$ ft. depth, which is fastened to the turbine shaft *C D*. The iron-shod brakes *E*, *F*, are tightened by two screw bolts of 2 sq. in. cross-section by means of the wrench *B*, and the end of the longer brake *F* is connected by an iron rod *K L* with the bent lever *K O H*, from whose horizontal arm *O H* the weight *G* is suspended. In order to prevent great vibrations of the instrument, an hydraulic "moderator" *N* is attached to the arm *O M*, and the deviation of *H M* from the horizontal is shown by the scale *Z*. The moderator consists essentially of a plate which moves up and down in the water-filled vessel *N* so that the water is forced to flow around the plate as it moves. In order to prevent overheating, two forked pipes *R* carry jets of water against the friction wheel.

FIG. 263.



In order to determine the effect of smaller forces, the method adopted by the author for model wheels may be used. Around a drum $B$, Fig. 264, which is fastened to the revolving shaft $A\,C$, whose power we wish to measure, two bands are passed, so that the two ends of each have opposite directions. These ends pass over guide rollers $C$, $C_1$ and $D$, $D_1$, and suspend weights $G$, $G$, and $G_1$, $G_1$. If, now, the weights $G$ and $G_1$ of each band are in equilibrium with the friction of that band, the entire force of rotation of the drum is $P = 2\,(G - G_1)$, where, of course, $G$ is the larger weight acting opposed to the direction of revolution.

FIG. 264.

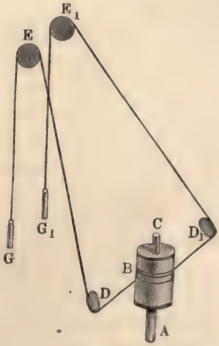


REMARK 1.—We may also determine directly the force of rotation of a shaft by weights suspended from the ends of a single band which the revolving shaft winds up. In my experiments upon models (see Weisbach's "Versuche über die Leistung eines einfachen Reactionsrades," Freiberg, 1851), I have, in order to diminish as much as possible the side pressure of the weights, made the revolving shaft wind up at once two equal weights, $G$ and $G_1$, Fig. 265, and for this purpose wound the band by means of the rollers $D$, $E$ and $D_1$, $E_1$, in opposite directions and upon opposite sides of the drum $B$.

The dynamometer, by means of which the axial force of steamship screws is determined, also belongs properly here. The shaft of the screw is connected with a lever whose longer arm is furnished with a spring dynamometer and recording apparatus, which records the work upon the surface of a revolving cylinder. (See "The Indicator and Dynamometer," etc., London, 1847.)

REMARK 2.—Egen treats of the various dynamometers in his "Untersuchungen über den Effect einiger Wasserwerke," etc. Also, Hülsse in the article "Bremsdynamometer" in the "Allgemeine Maschinenencyclopädie." In these two treatises will be found the complete literature of the subject. We have only to add here the new contributions in Dingler's Journal, Bände 88, 92, and

FIG. 265.



110. Especially worthy of note are the self-regulating dynamometers of Poncelet, Saint-Leger, etc., which by suitable wheel-work tighten or loosen the screws themselves according as the lever falls or rises. Upon spring dynamometers, we may also refer to "Notions fondamentales de Mécanique," par Morin, Paris, 1855; and upon dynamometers generally, Prechtl's "Technologische Encyclopädie," and Hachette, "Traité élémentaire des machines." Special treatises upon the subject are cited in the text. For the dynamometer with registering apparatus of Moison, Noury, and Matter, see "Civilingenieur," Bd. VIII., 1862.

§ **135.**—**Planimeter.**—The curved area obtained by the application of the recording apparatus, in experiments with the dynamometer, which gives the mechanical effect of a machine, may be measured by the planimeter. Among the various instruments of Ernst, Wetli, Hansen, Oppikofer, and Amsler, the last or the so-called polar planimeter of Amsler is one of the simplest. It is represented in Fig. 266. The instrument turns about the needle *C* which is driven into the table, while the pointer *B* is made to describe the circumference of the curve whose area is to be determined. The arms *A C* and *A B* are united by an axis *A*, and the prolongation *A D* of the arm *A B* carries a wheel *R* which runs upon the paper while the pointer *B* describes the outline of the area. In order to tell the number of revolutions of this wheel, it is graduated, and also, by means of an endless screw, turns a second graduated wheel *S*, which makes one revolution to ten of the first.

FIG. 266.

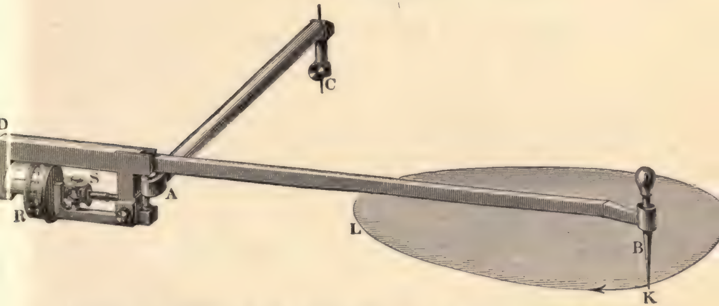
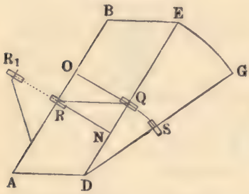


FIG. 267.



How the area of the curved figure described by the pointer may be deduced from the number of revolutions of the wheel *S*, may be shown in the following elementary manner: If a straight line *A B*, Fig. 267, is moved parallel to itself to the new position *D E*, the wheel *R* passes through the distance $R\,Q = A\,D = B\,E$, which is the resultant of the distances *R N* and *N Q*, of which the first is perpendicular to *A B*, and the second has the direction of *A B* or *D E*. In rolling upon the plane *A B D E* the distance $R\,N = \phi_1\, r$, where $\phi_1$ is the arc of revolution of the wheel for radius unity and $r$ its radius. But $A\,B \times R\,N = b\,\phi_1\, r = \phi_1\, b\, r$ is the area $P$ of the parallelogram *A E*, and hence

$$\phi_1 = \frac{P}{b\,r}$$

is a measure of this area.

If *D E* further turns about *D*, the wheel describes an arc *Q S*, and the line describes the sector *D E G* whose area is

$$S = \frac{D\,E \times E\,G}{2} = \tfrac{1}{2}\,\overline{D\,E}^2 \times \psi = \tfrac{1}{2}\,\psi\, b^2,$$

where $\psi$ is the arc for radius unity of the angle *E D G*. Therefore, the area *A B E G D* of the entire space is

$$F_1 = P + S = \phi_1\, b\, r + \tfrac{1}{2}\,\psi\, b^2.$$

If the angle of revolution of the wheel, in passing over *Q S*, is $\phi_2$, we have for the angle of revolution for the distance $R\,Q + Q\,S$,

$$\phi = \phi_1 + \phi_2, \qquad \text{and hence} \qquad \phi_1 = \phi - \phi_2,$$

or since, when the distance $A\,R = D\,Q = D\,S$ is denoted by $c$,

$$QS = \psi c = \phi_2 r, \ \phi_2 = \frac{c}{r}\psi, \ \phi_1 = \phi - \frac{c}{r}\psi,$$

and

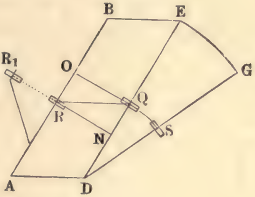
$$F_1 = \left(\phi - \frac{c}{r}\psi\right) b r + \tfrac{1}{2}\psi b^2$$

$$= \phi r b + \frac{\psi}{2}(b^2 - 2bc),$$

or

$$F_1 = b s_1 + \frac{\psi}{2}(b^2 - 2bc),$$

FIG. 268.



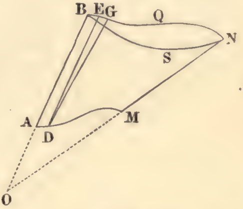
when $s_1 = \phi r =$ the entire distance of revolution of the wheel.

If the distances $AD = BE$ and $EG$ are very small, $ABEGD$ is the element of a figure such as $ABNM$, Fig. 269, described by any arbitrary motion of $AB$ in the plane of the paper, and in the formula

$$F_1 = b s_1 + \frac{\psi}{2}(b^2 - 2bc),$$

we must put for $\psi$ the arc for the entire angle $BON$ made by the two limiting positions $AB$ and $MN$ of the generating line, if we wish $F_1$ to be the area of the whole figure $ABQNM$. If the line $MN$ moves back to $AB$, it may describe the area

FIG. 269.



$$F_2 = b s_2 + \frac{\psi}{2}(b^2 - 2bc),$$

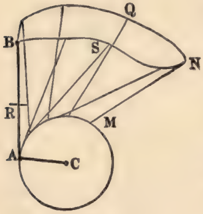
where $s_2$ is the arc of revolution of the wheel in the opposite direction; and if during backward motion the lower end of the generating line remains upon $AM$, we have between $BQN$ and $NSB$ an area $F$ which is the difference between $F_1$ and $F_2$. Hence

$$F = F_1 - F_2 = b(s_1 - s_2) = bs,$$

where $s$ denotes the difference of the arcs of revolution $s_1$ and $s_2$ given directly by the readings of the graduated wheel $S$, during the description of the figure $B\,Q\,N\,S\,B$.

In the Amsler planimeter the end $A$ of the line $A\,B$ describes a circular arc $A\,M$, Fig. 270, so that here also the area of the figure $B\,Q\,N\,S$ described by the point $B$ is proportional to the arc of revolution $s$ of the wheel $R$ and

FIG. 270.



$$\text{I. } F = b\,s.$$

This formula holds good also when the wheel, instead of being fixed to $A\,B$, is connected only with $A\,B$ as at $R_1$ in Fig. 268, only we must then understand by $c$, not the distance $A\,R_1$ but its projection $A\,R$ upon $A\,B$.

The last formula assumes that the point $A$ during description of the figure moves from $A$ to $M$ and back. If, however, this point goes round in the circle, as in Fig. 271 and Fig. 272, we must take into account also the area of the circle $C\,A\,M$. If the radius of this circle is $a$, we

FIG. 271.

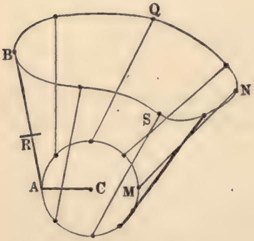
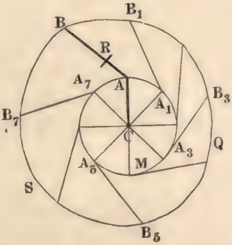


FIG. 272.



have, in the case of Fig. 271, where $C$ lies without the area $B\,Q\,N\,S$,

$$F = \pi\,a^2 + b\,s,$$

and in the case of Fig. 272, where $C$ lies within the area, and $A\,B$ makes a complete revolution, and therefore $\psi = 2\,\pi$,

$$F = \pi\,a^2 + \pi\,(b^2 - 2\,b\,c) + b\,s = \pi\,(a^2 + b^2 - 2\,b\,c) + b\,s.$$

The case of Fig. 271 assumes that $b > 2\,a$, therefore $a < \frac{1}{2}\,b$. If, therefore, as is generally the case, $a > \frac{1}{2}\,b$, the case of Fig. 271 does not occur. When in the second case the area $B\,Q\,S$ is enclosed by the circle $A\,M$, we have $b\,s$ negative, and hence

$$F = \pi\,(a^2 + b^2 - 2\,b\,c) - b\,s.$$

REMARK.—We may here refer to the planimeters of Ernst, Wetli, and Hansen, by Bauernfeind, Munich, 1853, as also "Mechanische Bestimmung des Flächeninhaltes, etc., ebener Figuren," by Amsler, Schaffhausen, 1856.

# CHAPTER II.

## THE POWER OF MEN AND ANIMALS, AND ITS RECIPIENT MACHINES.

§ 136.—**The Power of Animals.**—The working power of animals is, of course, not only different for animals of different species, but also for individuals of the same species. The power of animals of the same species depends upon individual constitution, age, health, temper, and management, as also upon the food, etc. We cannot regard all these varying elements, but for each species employed by man we assume a fair average specimen, judiciously applied to the work and suitably fed.

But the power of an animal depends also upon the force exerted, upon the speed and time of working. There is a certain mean effort, speed and time for which the work performed is a maximum. The greater the force exerted by an animal, the less the speed, and, inversely, the greater the speed the less the force exerted; indeed, there may be a maximum force for which the speed and therefore the work is zero, and, in like manner, a maximum speed for which the force and therefore the work is zero. We see, therefore, that animal powers should work with a certain mean force and mean velocity and for a certain average time, in order to obtain the greatest amount of work possible. It is a matter of observed fact that small deviations from the conditions necessary for the maximum effect are of little consequence. It is also a fact that animals produce a greater effect when they work with varying efforts and velocities than when these are constant. Less fatigue is experienced when pauses occur, which must be repeated more frequently the more the work actually performed in a unit of time differs from the mean amount of work.

The main point in the estimation of the animal powers is the *day's work*. If we compare this with the daily maintenance, and with the daily interest upon the cost, we may obtain a measure for the comparison of the values of different animal powers.

§ **137.**—The manner and means of employing the power of men and animals are very different. Animals work either with or without the intervention of machines; men with the hands or feet, or both together; animals with the feet alone. For the very many methods employed, the degree of fatigue is not proportional to the work performed. Many operations fatigue more than others, or, what amounts to the same thing, the mechanical effect is less in some methods of applying labor than in others. Thus, all labor cannot be measured by the same standard; the work done in the transport of burdens on a horizontal road cannot be referred to the same standard as the raising of a weight. According to the ideas already acquired, the mechanical effect performed in the transport of burdens on a horizontal road is zero, because there is no space described in the direction of the force (Vol. I., Art. 83), which acts vertically, while, in lifting a weight, the work done is the product of the weight into the distance through which it has been raised. It is true that walking or carrying fatigues as well as lifting—*i. e.*, the capacity for work is diminished by one kind of labor as well as the other, and therefore a certain "day's work" is to be attributed to one as well as to the other, although they are in their nature essentially different. According to experiment, a man can walk, unburdened, for 10 hours a day, at $4\frac{3}{4}$ ft. per second (about $3\frac{1}{4}$ miles per hour). If we assume his weight at 140 lbs., we have for his day's labor, $140 \times 4 \cdot 75 \times 60 \times 60 \times 10 = 23940000$ ft. lbs.

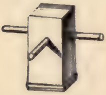
If the man carries 80 lbs. upon his back, he can walk for 7 hours daily with a speed of 2·4 ft. per second, and, apart from his weight, his "day's work" is then $80 \times 2 \cdot 4 \times 60 \times 60 \times 7 = 4838400$ ft. lbs.

A horse will carry 240 lbs. 10 hours daily, walking 3½ ft. per second, and the day's work is therefore $240 \times 3 \cdot 5 \times 60 \times 60 \times 10 = 30240000$ ft. lbs., or more than six times as much as a man doing the same kind of work. If the horse carries only 160 lbs., he can trot 7 hours daily at a speed of 7 ft. per second, and the day's work is then $160 \times 7 \times 60 \times 60 \times 7 = 28224000$. The work done in raising burdens is much less, for in this case we understand work in its proper sense, or the distance is measured in the direction of the force.

If a man, unburdened, ascend a flight of steps, then, for a day's work of 8 hours, the vertical velocity is 0·48 ft., and hence his day's work is $140 \times 0 \cdot 48 \times 60 \times 60 \times 8 = 1935360$ ft. lbs. Thus a man can go over 12½ times the space horizontally that he can vertically.

In the construction of a dam, the author observed that four practised men raised a rammer, Fig. 273, weighing 112 lbs., 34 times per minute, 4 ft. high, and after a spell of 260 seconds, rested 260 seconds; so that, on the whole, the day's work was 5 hours. The day's work of each man was then $\frac{112}{4} \times 34 \times 60 \times 5 \times 4 = 1142400$ ft. lbs.

FIG. 273.



REMARK 1.—More detailed information upon animal power will be found in the "Ingenieur." We also give further on, the work of animals by the aid of machines, for each machine treated of.

REMARK 2.—The work of men and animals is by no means completely determined. The work of unpractised laborers or men working under disadvantageous circumstances, such as great heat, rain, etc., may be only half that of well-trained hands. The first complete investigations upon the effect of animal motors were made by Coulomb, in his "Théorie des machines simples." Desaguliers ("Cours de Physique expérimentale") and Schulze ("Abhandlungen der Berliner Akademie," 1783) had previously investigated the subject. In more recent time, the experiments of Coulomb have been extended by many others. See Hachette, "Traité élémentaire des machines." Bouguer, Euler, and Gerstner have sought to deduce the laws of animal power. The problem, however, can by no means be considered as solved, even by Gerstner (Mechanik, Bd. I.).

§ **138.—Formulæ.**—Force and velocity in the application of animal power stand in the most intimate relation, but nevertheless the law of their dependence is by no means known, and is still less deducible *à priori.* The empirical formulæ of Bouguer and Euler can only be considered as approximations. If $K_0$ is the greatest force which an animal can exert without velocity, and $c_0$ the greatest velocity without the exertion of force, we have for any other velocity $v$, according to Bouguer, the force

$$P = \left(1 - \frac{v}{c_0}\right) K_0,$$

and according to Euler,

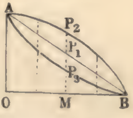
$$P = \left(1 - \frac{v^2}{c_0^{\,2}}\right) K_0,$$

and also

$$P = \left(1 - \frac{v}{c_0}\right)^2 K_0.$$

Of these three formulæ, the first is the simplest, and, according to Gerstner, that which best agrees with observation. According to the observations of Schulze and others, again, the third formula seems most consistent with experiment. If we take $v$ as the abscissa and $P$ as the ordinate of a curve, the first formula corresponds to a straight line $A\,B$, Fig. 274, the second to a parabola $A\,P_2\,B$ concave to the axis $O\,B$, and the third to a parabola $A\,P_3\,B$ convex to the axis $O\,B$, and the ordinate $M\,P_1$ to the straight line lies always between the ordinates $M\,P_2$ and $M\,P_3$ of these two curves. Thus, to the abscissa $O\,M = v = \frac{1}{2}\,c_0$ the corresponding ordinate is $M\,P_1 = \frac{1}{2}\,K = \frac{1}{2}\,O\,A$, while $M\,P_2 = \frac{3}{4}\,K = \frac{3}{4}\,O\,A$ and $M\,P_3 = \frac{1}{4}\,K = \frac{1}{4}\,O\,A$. The formula of Bouguer thus gives values of the force which lie between those given by the formulæ of Euler, and we may at least make use of it until some special reason can be given for preferring one of the latter. If we put for the maximum values

FIG. 274.



$K_0$ or $c_0$ in Bouguer's formula, the mean values $K = \frac{1}{2} K_0$ and $c = \frac{1}{2} c_0$, we have the formula first given by Gerstner,

$$P = \left(1 - \frac{v}{2c}\right) 2K,$$

or

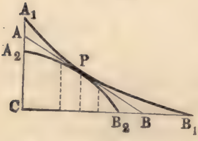
$$P = \left(2 - \frac{v}{c}\right) K,$$

or inversely

$$v = \left(2 - \frac{P}{K}\right) c.$$

Even although we may not be able to depend upon this formula as accurate for extreme values of $v$ and $P$, we may at least expect that for values not deviating very much from the mean, they are sufficiently exact, especially since for equal values of $c$ and $K$, both the curves $A_1 P B_1$ and $A_2 P B_2$, Fig. 275, for Euler's formulæ are tangent at $P$ to the straight line for Bouguer's formula.

FIG. 275.



The mechanical effect per second is therefore

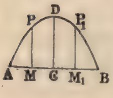
$$P v = \left(2 - \frac{v}{c}\right) v K.$$

Since $\left(2 - \frac{v}{c}\right) v K = \left(2c - v\right) v \frac{K}{c}$, we have, as in Vol. I., Art. 500, the mechanical effect greatest when $v = c$ or when $P = K$; that is, *when the velocity and force have their mean values*, or

$$L = P v = K c.$$

If we try to get a greater or less velocity, or a greater or less force, we obtain a mechanical effect $L = P v$ less than $K c$. If, again, we take the velocity as abscissa and the effect as ordinate, we obtain a parabola $A D B$, as in Fig. 276, and see at once that both for $A M < A C$ and for $A M_1 > A C$, we have ordinates $M P$, $M_1 P_1$ less than

FIG. 276.



for $A\ C = c$. For $v = \frac{c}{2}$, as also for $v = \frac{3}{2}\ c$, we have, for example,

$$L = \tfrac{3}{4}\ K c, \qquad \text{therefore} \qquad M\ P = M_1\ P_1 = \tfrac{3}{4}\ C\ D.$$

According to Gerstner, we have for the *draught* of animals the following table:

| Animals. | Weight. | Mean effort $K$ in lbs. | Mean velocity $c$ in ft. per second. | Mean period of day's work in hours. | Mechanical effect per sec. in ft. lbs. | Mechanical effect per day in ft. lbs. |
|---|---|---|---|---|---|---|
| Man....... | 150 | 30 | 2·5 | 8 | 75 | 2160000 |
| Horse..... | 600 | 120 | 4·0 | 8 | 480 | 13824000 |
| Ox......... | 600 | 120 | 2·5 | 8 | 300 | 8640000 |
| Ass........ | 360 | 72 | 2·5 | 8 | 180 | 5184000 |
| Mule....... | 500 | 100 | 3·5 | 8 | 350 | 10080000 |

EXAMPLE 1.—According to the above table, a man working with a mean velocity of 2½ ft. per second and mean effort of 30 lbs., accomplishes daily a mechanical effect of 2160000 ft. lbs. If, however, he be forced to work with a mean velocity of 3 ft. per second, the effort will be only

$$P = \left(2 - \frac{3}{2\cdot5}\right)30 = 24 \text{ lbs.},$$

and his daily effect will be only 24 × 3 × 8 × 60 × 60 = 2073600 ft. lbs.

EXAMPLE 2.—If a horse be obliged to pull with an effort of 150 lbs., he can only do it with the velocity

$$v = \left(2 - \frac{150}{120}\right) 4 = 3 \text{ ft. per second},$$

and thus his mechanical effect is only 3 × 150 = 450 ft. lbs. per second, or 30 ft. lbs. less than when working at the mean velocity of 4 ft. per second, and with a draught of 120 lbs.

REMARK.—Fourier gives in the "Annales des ponts et chaussées," 1836, a complicated formula for the mechanical effect of horses. See also Crelle's "Journal der Baukunst," Band XII., 1838.

§ **139.**—We can also compute, according to Gerstner, the mechanical effect of animals mounting inclined planes. If $G$ is the weight of the animal, $Q$ the load, and $\alpha$ the angle of inclination of the plane, then the effort along the plane is $(Q + G)\ sin.\ \alpha$ (Vol. I., Art. 146), and hence

$$\left(2 - \frac{v}{c}\right) K = (Q + G)\ sin.\ \alpha.$$

We can from this determine the load $Q$, and also the angle $\alpha$ corresponding to a given load. Thus,

$$sin.\ \alpha = \frac{\left(2 - \frac{v}{c}\right)K}{Q + G},$$

or for $Q = 0$ and $v = c$, *i. e.*, when the animal is unloaded and moves with the mean velocity;

$$sin.\ \alpha = \frac{K}{G}.$$

But the weight of an animal is almost always five times its mean effort, and hence $sin.\ \alpha = \frac{1}{5}$, or $\alpha = 11\frac{1}{2}°$. The angle of inclination of a plane upon which an animal moves with mean effort is then $11\frac{1}{2}°$, corresponding to a rise of one foot in every five feet, nearly.

REMARK.—In walking upon a horizontal plane $HR$, Fig. 277, the body turns about the point $C$, while the centre of gravity of the body rises through the distance $DE = h$. If $CA = CB = l$ and $CH = CR = s$, we have

$$DE = \frac{\overline{AD}^2}{2\,AC}, \text{ or } h = \frac{s^2}{8\,l}.$$

If, now, $G$ is the weight of the man and $Q$ the load, we have at each step the mechanical effect

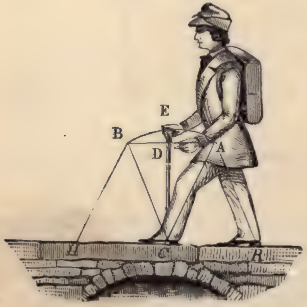
$$L = (G + Q)\,h = \frac{(G + Q)\,s^2}{8\,l},$$

and hence the corresponding effort

$$P = \frac{L}{s} = \frac{(G + Q)\,s}{8\,l}.$$

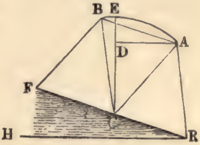
If we take $l = 3$ ft. and $s = 2$ ft., we have

FIG. 277.



$$P = \frac{2\,(G + Q)}{8 \times 3} = \tfrac{1}{12}\,(G + Q) = 0.08333\,(G + Q).$$

FIG. 278.



For $Q = 0$ and $G = 150$ lbs., we have $P = \frac{1}{12}\,G = 12 \cdot 5$ lbs. Therefore the mechanical effect while walking along a horizontal plane through a distance $s$ is equal to that due to a vertical rise of $\frac{1}{12}\,s$. The exertion in walking on a level is then for equal distances the same as the raising of 12·5 lbs.

In climbing an inclined plane $FR$, Fig. 278, if $\alpha$ is the angle of inclination $FRH$, and $\beta$ the angle $A\,C\,B$, we have for the height $D\,E$ for each step,

$$DE = h = CE - CD = CE\,(1 - cos.\,A\,C\,D) = l\left[1 - cos.\left(\alpha + \frac{\beta}{2}\right)\right] =$$
$$l\left(1 - cos.\,\alpha\, cos.\,\frac{\beta}{2} + sin.\,\alpha\, sin.\,\frac{\beta}{2}\right) = l\left(\frac{s^2}{8\,l^2} + \frac{s}{2\,l}\,sin.\,\alpha\right) = \frac{s}{2}\left(\frac{s}{4\,l} + sin.\alpha\right).$$

The mechanical work for each step is then

$$L = (G + Q)\,h = (G + Q)\left(\frac{s}{4\,l} + sin.\,\alpha\right)\frac{s}{2},$$

and the mean effort is,

$$P = \tfrac{1}{2}\,(G + Q)\left(\frac{s}{4\,l} + sin.\,\alpha\right).$$

In descending the plane, $\alpha$ is negative, and hence

$$P = \tfrac{1}{2}\,(G + Q)\left(\frac{s}{4\,l} - sin.\,\alpha\right).$$

For $sin.\,\alpha = \frac{s}{4\,l}$ we have the effect zero. If we assume again $l = 3$ and $s = 2$ ft., we have $sin.\,\alpha = \frac{1}{6} = 0.1666$ or $\alpha = 9\frac{1}{2}$ degrees, the angle of inclination for which at any rate the descent is easiest. This corresponds nearly to a fall of one foot in every six feet horizontal.

If the angle $\alpha = \frac{\beta}{2}$, we have for the effort in ascent,

$$P = \frac{(G + Q)\,s}{4\,l}.$$

and if $\alpha > \frac{\beta}{2}$, *i. e.*, $> \frac{s}{2\,l}$, or if $\alpha > \frac{1}{3}$, and hence $\alpha^\circ > 19$ degrees, we have simply

$$P = (G + Q)\,sin.\,\alpha.$$

§ **140.—Work by the Aid of Machines.**—If, according to Gerstner, the time of each shift or day's work $z$ has the same influence upon the amount of work done as the velocity, we have for the effort,

$$P = \left(2 - \frac{v}{c}\right) \left(2 - \frac{z}{t}\right) K,$$

and hence the daily effect produced is

$$L = \left(2 - \frac{v}{c}\right) \left(2 - \frac{z}{t}\right) K v z.$$

There can be no doubt that the effect is greatest and equal to $Kct$, when the animal works not only with the mean velocity and effort, but also within the mean of the time. We must, however, keep in mind that the above formula holds good only for such values of $v$, $z$ and $P$ as deviate but little from the mean values $c$, $t$ and $K$.

Maschek recommends the simpler expression,

$$P = \left(3 - \frac{v}{c} - \frac{z}{t}\right) K,$$

which is certainly more convenient for calculation. (See "Neue Theorie der menschlichen und thierischen Kräfte," etc., by F. J. Maschek, Prag.)

In general, eight to ten hours per day is a good average day's work, and hence, disregarding the factor $\left(2 - \frac{z}{t}\right)$, we have for the day's work,

$$L = \left(2 - \frac{v}{c}\right) K v z.$$

If, now, the animal work by the aid of a machine, its effort may be divided into a force $P_1$, producing useful work, and a force $P_2$, exerted in overcoming prejudicial resistances, or $P = P_1 + P_2$, both resistances being reduced to the point of application of the effort. Also in general, as we shall see, the prejudicial resistance $P_2$ may be divided into a constant part $R$ due to the machine alone, and into a variable part $\delta P_1$, depending upon the useful effect of the machine, where $\delta$ is an experimental coefficient. We have therefore,

$$P_2 = R + \delta P_1,$$

and hence

$$P = (1 + \delta) P_1 + R,$$

or

$$\left(2 - \frac{v}{c}\right) K = (1 + \delta) P_1 + R.$$

The total effect per second is now

$$Pv = \left(2 - \frac{v}{c}\right) K v = (1 + \delta) P_1 v + R v,$$

and therefore we find for the useful effect obtained,

$$P_1 v = \frac{(2K - R) v - \dfrac{K v^2}{c}}{1 + \delta} = \left[\left(2 - \frac{R}{K}\right) c - v\right] v \frac{K}{(1 + \delta) c}.$$

In order that this effect may be as great as possible (Art. 138), we must have

$$v = \tfrac{1}{2}\left(2 - \frac{R}{K}\right) c = \left(1 - \frac{R}{2K}\right) c,$$

or the velocity must be less than the mean velocity $c$, and, in fact, smaller the greater the constant portion $R$ of the prejudicial resistance. The corresponding effort is accordingly,

$$P = \left(1 + \frac{R}{2K}\right) K = K + \frac{R}{2},$$

or greater than the mean force $K$. The useful effort is, on the other hand,

$$P_1 = \frac{K - \frac{R}{2}}{1 + \delta},$$

while we have for the total effect exerted, or work performed by the animal,

$$P v = \left[1 - \left(\frac{R}{2K}\right)^2\right] K c,$$

and for the useful effect or work obtained,

$$P_1 v = \left(1 - \frac{R}{2K}\right)^2 \frac{Kc}{1 + \delta},$$

and finally for the efficiency of the machine,

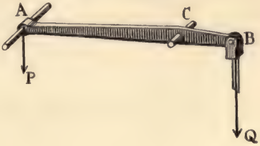
$$\eta = \frac{\left(1 - \frac{R}{2K}\right)^2}{1 + \delta}.$$

EXAMPLE.—If in a machine set in motion by two horses, the constant prejudicial resistance $R$ due to the machine alone, reduced to the point of application of the motive power, is 60 lbs., we have for the velocity at which the horses should work when $K = 2 \times 120 = 240$ lbs., and $c = 4$ ft., $v = \left(1 - \frac{60}{480}\right) c = \frac{7}{8} \times 4 = 3 \cdot 5$ ft. Further, the effort of the horses is $P = 240 + \frac{60}{2} = 270$ lbs., or that of one horse is 135 lbs. If, now, the variable part of the prejudicial resistance is 15 per cent of the useful resistance, then $\delta = 0 \cdot 15$, and hence the useful resistance overcome is $P_1 = \frac{240 - 30}{1 \cdot 15} = 182 \cdot 5$ lbs., the useful work performed is $P_1 v = 182 \cdot 5 \times 3 \cdot 5 = 638 \cdot 75$ ft. lbs. per second, or 38325 ft. lbs. per minute, or 1·16 horse-power, and finally the efficiency of the machine is

$$\eta = \frac{(\frac{7}{8})^2}{1 \cdot 15} = 0 \cdot 67.$$

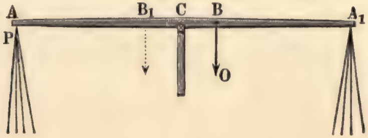
§ **141.—The Lever.**—Animal powers are applied either by means of the lever or of the wheel and axle. The latter may be either horizontal, or vertical, or inclined. Let us first consider the lever as a machine for receiving and transmitting man-power. The general theory of this machine is given in Vol. I., Arts. 135 and 187. The lever may be either single, as *A C B*, Fig. 279, or double, as *A C B* $A_1$, Fig. 280.

FIG. 279.



The first has but one arm *C A* for the application of the power, while the second has two—viz., *C A* and *C* $A_1$. We may produce by means of the lever an oscillating circular motion, and it is therefore advantageously applied in those cases where a reciprocating up-and-down motion is desired, as in pumps. For the application of man-power, handles may be attached in the desired number and length. As the strength can be better exerted in pulling down than in lifting, it is usual to make the down-stroke the working-stroke, and counter-weights may be attached to aid the up-stroke, or the double lever may be used, upon which the workers alternately pull downwards. When the down-stroke is the effective stroke, the handles may often be re-

FIG. 280.



placed by ropes. Levers are also sometimes moved by the tread of the feet.

In order to avoid too great a change of direction during one stroke, the motion of the lever is confined to an arc of not more than 60°, and in order to facilitate the application of the power, the space passed through at each stroke is adapted to the length of arm of the workers, or is about 2½ to 3½ ft., while the handles should not come within from 3 to 3½ ft. from the floor. According to experiments, a man can work 8 hours per day upon a lever with the effort $K = 12$ lbs. and velocity $c = 2 \cdot 5$ ft. Therefore the mechanical effect per second is

$$L = 12 \times 2 \cdot 5 = 30 \text{ ft. lbs.},$$

or daily,

$$K\,c\,t = 30 \times 8 \times 3600 = 864000 \text{ ft. lbs.}$$

It is necessary, in the arrangement of a lever, to take care that the men may act with the ascertained mean effort and velocity, or rather that the effective effort shall exceed the mean effort by only one half of the constant prejudicial resistance.

The lever itself is subject to only one prejudicial resistance—viz., the friction upon the fulcrum. If $D$ is the pressure upon the fulcrum due to the weight of the lever and the power and resistance, $r$ the radius of the fulcrum, $\phi$ the coefficient of friction, and $a$ the lever arm $C\,A$ of the power, we have for the axle friction reduced to the point of application of the power,

$$F = \frac{\phi\, r}{a} D.$$

As, however, $\phi$ and $\frac{r}{a}$ are generally small fractions, the friction $F$ is so small that in most cases it may be neglected in comparison with the other resistances.

If we conceive at the point $B$ a useful resistance $Q$ and a prejudicial resistance $\delta Q + W$, and denote the lever arm $CB$ by $b$, we have for the moment of the power,

$$Pa = [(1 + \delta) Q + W] b,$$

and hence for the power itself

$$P = \frac{b}{a}[(1 + \delta) Q + W].$$

But that the power of the men may be most advantageously applied, we have also

$$P = K + \frac{b}{a}\frac{W}{2},$$

and hence

$$\frac{a}{b} K = (1 + \delta) Q + \frac{W}{2};$$

the ratio of the lever arms to be employed is therefore

$$\frac{a}{b} = \frac{(1 + \delta) Q + \frac{1}{2} W}{K}.$$

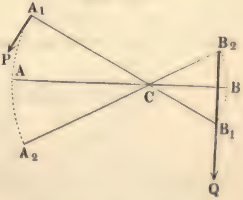
REMARK.—The lever arms during the stroke are in general to a certain extent variable, and we should therefore find and insert in the calculation their mean values. If the arm $CB$, Fig. 281, is horizontal at half stroke, and if the stroke angle $B_1 C B_2 = \beta^\circ$, we have for the height $B_1 B_2$, through which the resistance is overcome,

$$s = 2 b \sin. \frac{\beta}{2};$$

hence the work for one stroke is,

$$2 b \sin. \frac{\beta}{2} Q.$$

FIG. 281.



If, however, the resistance were constant during the stroke with the lever arm $CB = b$, the distance passed over would be the arc $B_1 B B_2 = \beta b$, and hence the resistance would be

$$Q_1 = \frac{2\,b\,sin.\frac{\beta}{2}}{\beta\,b}\,Q = \frac{2\,sin.\,\frac{1}{2}\beta}{\beta}\,Q,$$

and its moment would be

$$Q_1\,b = \frac{2\,sin.\,\frac{1}{2}\beta}{\beta}\,Q\,b.$$

Inversely, we may also assume that the resistance $Q$ acts during a stroke with the lever arm

$$\frac{2\,b\,sin.\,\frac{1}{2}\beta}{\beta}.$$

For $\beta = 60°$, this lever arm is

$$\frac{b}{arc.\,60°} = \frac{b}{1\cdot0472} = 0\cdot955\,b,$$

or 5½ per cent less than $b$, while for a smaller angle of stroke the difference is still less.

EXAMPLE.—What proportion of arms should be chosen for a lever, in order that for a useful resistance $Q$ of 160 lbs. and a prejudicial resistance,

$$Q_2 = 0\cdot15\,Q + 55 = 0\cdot15 \times 160 + 55 = 79 \text{ lbs.},$$

four men may work to the greatest advantage?

We have

$$K = 4 \times 12 = 48 \text{ lbs.},$$

hence,

$$\frac{a}{b} = \frac{1\cdot15 \times 160 + \frac{1}{2} \times 55}{48} = \frac{211\cdot5}{48} = 4\cdot4.$$

If, now, during each stroke, the resistance is overcome through one foot, the power must act through 4·4 ft., and if we assume the stroke angle at 50° we have for the desired lever arms,

$$b = \frac{s}{2\,sin.\frac{\beta}{2}} = \frac{0\cdot5}{sin.\,25°} = 1\cdot183 \text{ ft.},$$

and

$$a = 4\cdot4\,b = 4\cdot4 \times 1\cdot183 = 5\cdot20 \text{ ft.}$$

The effort or power necessary is then

$$P = \frac{160 + 79}{4\cdot4} = 54\cdot32 \text{ lbs.},$$

and hence the effort of each man is 13·58 lbs., while the efficiency of the lever is

$$\eta = \frac{\left(1 - \frac{55}{2 \times 4{\cdot}4 \times 48}\right)^2}{1{\cdot}15} = \frac{(1 - 0{\cdot}13)^2}{1{\cdot}15} = 0{\cdot}658.$$

If, then, the four men can perform daily 1075860 × 4 = 4303440 ft. lbs. of work, by this machine they will perform only 0·658 × 4303440 = 2800000 ft. lbs. of useful work.

§ **142.**—**Windlass.**—The best manner of applying the power of men is by means of the wheel and axle, which in this case receives the name of windlass (Fr. treuil, tour; Ger. Haspel). This machine consists of a horizontal axle upon whose circumference the resistance acts, and of a series of handles, or of a crank, for the application of the power. We may distinguish three kinds of windlass—viz., the crank, the spike, and the wheel and spoke. In the crank windlass the power is applied to the crank *C A D*, Fig. 282. The spike windlass is worked by hand spikes inserted in the axle *C O*, Fig. 283, and in the wheel and spoke we have a complete wheel and axle with radial handles or spokes, Fig. 284. In the first, the point of application of the power remains

FIG. 282.

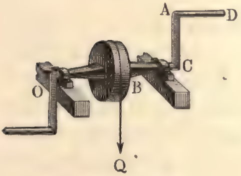


FIG. 283.

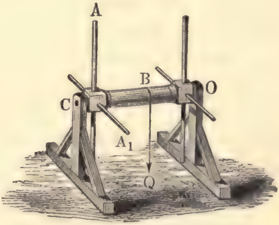
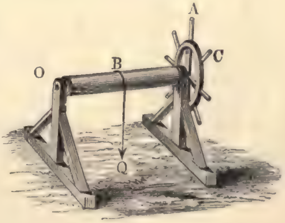


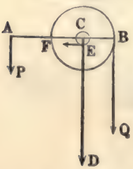
FIG. 284.



always the same; in the two last, the worker acts hand over hand. These latter are used when it is required to overcome great resistances in a short time and at long intervals, as in raising building materials and parts of machines, working the tiller of a ship, etc. The crank is used for ordinary continuous work.

In order that the workman may produce the best result with the crank-handled windlass, the length of lever should not exceed 16 to 18 inches, corresponding to his length of arm, and the axle, for men of average height, should not be more than 36 to 39 inches above the ground. The handle may be adapted to one, two, or more men as desired. As a man works with less fatigue when pushing and pressing than pulling and lifting, the effort is not the same for each point of the revolution, and it is therefore well in double-handled windlasses to set the handles 180° apart, and, when more handles are used, to distribute them at equal distances around the axle.

FIG. 285.



The day's work of a man with the crank has been found to be 1175040 ft. lbs., with a mean effort $K = 16$ lbs., and mean velocity $c = 2{\cdot}4$ ft., and length of day 8 hours. The calculation for the windlass is the same as for the wheel and axle. If the resistance $Q$, Fig. 285, acts with the lever arm $CB = b$, the force $P$ with the lever arm $CA = a$, we have $Pa = Qb$, and hence the power corresponding to a given resistance is

$$P = \frac{b}{a} Q.$$

If, now, $D$ is the pressure upon the journals, and $r$ the radius of the journal $CE$, we have more exactly

$$Pa = Qb + \phi D r,$$

and hence,

$$P = \frac{b}{a} Q + \frac{r}{a} \phi D.$$

If the resistance consists, besides the friction $\frac{r}{a}\phi D$, of the useful resistance $Q_1$, the constant prejudicial resistance $W$ and the variable prejudicial resistance $\delta Q_1$, if, therefore,

$$Q = (1 + \delta) Q_1 + W,$$

we have

$$P = \frac{b}{a}\Big[(1 + \delta) Q_1 + W\Big] = K + \frac{b}{a}\frac{W}{2},$$

and hence must have

$$\frac{a}{b} = \frac{(1 + \delta) Q_1 + \frac{1}{2} W}{K}.$$

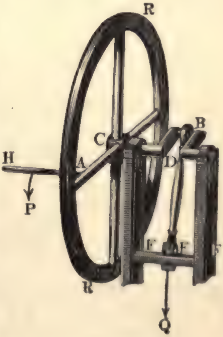
But as the crank has a prescribed length of 16 to 18 inches, the lever arm $b$ of the resistance must be determined from this, or

$$b = \frac{K a}{(1 + \delta) Q_1 + \frac{1}{2} W},$$

in order that the workman may act to the best advantage.

When the resistance $Q$ is variable, as, for example, when it acts upon another crank, $D B$, Fig. 286, it is well to furnish the axle $C D$ with a fly wheel $R R$, which by its inertia may in some measure counterbalance the variability of the necessary force $P$.

Fig. 286.



In this case, we may fix the handle $A H$ to an arm of the wheel, which then forms with it a veritable crank. The resistance $Q$ is here applied first to a cross-head $F F$, which moves in guides, and is attached to the crank by the crank rod $B E$.

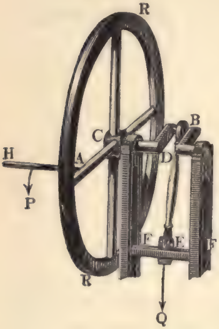
If here, again, $a$ is the length of the arm $C A$, and $b$ the length of the arm $D B$, we have during a half revolution the distance passed over by the power $\pi a$, and by the resistance $2b$, and hence, neglecting the prejudicial resistances,

$$P \pi a = Q\, 2b,$$

therefore, the mean force of revolution is

$$P = \frac{2}{\pi}\frac{b}{a} Q.$$

FIG. 287.



EXAMPLE.—Upon a double-handled windlass acts a resistance, $Q$, of 200 lbs., of which, however, only 150 lbs. is useful, while 30 lbs. is the constant, and 20 lbs. is the variable prejudicial resistance. The lever arm of the resistance is 4 inches, that of the power 18 inches, the radius of axis ½ an inch, the coefficient of friction $\phi = 0 \cdot 1$, and the weight of the machine is 80 lbs. Required the mechanical effect of this machine.

The entire power is, if we take the pressure upon the journals,

$$D = 200 + 80 = 280 \text{ lbs.};$$

$$P = \frac{4}{18} \times 200 + 0 \cdot 1 \times \frac{1}{2 \times 18} \times 280 = 44 \cdot 44 + 0 \cdot 78 = 45 \cdot 22 \text{ lbs.},$$

and hence the force exerted by one man is 22·61 lbs., and, according to Gerstner's formula, the velocity of the power or of the handle is

$$v = \left(2 - \frac{P}{K}\right) c = \left(2 - \frac{22 \cdot 61}{16}\right) \times 2 \cdot 4 = 1 \cdot 408 \text{ ft.},$$

while that of the resistance is

$$w = \frac{a}{b}\, v = \frac{2}{9} \times 1 \cdot 408 = 0 \cdot 313 \text{ ft.},$$

and the useful effect per second is then

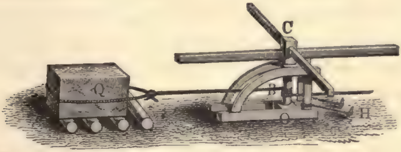
$$Q_1\, w = 0 \cdot 313 \times 150 = 46 \cdot 95 \text{ ft. lbs.},$$

or, daily, 1352160 ft. lbs. Finally, the efficiency, since the men can perform 2 × 1175040 = 2350080 ft. lbs., is

$$\eta = \frac{1352160}{2350080} = 0 \cdot 575.$$

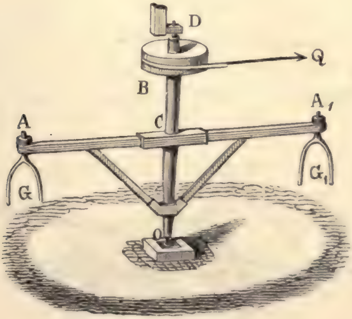
§ **143.—Vertical Capstan.**—The vertical capstan (Fr. cabestan; Ger. stehende Welle) may be set in action either by men or animals. We may distinguish the land capstan, ship capstan, and horse capstan. The land capstan, Fig. 288, is portable and serves ordinarily for moving great weights along the ground. It consists of an axle, *C O*, and of four arms, as *C A*, etc. The frame is fastened to pins at *H* by chains or ropes. The ship capstan is not essentially different.

Fig. 288.



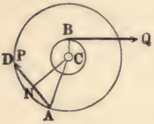
The horse windlass or *whim-gin* (Fr. baritel; Ger. Göpel) is a larger vertical axle, especially serviceable for the lifting of weights, as, for instance, in mines. It is moved either by men or animals, and is called, in the first case, a hand-gin; in the second, a horse-gin. The animals go round in a path, either pushing or pulling the arms. In Fig. 289

Fig. 289.



we have such a machine. $DO$ is the axle, which rests upon a pivot at $O$, and $ACA_1$ is a double lever with forked shafts, $G$, $G_1$. These last fit over the backs of the horses, and the harness is attached to them. The resistance $Q$ acts upon a drum or upon a toothed wheel at $B$, either directly or indirectly. The length of arms $CA$ is made as great as possible, in order that the animals may have a large circle. In the hand windlass these levers are made from 8 to 12 ft.; in the horse windlass from 20 to 30 ft. long. The line of traction should also be as nearly horizontal as possible, and hence the levers adapted to the height of the animals. In the arrangement of Fig. 289, the horses work very nearly at right angles to the levers; but if the horses be attached by traces to a cross-bar, the direction of the traction makes an angle with the lever. From the length of beam $CA = a$, Fig. 290, and the length of traces $AD = d$, we have for the lever arm of the horses

FIG. 290.



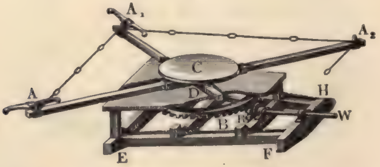
$$CN = a_1 = \sqrt{a^2 - \frac{d^2}{4}},$$

or approximately

$$a_1 = a - \frac{d^2}{8a}.$$

In Fig. 291 we have a portable capstan for country use. It consists of four arms $CA$, $CA_1$, $CA_2$ . . ., of which one is broken off in the figure, and rests upon a frame $EFH$ directly

FIG. 291.



in contact with the ground. The cross-bars $A$, $A_1$, to which the horses are hitched, are pivoted upon vertical bolts, and the arms are connected by chains. Upon the vertical axle to which the arms are attached by an iron cross, is a large toothed wheel $BD$, engaging with a smaller one $R$, which turns the horizontal shaft $RW$ by which the power is transmitted to the working point. The vertical axle passes through a collar and rests below upon a footstep.

It is found by experiment that a man can work 8 hours per day on a capstan with an effort $K = 24$ lbs. and velocity of 2 ft., and can therefore perform in a day a work of $24 \times 2{\cdot}0 \times 28800 = 48 \times 28800 = 1382400$ ft. lbs.

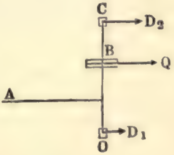
A horse in a day of 8 hours exerts a force of 90 lbs., with a velocity of 3 ft. per second, or $90 \times 3 \times 28800 = 270 \times 28800 = 7776000$ ft. lbs.

The power is as for any wheel and axle,

$$P = \frac{b}{a} Q.$$

In consequence of the friction upon the footstep and periphery of the pivots at top and bottom, this power must be somewhat greater. If $G$ is the weight of complete capstan and $r_1$ the radius of the pivot, we have for the statical moment of the friction on the footstep (Vol. I., Art. 188), $\frac{2}{3} \phi G r_1$. In general, the point of application $B$ of the resistance does not lie at the centre between the pivots $C$ and $O$, Fig. 292, but is nearer to one than the other. They therefore bear unequal pressures, and their dimensions are of course unequal also. If the point $B$ is distant from the lower pivot by $BO = l_1$ and from the upper by $BC = l_2$, and if the entire length of axle $l_1 + l_2 = l$, we have the pressure upon the lower

FIG. 292.



$$D_1 = \frac{l_2}{l} Q,$$

and for the pressure upon the upper

$$D_2 = \frac{l_1}{l} Q.$$

Hence the friction is

$$\phi D_1 r_1 + \phi D_2 r_2 = \frac{r_1 l_2 + r_2 l_1}{l} \phi Q,$$

and

$$P a = Q b + \tfrac{2}{3} \phi G r_1 + \phi Q \frac{r_1 l_2 + r_2 l_1}{l}.$$

REMARK 1.—French authors assert that a horse going at a trot can work daily 4½ hours, exerting an effort of 30 kilogrammes = 66 lbs., at a velocity of 2 meters = 6·6 ft., and therefore performing per day 7056720 ft. lbs. If we apply Gerstner's formula, and put $K = 120$ lbs., $c = 4$ ft., $v = 6 \cdot 6$ ft., $t = 8$ hrs., and $z = 4\frac{1}{2}$ hrs., we get the power,

$$P = \left(2 - \frac{6 \cdot 6}{4}\right) \left(2 - \frac{4 \cdot 5}{8}\right) 120 = 60 \text{ lbs.},$$

and therefore the day's work is 60 × 6·6 × 4·5 × 3600 = 6415200 ft. lbs., or pretty nearly the above result. If, however, we take the velocity 2·9, we obtain a much greater force—viz.,

$$\left(2 - \frac{2 \cdot 9}{4}\right) 120 = 153 \text{ lbs.},$$

and therefore per day 12778560 ft. lbs.

REMARK 2.—The forces of the horses when acting upon opposite arms do not increase the first friction, but if they are attached to one arm, their force contributes to increase this friction somewhat. Thus, according to a treatise by the author in the "Polytechnische Mittheilungen," Band I., we must put in the place of $Q$,

$$Q \left[1 + \tfrac{1}{4} \left(\frac{P}{Q}\right)^2\right] = Q \left[1 + \tfrac{1}{4} \left(\frac{b}{a}\right)^2\right],$$

and hence,

$$D_1 = \frac{l_2}{l} \left[1 + \tfrac{1}{4} \left(\frac{b}{a}\right)^2\right] Q \text{ and } D_2 = \frac{l_1}{l} \left[1 + \tfrac{1}{4} \left(\frac{b}{a}\right)^2\right] Q,$$

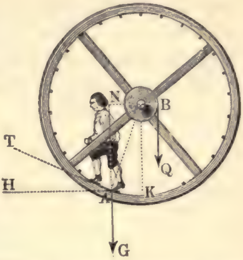
so that the moment of the friction is

$$F = \left[1 + \tfrac{1}{4} \left(\frac{b}{a}\right)^2\right] \left(\frac{r_1 l_2 + r_2 l_1}{l}\right) Q.$$

The same holds good for a man windlass.

**§ 144. Tread-Mill.**—The weight of men and animals is sometimes used as the motive power of machines, the effort or moving force being caused by climbing upon the periphery of a wheel. Such machines are called in general *tread-mills* (Fr. treuils à tambour; Ger. Treträder), and are of very various construction. The wheel consists of two crowns attached to the axle by arms and connected with each other by flooring. The worker acts either upon the interior or exterior circumference, cross-pieces or steps being placed upon the flooring (Figs. 293 and 294), at suitable intervals.

FIG. 293.



In Fig. 295, we have a wheel with only one crown, provided with rounds or pegs instead of steps, upon which the men climb as upon a ladder. The worker acts at about the middle of the wheel, so that his entire weight *G* acts with the lever arm $CA = a$, greater than the radius of the wheel itself. In the other two wheels, on the other hand, he acts at a point at an acute angle $ACK = \alpha$ from the top or bottom of the wheel, and hence the lever arm of his weight is less than the radius $CA = a$, or

FIG. 294.

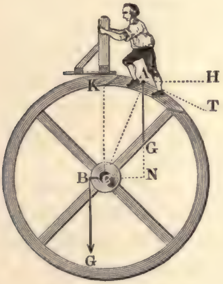
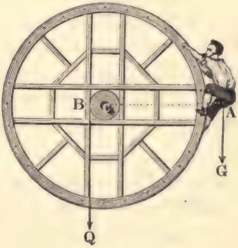


FIG. 295.



$$CN = a_1 = \overline{CA}\ sin.\ CAN = a\ sin.\ \alpha.$$

The fatigue of the laborer upon the last wheel is, however, greater than upon the other two. In the one case, the effort is that of climbing a vertical ladder, and in the others of ascending a plane inclined at the angle $TAH = CAN = \alpha$. In the case of the ladder, the effort is therefore $G$, while in the others it is $G \sin. \alpha$.

If the resistance $Q$ acts with the lever arm $CB = b$, we have for the one case

$$G a = Q b,$$

and for the other two cases,

$$G a \sin. \alpha = Q b,$$

or, introducing the effort or power $P$, we have for all three machines, as for the wheel and axle,

$$P a = Q b.$$

Mathematically considered, therefore, tread-mills possess no advantage over the windlass or capstan. A man, however, can produce a greater daily effect than by other machines, and they may often therefore be advantageously applied. The use of animals is not advantageous, not only because four-footed beasts perform less work in climbing, but also because they can be applied only with difficulty and are in danger of injury by accident.

It is found by experiment that a man can work 8 hours daily with a force of 128 lbs. and a velocity of 0·48 ft. upon a tread-mill, when acting near the middle, but only with an effort of $25\frac{5}{8}$ lbs. and a velocity of $2\frac{1}{4}$ ft. when acting at 24° from the top or bottom. The work obtained in the first case is therefore daily,

$$128 \times 0{\cdot}48 \times 28800 = 1769000 \text{ ft. lbs.},$$

and in the second,

$$25\tfrac{5}{8} \times 2\tfrac{1}{4} \times 28800 = 1663000 \text{ ft. lbs.}$$

Horses and other cattle produce less effect with such machines than with the vertical axle or gin.

A part of the advantage which tread-mills have over the windlass or capstan is neutralized by the axle friction, which in this case is much greater on account of the greater weight. If $n\,G$ is the weight of the laborers, $G_1$ the weight of the machine, and if the resistance $Q$ acts vertically downwards, we have for the pressure upon the journals,

$$D = n\,G + G_1 + Q,$$

and if $r$ is the radius of the journals, we have for the moment of the force of friction,

$$\phi\,(n\,G + G_1 + Q)\,r$$

and hence,

$$n\;G\,a\,sin.\;\alpha = Q\,b + \phi\;(n\,G + G_1 + Q)\,r.$$

If the resistance is given, we can find the angle $\alpha$, viz.,

$$sin.\;\alpha = \frac{Q\,b + \phi\,(n\;G + G_1 + Q)\,r}{n\,G\,a},$$

or the number of men,

$$n = \frac{Q\;b + \phi\;(G_1 + Q)\,r}{G\,(a\;sin.\;\alpha - \phi\;r.)}.$$

The men work to best advantage when for a constant prejudicial resistance $W$ their effort is

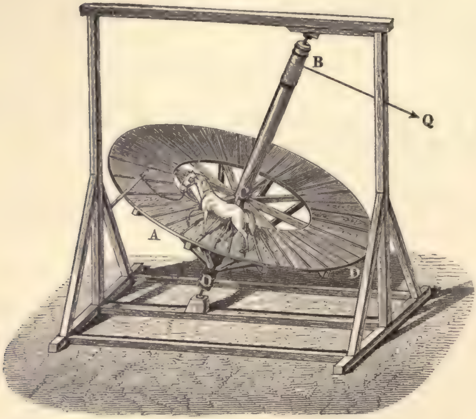
$$n\,P = n\;G\;sin.\,\alpha = n\;K + \frac{b}{a}\cdot\frac{W}{2},$$

therefore

$$sin.\;\alpha = \left(K + \frac{b}{a}\cdot\frac{W}{2\,n}\right) \div G.$$

§ **145.—Tread Plane.**—For farming purposes, such an arrangement as shown in Fig. 296 is sometimes used. The horse or ox works upon such an inclined plane only a short spell. The machine has the advantage that the animal may be left without a driver. The action of the animal is precisely the same as in the tread-mill, when he acts near the horizontal diameter. The machine consists

FIG. 296.



of an axle *B O*, whose axis deviates 20 to 25° from the vertical, and of a revolving plate *A C D* of 20 to 25 ft. radius, furnished with radial slats to afford foothold. This plate or plane is at right angles to the axle, and has therefore an inclination of 20 to 25° with the horizon. If the animal acts at the distance $C A = a$ of the horizontal radius *C A* from the axle, and if the angle of inclination is $\alpha$, the force of rotation is

$$P = G \sin. \alpha,$$

and hence just as for the tread-mill,

$$P a = G a \sin. \alpha.$$

If, now, the resistance acts with the lever arm $b$, we have its moment $= Q b$. If, further, $G_1$ is the weight of the machine and $r$ the radius of its pivot, we have for the moment of the friction on the footstep,

$$\tfrac{2}{3}\,\phi\,(G + G_1)\,cos.\,\alpha \cdot r,$$

and the moment of the friction upon the periphery of the pivots,

$$\phi\,[(G + G_1)\,sin.\,\alpha + Q]\,r\,;$$

because the weight $G + G_1$ is resolved into the components $(G + G_1)\,cos.\,\alpha$ in the direction of the axis, and $(G + G_1)\,sin.\,\alpha$ in the direction of the plane, while $Q$ acts opposite to this last. Whence we have

$$G\,a\,sin.\,\alpha = Q\,(b + \phi\,r) + \phi\,(G + G_1)\,(\tfrac{2}{3}\,cos.\,\alpha + sin.\,\alpha)\,r.$$

Since the component $G\,cos.\,\alpha$ of the weight $G$ in the direction of the axis $B\,O$ acts beyond the centre, it causes not only a pressure upon the axis but also a couple (Vol. I., Art. 133), which tends to turn the plate in the plane $A\,B\,C$, and increases somewhat the side pressures at $B$ and $O$. This increase is for ordinary dimensions and weight so small that it may be disregarded.

We may mention here the so-called horse-power engine (Tretbrücke) in which the horse works upon an inclined plane which forms an endless chain. (See the article "Tretrad" in Precht's Encyclopædia, also Whitworth: "Report on the New York Industrial Exhibition of 1853.")

EXAMPLE.—Required the number of men necessary to raise 900 lbs. by means of a tread-mill of 20 ft. diameter, the lever arm of the resistance being 0·75 ft. If we estimate the weight of the loaded wheel at 5000 lbs., the radius of the pivot at 2 inches, and the coefficient of friction at 0·075, we have

$$Q\,b + \phi\,(G + G_1)\,r = 0{\cdot}75 \times 900 + 0{\cdot}075 \times \tfrac{1}{6} \times 5000 = 675 + 62\tfrac{1}{2} = 737\tfrac{1}{2} \text{ ft. lbs.},$$

and hence the necessary power at the circumference of the wheel is

$$P = \frac{737{\cdot}5}{10} = 73{\cdot}75 \text{ lbs.}$$

Now, a man at about 24° from the top exerts a force of about $25\frac{2}{3}$ lbs. The required number of men is then

$$n = \frac{73 \cdot 75}{25\frac{2}{3}} = 3.$$

These men could produce $3 \times 1663000 = 4989000$ ft. lbs. per day of 8 hours, and therefore could raise $Q = 900$ lbs. daily through the distance $\frac{4989000}{900} = 5543\frac{1}{3}$ ft., or if the load had only to be raised 200 ft., the three men could raise $\frac{5543\frac{1}{3}}{200} = 28$ times 900 lbs. to this height.

# CHAPTER III.

## THE COLLECTION AND SUPPLY OF WATER FOR POWER.

§ **146.—Water Conduits.**—Water intended to serve as power (Fr. l'eau motrice; Ger. Aufschlagewasser), or to move machines, is collected from brooks and rivers, often also from lakes and ponds, and but seldom directly from springs. In most cases, the machine cannot be erected directly at the spot where the water is obtained, but the latter must be brought to the machine by a conduit. Such conduit may be an open channel or canal, or a pipe. Open channels include canals, ditches, and races. Canals are large and generally navigable, ditches smaller and never navigable, and formed of masonry, stones, earth or sand, while races are open channels formed of wood, iron, or stone. Pipe conduits (Fr. tuyaux de conduite; Ger. Röhrenleitungen) consist of cylindrical or prismatic tubes of iron, wood, clay, stone, glass, etc. They are usually made use of only for small quantities of water. They possess the advantage over open channels, that they admit of any rise or fall, while open canals must always descend from the point of supply. By pipe conduits, therefore, valleys and hills may be passed without aqueducts or tunnels.

§ **147.—Weirs or Dams.**—The running water from which the supply for a machine is obtained is furnished either by brooks (Fr. ruisseaux; Ger. Bäche) or rivers (Fr. rivières; Ger. Flüsse). The *vis viva* of the running water—having a velocity of from 1 to 6 ft.—is seldom sufficient to drive the machine, and in order to increase it, or to cause the water to act by means of its weight, it is necessary to dam it up and thus to create a head or fall (Fr. chute; Ger. Gefälle). The water is dammed up by weirs, bars, or dams (Fr. barrages; Ger. Wehre). We may distinguish overflow weirs and sluice weirs. In the first, the water flows freely over the sill or top edge, while in the second, movable sluice boards dam the water above the sill. In general, by means of the overflow weir, the water or a portion of it is constrained to enter a side canal near it, which conducts it to the machine; while by the sluice weir an increased *vis viva* is obtained, and the water is directly applied to the machine immediately below the weir.

In large rivers and streams, dams are often constructed which do not extend completely across. Such dams may be called incomplete weirs, in contradistinction to complete weirs which extend from side to side. Bridge piers, dykes, and other constructions which diminish the passage for the stream, may be regarded as incomplete weirs (Fr. barrage discontinus; Ger. lichte Wehre).

Overflow weirs may also be classed as complete and incomplete. The first rises above the surface of the water in the stream below it, so that the water flows freely over, while the second has its sill completely submerged, so that a portion of the water flowing over undergoes a resistance from the water below the weir.

§ **148.—Swell or Backwater.**—By all of the above constructions, there is caused a swell (Fr. remou; Ger. Stauung), *i.e.*, an elevation of the water back of the weir and a corresponding decrease in its velocity. The height and amplitude of this swell are of especial importance. The first is the height of the surface of the backed-up water above the surface of the freely flowing water immediately over the weir; the latter is the distance backwards from the weir to which the swell extends. It is a very important problem to determine the relation between the height of swell and the dimensions of the weir, the law according to which the swell varies with the distance up stream, and at what point therefore this swell becomes imperceptible.

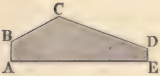
The knowledge of these relations is necessary, not only because by too great a swell, floods may be caused up stream, but also because other establishments up stream may be injured by being deprived of a portion of their fall by the backwater. For these reasons, the height of overflow sill is fixed by law and indicated by water-markers (Fr. marqueurs; Ger. Aichpfähle). The marker often has a scale attached, to show the level of the water.

The water, increased in velocity by flowing over a dam or by constraint of partial or incomplete weirs, acquires, before it assumes a uniform velocity corresponding to the inclination of the river bed, a waving, eddying motion, by which its power is diminished. The increased velocity and eddying motion of the water acts very often injuriously upon the foot of the dam or weir, which then often requires to be protected by special arrangements.

The quantity of water furnished by a brook or river is different at different times, so that we have *high water* for a short time after heavy rainfalls, *mean water*, which occurs generally in autumn and at other times—on the whole, about half of the year—and *low water* for a short time in summer. It is of importance to know at least the mean and low water of the brook, in order that the machine and also the weir and canals may be properly arranged and constructed. For this purpose, measurements of the flow at different times must be made, according to the methods given in Vol. I., Arts. 480, 481, etc. The water should in any case not be backed up by the dam or weir to such an extent as to cause during high water, flooding of the surrounding region.

§ **149.—Construction of Weirs.**—The overflow weir is the most important means of obtaining water-power. It consists generally of a dam built square across the stream, or in two inclined parts, the angle pointing up the stream and rounded off, or with a short intermediate dam at the centre. It is also sometimes made curved, with the convex side up stream. It is made of wood, stone, or of both. It can but seldom be founded upon solid rock, but must generally be built upon a pile foundation. The cross-section of wooden or other dams is of the form, more or less, of a five-sided figure *A B C D E*, Fig. 297, in which *A B* is the *breast*, *B C* the *front slope*, *C D* the *apron*, *D E* the *back*, *E A* the *sole*, and *C* the *sill* or *saddle*. The cross-section of stone weirs is composed of curved lines, approximating more or less to a five-sided figure, so as to facilitate the flow of the water.

FIG. 297.



An *incomplete* overflow weir consists, as shown in Fig. 298, of a row of piles *D* driven square across the stream, and surmounted by a saddle beam *C;* of sheeting piles *E* in front of the first piles, of a second row of piles *F* below, and a stone casing *G* between. The *complete* overflow weir shown in Fig. 299 rests upon a pile foundation with grillage

FIG. 298.

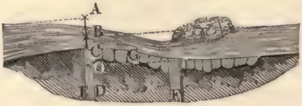
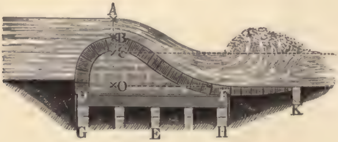


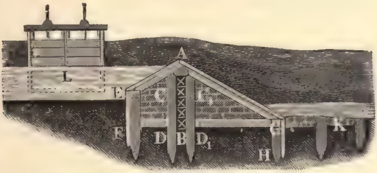
FIG. 299.



*D E F*, with two walls of sheeting piles at *G* and *H*, and is composed of large stones laid in hydraulic mortar. The outline is curved as shown in *D B F*, and in order to protect the foot *H K*, it is cased with large stones bounded by a row of piles at *K*. The intervals between the piles *G E H* may also be cleared out as far as possible and rammed with concrete.

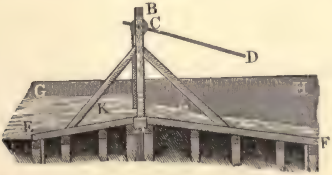
The construction of a wooden weir is shown in Fig. 300. *A B* is a wall of beams lying one upon the other; *C D* and *C*, *D*, are rows of piles driven upon either side; *E F* and *G H*, two others lined upon the exterior with a close wall of sheeting piles and surmounted with sills at *E* and *G*; *C E* and *C*, *G* are rafters joining the saddle beam *A* with these sills and covered with planking. The interior is filled with stone, clay, or concrete, and the foot of the apron is continued on to *K* and protected with large stones. At *L* we have represented the sluice gate for drawing off the water when desired.

FIG. 300.



In Fig. 301, finally, we have represented a *sluice weir*; *A* is the sill, *A B* are the guide posts with grooves in which the gate works. The devices for raising the gate are manifold. In the figure, we have a sort of capstan arrangement, and the gate hangs from it by a chain. From the sill *A* the front and rear race boards *A E* and *A F* incline downwards, and both rest upon a pile grillage. In order to prevent the water from leaking in, the piles are enclosed in a tight sheathing, and the intervals between them may also be

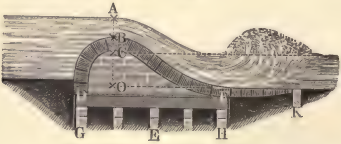
FIG. 301.



cleared out and well rammed with concrete. Upon both sides we have the side walls *G H*, formed of stout planking spiked to long piles. The centre guide posts are also furnished with braces *K*, *L*, of which the upper one, *K*, serves also as an ice breaker.

§ **150.—Height of Swell.**—By the aid of the hydraulic formulæ, already deduced in Vol. I., we may readily determine the dimensions of the swell caused by the dam. If for a complete overflow dam, Fig. 302, $h$ is the head $A\ B$, $b$ the breadth, and $k$ the height due to the velocity $c$ of the water as it flows up to the weir, or $k = \frac{c^2}{2g}$; we have for the discharge over the weir (Vol. I., Art. 416),

FIG. 302.



$$Q = \tfrac{2}{3}\,\mu\, b \sqrt{2\,g}\,[(h + k)^{\frac{3}{2}} - k^{\frac{3}{2}}].$$

If inversely the discharge $Q$ is known, we have for the corresponding head at the sill,

$$h = \left(\frac{\frac{3}{2}\,Q}{\mu\, b \sqrt{2\,g}} + k^{\frac{3}{2}}\right)^{\frac{2}{3}} - k.$$

In order, therefore, for a given rise or height of swell $A\ C = h_1$, to find the corresponding weir height $B\ O = x$, we put

$$A\ C + C\ O = A\ B + B\ O,$$

or, designating the original depth of water down stream $C\ O$, by $a$, we have

$$h_1 + a = h + x,$$

and hence,

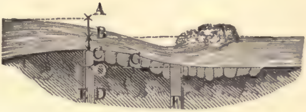
$$x = a + h_1 - h.$$

For a considerable backwater or rise, say $x$ at least 2 feet, we may disregard the head $k$ due to the velocity $c$ of the water as it comes to the weir, and put

$$x = a + h_1 - \left(\frac{\frac{3}{2}\,Q}{\mu\, b\, \sqrt{2\,g}}\right)^{\frac{2}{3}}.$$

According to experiments by the author, the coefficient $\mu$ may be taken equal to 0·80.

FIG. 303.



The calculation for the incomplete weir, Fig. 303, is more complicated, because in this case two different discharges are combined. Thus the head $A\,C = h$ is greater than the swell head $A\,B = h_1$, and therefore only the water above the level $B$ flows away freely, while the water below $B$ flows away under the head $A\,B = h_1$.

Hence, the discharge through $A\,B$ is

$$Q_1 = \tfrac{2}{3}\,\mu\, b\, \sqrt{2\,g}\,\left[(h_1 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right],$$

while that through $B\,C = h - h_1$ is

$$Q_2 = \mu\, b\,(h - h_1)\,\sqrt{2\,g}\,(h_1 + k)^{\frac{1}{2}},$$

and, therefore, the entire discharge is

$$Q = Q_1 + Q_2 = \mu\, b\, \sqrt{2\,g}\,\left(\tfrac{2}{3}\left[(h_1 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right] + (h - h_1)\,(h_1 + k)^{\frac{1}{2}}\right).$$

From the discharge $Q$ and the height $h_1$ to which the water is raised, we have the height of the water above the saddle,

$$h = h_1 + \frac{Q}{\mu\, b\, \sqrt{2\,g}\,(h_1 + k)^{\frac{1}{2}}} - \frac{2}{3}\,\frac{(h_1 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}}{(h_1 + k)^{\frac{1}{2}}},$$

whence we have the height of weir,

$$C\,O = x = a + h_1 - h.$$

For small values of $k$ we may have the simpler expression,

$$x = a + \tfrac{2}{3}\,h_1 - \frac{Q}{\mu\, b\, \sqrt{2\,g\,h_1}}.$$

The weir is incomplete when $h > h_1$, or when

$$Q > \tfrac{2}{3}\,\mu\, b\, \sqrt{2\,g}\,\left[(h_1 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right].$$

If the weir is circular or curved, we must insert in place of $b$ the length of the curve, and in the value of

$$k = \frac{c^2}{2\,g} \quad \text{make} \quad c = \frac{Q}{b\,(a + h_1)}.$$

EXAMPLE.—A brook 30 ft. wide and 3 ft. deep runs 310 cubic ft. of water per second. It is required to raise it 4½ ft. higher. What is the necessary height of weir? Since the backwater or swell is rather large, we may use the simpler formula,

$$x = a + h_1 - \left(\frac{3\,Q}{2\,\mu\, b\, \sqrt{2\,g}}\right)^{\frac{2}{3}}.$$

In this case, we have $a = 3$, $h_1 = 4{\cdot}5$, $Q = 310$, $b = 30$, $\mu = 0{\cdot}80$, and $\sqrt{2\,g} = 8{\cdot}02$. Hence,

$$x = 3 + 4{\cdot}5 - \left(\frac{3 \times 310}{2 \times 0{\cdot}8 \times 30 \times 8{\cdot}02}\right)^{\frac{2}{3}} = 5{\cdot}7 \text{ ft.},$$

and therefore the weir is a complete one, as was assumed.

If the water is to be raised only 2 ft., we would have by the last formula,

$$x = 3 + 2 - 1{\cdot}82 = 3{\cdot}2 \text{ ft.},$$

and therefore the weir is still complete.

If, finally, the swell is only 1½ ft., the dam would not rise above the level of the water down stream, or the natural level of the water in the stream, and hence the weir is submerged or incomplete. If we apply the more accurate formulæ in this case, and put

$$k = \frac{c^2}{2\,g} = 0{\cdot}0155\left(\frac{Q}{(a + h_1)\,b}\right)^2 = 0{\cdot}0155\left(\frac{310}{4{\cdot}5 \times 30}\right)^2 = 0{\cdot}0155 \times 5{\cdot}27 = 0{\cdot}084 \text{ ft. and } \mu = 0{\cdot}80,$$

we get

$$h - h_1 = \frac{310}{0 \cdot 8 \times 30 \times 8 \cdot 02 \sqrt{1 \cdot 584}} - \frac{2}{3} \frac{(1 \cdot 584)^{\frac{3}{2}} - (0 \cdot 084)^{\frac{3}{2}}}{(1 \cdot 584)^{\frac{1}{2}}}$$

$$= 1 \cdot 28 - 1 \cdot 06 + 0 \cdot 01 = 0 \cdot 23 \text{ ft.}$$

The saddle must then be about ¼ foot or 3 inches below the surface of the water, and therefore the weir itself has the height

$$x = a + h_1 - h = 3 - 0 \cdot 25 = 2 \cdot 75 \text{ ft.}$$

§ **151.—Swell for Sluice Weirs.**—The height and amplitude of the backwater or swell in the case of a sluice weir are to be determined according to the theory for discharge through sluices. Here three cases may occur. The water may flow through freely, or it may flow under water, or it may flow part freely and part under water. In the case of free flow, as shown in Fig. 301, the velocity of discharge depends only upon the head $h$ measured from the centre of the sluice-opening to the surface of the water. If, therefore, $a_0$ is the height of the opening, $b$ the breadth, we have,

$$Q = \mu\, a_0\, b \sqrt{2\, g\, h},$$

or

$$h = \frac{1}{2\, g} \left( \frac{Q}{\mu\, a_0\, b} \right)^2,$$

or with reference to the head $k$, due to the velocity of the approaching water,

$$h = \frac{1}{2\, g} \left( \frac{Q}{\mu\, a_0\, b} \right)^2 - k.$$

For the height of opening, we have, therefore,

$$a_0 = \frac{Q}{\mu\, b \sqrt{2\, g\, h}},$$

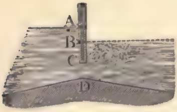
or when the swell $h_1$ above the sill is given,

$$a_0 = \frac{Q}{\mu\, b \sqrt{2\, g \left( h_1 - \frac{a_0}{2} \right)}}.$$

Experiments by the author give for this case, $\mu = 0 \cdot 60$.

If the down-stream water rise above the sluice-opening, as in Fig. 304, we have for $h$ the difference of level $A\ B$, and hence

FIG. 304.

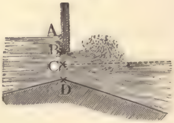


$$a_0 = \frac{Q}{\mu\, b \sqrt{2\, g\, h}}$$

where $h$ has this new value.

If, finally, the level of the down-stream water lies within the sluice-opening, a part of the water runs through freely and a part runs under water. If the height of swell or the difference of level $A\ C$ between the two surfaces, Fig. 305, is $h$, the height $B\ C$ of the upper side of the opening above the down-stream surface is $a_1$, and the height $C\ D$ of this surface above the sill is $a_2$, the discharge for the first part is

FIG. 305.



$$Q_1 = \mu\, a_1\, b \sqrt{2\, g \left(h - \frac{a_1}{2}\right)},$$

and for the second,

$$Q_2 = \mu\, a_2\, b \sqrt{2\, g\, h}\,;$$

hence the entire discharge is

$$Q = Q_1 + Q_2 = \mu\ b\ \sqrt{2\, g} \left(a_1 \sqrt{h - \frac{a_1}{2}} + a_2 \sqrt{h}\right).$$

From the discharge $Q$, height of swell $h$, and distance $a_2$ of sill below the surface, we have the distance of the sluice board above this surface,

$$a_1 = \left(\frac{Q}{\mu\ b \sqrt{2\, g}} - a_2 \sqrt{h}\right) \div \sqrt{h - \frac{a_1}{2}}.$$

EXAMPLE I.—How high must the sluice board, Fig. 301, be raised in order to discharge 250 cubic ft., the breadth of opening being $b = 24$ ft., and height $h_0$ of water above sill being 5 ft.? For free discharge,

$$a_0 = \frac{250}{0{\cdot}6 \times 24 \times 8{\cdot}02 \sqrt{5 - \frac{a_0}{2}}} = \frac{2{\cdot}16}{\sqrt{5 - \frac{a^0}{2}}}.$$

Approximately, $a_0 = 1$, hence,

$$\sqrt{5 - \frac{a_0}{2}} = \sqrt{4{\cdot}5} = 2{\cdot}121\,;$$

therefore the height required is more exactly

$$a_0' = \frac{2{\cdot}16}{2{\cdot}121} = 12{\cdot}22 \text{ inches.}$$

2. How high must the sluice be raised in Fig. 304 in order to discharge 120 cubic ft. per second under a head of 1·5 ft. for a breadth of opening of 30 feet? Here we have discharge under water, and hence,

$$a_0 = \frac{120}{0{\cdot}6 \times 30 \times 8{\cdot}02\sqrt{1{\cdot}5}} = 0{\cdot}678 \text{ ft.} = 8{\cdot}14 \text{ inches.}$$

3. Required the quantity of water for a sluice as in Fig. 305, whose breadth $b = 18$ ft., and height $BD = a_1 + a_2 = 1{\cdot}2$ ft., when the head $AC = h = 2$ ft., and the height of surface above sill, $a_2 = 0{\cdot}5$ ft. Here we have

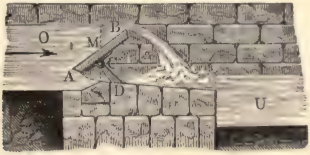
$$\mu b \sqrt{2g} = 0{\cdot}6 \times 18 \times 8{\cdot}02 = 86{\cdot}6.$$

Further, $a_2\sqrt{h} = 0{\cdot}5\sqrt{2} = 0{\cdot}707$, and $a_1\sqrt{h - \frac{a_1}{2}} = 0{\cdot}7\sqrt{1{\cdot}65} = 0{\cdot}899$; therefore the quantity of water required is,

$$Q = 86{\cdot}6\,(0{\cdot}707 + 0{\cdot}899) = 86{\cdot}6 \times 1{\cdot}606 = 139{\cdot}08 \text{ cubic ft.}$$

REMARK.—If we put a sluice arrangement above the sill of an overflow weir, we have a combined sluice overflow weir. We may also have so-called movable weirs, where the height of the overflow sill may be changed at will, to accommodate it to high and low water. The simplest weir of this character is the *beam weir*, where the wall causing the swell is composed of beams placed one over the other. We have also the *needle weir*, where this wall consists of vertical posts, the so-called needles, whose upper ends are joined by a strong rope, and which rest against a solid frame. The movable weir, in the proper sense, consists of a sluice gate which for high water opens of itself and for low water closes. A simple weir of this kind is shown in Fig. 306. $O$ is the surface of the water in front and $U$ that behind; $AB$ a gate

FIG. 306.



turning about $C$, which takes a vertical position and rests against $D$ when the water in front sinks to a certain point, and which turns and opens as this surface rises. If the surface reaches to $B$, the centre of action $M$ of the pressure upon $A\,B$ is distant $B\,M = \frac{2}{3}\,B\,A$ from $B$ (Vol. I., Art. 358). The axis $C$ must then be so placed that it is twice as far from $B$ as from $A$. Then the gate will open when the water rises above $B$, and will shut when it falls below. Of this class is the sluice of Chaubart (see "Civilingenieur," Bd. III., 1857).

Movable weirs have the advantage over the simple overflow, that the excessive swell which may occur during high water, by which floods may be caused, is avoided.

§ **152.—Discontinuous Weirs—Backwater.**—The extent of the backwater in the case of incomplete weirs, such as bridge piers, jetties, etc., may be determined in very much the same way as for overflow weirs. In the incomplete weir $B\,E$, Fig. 307, backwater is caused by the contraction of the stream from $A\,C$ to $B\,C$. Thus, if the side canal $D$ is closed, the entire flow $Q$ must pass through the contracted area $B\,C$. If we represent the breadth $B\,C$ by $b$, the height of swell $A\,B_1$, Fig. 308, by $h$, and the depth $B_1\,C_1$ of the water below the weir, or *under-water*, by $a$, we have for the quantity of water flowing freely above the under-water,

FIG. 307.

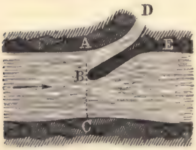
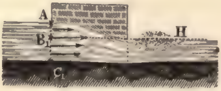


FIG. 308.



$$Q_1 = \frac{2}{3}\mu\, b\, \sqrt{2\,g\,h^3},$$

and for the under-water discharge,

$$Q_2 = \mu\, b\, a\, \sqrt{2\,g\,h};$$

hence the entire discharge is

$$Q = \mu b \sqrt{2gh} \left(\tfrac{2}{3} h + a\right).$$

The breadth of efflux corresponding to a given height of swell is therefore

$$b = \frac{Q}{\mu \left(\tfrac{2}{3} h + a\right) \sqrt{2gh}}.$$

If the height of swell $h$ is small, or the velocity of the water great, we must also consider the velocity of the approaching water. Thus let $k$ be the height due to the velocity of the approaching water, then

$$Q_1 = \tfrac{2}{3} \mu b \sqrt{2g} \left[(h+k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right],$$

and

$$Q_2 = \mu b a \sqrt{2g(h+k)}.$$

The whole discharge is therefore

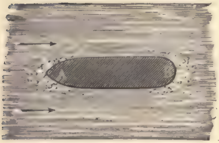
$$Q = \mu b \sqrt{2g} \left(\tfrac{2}{3} \left[(h+k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right] + a (h+k)^{\frac{1}{2}}\right),$$

or inversely,

$$b = \frac{Q}{\mu \sqrt{2g} \left(\tfrac{2}{3} \left[(h+k)^{\frac{3}{2}} - k^{\frac{3}{2}}\right] + a (h+k)^{\frac{1}{2}}\right)}.$$

During the free flow of water in river channels, the velocity is greatest at the surface and diminishes towards the bottom (Vol. I., Art. 470), but if the water meets with obstruction of any kind, the case is different. In such case, the velocity increases from the surface of the upper-water to that of the under-water, and from thence diminishes downwards, but very slightly. We have thus a change of velocity as indicated by the arrows in Fig. 308. The reason of the above is evident, because the water above the under-water surface flows under a head which increases from 0 to $h$, while below the under-water surface the water flows away under the constant pressure $h$, and again for unimpeded motion, the head at all depths $h = 0$.

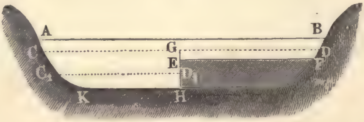
FIG. 309.



The above formulæ apply to the case of *bridge piers*, if we denote by $b$ the sum of the distances between the piers. In order to avoid as much as possible the injurious effect of the eddying motion of the water between and behind the piers, upon the piers and upon the bottom, the pier $A\,B$, Fig. 309, is rounded or sharpened in front and behind. If the front is rounded or presents a very obtuse angle, we may take $\mu = 0{\cdot}90$; if, however, the angle is acute, we have $\mu = 0{\cdot}95$, and if elliptical, or, as in Fig. 309, formed of two circular arcs, we may have $\mu = 0{\cdot}97$, or nearly 1. (See Gauthey's "Traité de la construction de ponts.")

REMARK.—When a jetty or other construction which contracts the stream does not rise above the surface, the whole flow may be divided into three parts. If the top of the dam $E\,F$, Fig. 310, lies below the under-water surface $C\,D$, and if $h$ is the height of swell, and $b$ the breadth $A\,B$ of the entire cross-section, we have for the quantity of water flowing through the cross-section $A\,B\,C\,D$,

FIG. 310.



$$Q_1 = \tfrac{2}{3}\,\mu\, b \sqrt{2\,g}\,[(h + k)^{\frac{3}{2}} - k^{\frac{3}{2}}].$$

Secondly, the remaining part above the dam and under the constant head $h$, is

$$Q_2 = \mu\, b_1 (a - a_1) \sqrt{2\,g\,(h + k)},$$

where $a$ is the depth $G\,H$ of the under-water, $a_1$ the height $E\,H$ of the dam, and $b_1$ its breadth $E\,F$.

Finally, the remaining part beyond the end of dam which has the constant head $h$ is

$$Q_3 = \mu\, b_2\, a \sqrt{2\,g\,(h + k)}.$$

We have, therefore,

$$Q = Q_1 + Q_2 + Q_3 = \tfrac{2}{3}\,\mu\, b \sqrt{2\,g}\,[(h + k)^{\frac{3}{2}} - k^{\frac{3}{2}}] + \mu\,(b\,a - b_1\,a_1) \sqrt{2\,g\,(h+k)},$$

and hence we can find for a given height of swell the corresponding height or breadth of the dam.

If, however, the under-water surface is $C_1 D_1$, or if the dam rises above the under-water, we have,

$$Q = \tfrac{2}{3}\,\mu\, b_1 \sqrt{2g}\,[(a + h - a_1 + k)^{\frac{3}{2}} - k^{\frac{3}{2}}] + \tfrac{2}{3}\,\mu\, b_2 \sqrt{2g}\,[(h + k)^{\frac{3}{2}} - k^{\frac{3}{2}}]$$
$$+ \mu\, a\, b_2 \sqrt{2g\,(h + k)}.$$

EXAMPLE.—What length must the dam $B\,E$ (Fig. 307) have, in order that the stream which is 550 ft. wide, 8 ft. deep, and which discharges 14000 cubic ft., may be raised by ¾ of, a foot higher?

Here we have

$$k = 0{\cdot}0155 \left(\frac{14000}{550 \times 8}\right)^2 = 0{\cdot}0155 \times 3{\cdot}18^2 = 0{\cdot}156\,;$$

and if we assume now $\mu = 0{\cdot}9$, we have for the breadth of the contracted stream,

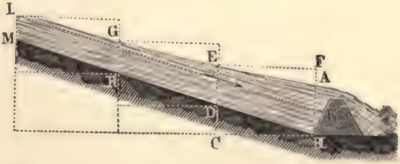
$$B\,C = b = \frac{14000}{0{\cdot}9 \times 8{\cdot}02\,[\frac{2}{3}(0{\cdot}906^{\frac{3}{2}} - 156^{\frac{3}{2}}) + 8 \times 0{\cdot}906^{\frac{1}{2}}]}$$
$$= \frac{14000}{7{\cdot}218\,(0{\cdot}522 + 7{\cdot}608)} = \frac{14000}{7{\cdot}218 \times 8{\cdot}13} = 238{\cdot}5 \text{ ft.}$$

Hence the desired length of dam is

$$A\,B = b_1 = 550 - 238{\cdot}5 = 311{\cdot}5 \text{ ft.}$$

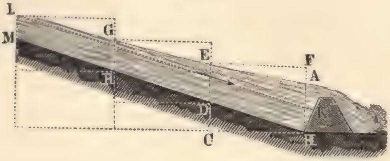
§ **153.—Extent of Swell or Backwater.**—We have now to consider the other important question, according to what law does the height of swell diminish with the distance from the dam? Without resorting to special theories, we may here directly apply the theory of the variable motion of water in river channels, as given in Vol. I., Arts. 477 and 478.

FIG. 311.



Let us conceive the length of backwater from the dam $A\,B\,K$, Fig. 311, divided into portions, and calculate each portion separately. If, now, $a_0$ is the depth $AB$ at the dam, Fig. 312, $a_1$ the depth $DE$ at the beginning of the portion $ABDE$, $F_0$ the cross-section of the flowing water at the dam, $F_1$ the cross-section at $DE$, $Q$ the quantity of flow, $p$ the mean circumference of the water profile for this length, and $\alpha$ the angle of inclination $DBC$ of the river bed, then, according to Vol. I., Art. 478, the corresponding length of the first division $BD$ is,

FIG. 312.



$$l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{sin.\ \alpha - \zeta \frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g}}.$$

If $a_2$ is the depth of water $GH$ at beginning of the second section $DEGH$, $F_2$ the cross-section, and $p_1$ the mean circumference of the water profile of this portion, we have for the length $DH$,

$$l_1 = \frac{a_1 - a_2 - \left(\frac{1}{F_2^2} - \frac{1}{F_1^2}\right)\frac{Q^2}{2g}}{sin.\ \alpha - \zeta \frac{p_1}{F_1 + F_2}\left(\frac{1}{F_1^2} + \frac{1}{F_2^2}\right)\frac{Q^2}{2g}}.$$

If we thus continue, assuming arbitrary decreases of depth $a_0 - a_1$, $a_1 - a_2$, etc., and calculating the section $F_1$, $F_2$, etc., and the mean perimeters, we obtain by the above formulæ the corresponding lengths $l$, $l_1$, $l_2$, etc., or the distances $l$, $l + l_1$, $l + l_1 + l_2$, etc., from the dam.

To find the depth $y$ corresponding to a given distance $x$, we may either interpolate in the values $l$, $l + l_1$, $l + l_1 + l_2$, etc., just found as above, or may use the approximate formula likewise given in Vol. I., Art. 478, viz.,

$$a_0 - a_1 = \frac{\left(\sin.\,\alpha - \zeta \dfrac{p_0}{a_0\, b_0} \cdot \dfrac{v_0^2}{2\,g}\right)}{1 - \dfrac{2}{a_0} \cdot \dfrac{v_0^2}{2\,g}}\, l.$$

If we put instead of $b_0$, the breadth, and instead of $p_0$, the perimeter, and for $v_0$, the velocity at the weir, this formula gives the decrease of swell height $(a_0 - a_1)$ in the first short division $l$, and for the next division we have the decrease,

$$a_1 - a_2 = \frac{\left(\sin.\,\alpha - \zeta \dfrac{p_1}{a_1\, b_1} \cdot \dfrac{v_1^2}{2\,g}\right)}{1 - \dfrac{2}{a_1} \cdot \dfrac{v_1^2}{2\,g}}\, l_1, \text{ etc.,}$$

and, finally, for a given distance $l + l_1 + l_2 + \ldots$ we may find the depth $a_0 - (a_0 - a_1) - (a_1 - a_2) - \ldots$

EXAMPLE 1.—In a stream 80 ft. wide and 4 ft. deep, which discharges 1400 cubic ft. per second, a dam is to be built which shall back up the water 3 ft. Required the heights of swell above the dam.

Without the dam, the velocity of the water is

$$c = \frac{1400}{80 \times 4} = \frac{35}{8} = 4{\cdot}375 \text{ ft.,}$$

and hence, according to the table in Vol. I., Art. 476, the coefficient of resistance is $\zeta = 0{\cdot}00747$, and the inclination of the channel is

$$\sin.\,\alpha = 0{\cdot}00747 \cdot \frac{p}{F} \cdot \frac{c^2}{2g}.$$

If, now, $p = 84$, $F = 80 \times 4 = 320$, $c = 4{\cdot}375$, and $\frac{1}{2g} = 0{\cdot}0155$, we have

$$\sin.\,\alpha = 0{\cdot}00747 \cdot \frac{84}{320} \times 0{\cdot}0155\,(4{\cdot}375)^2 = 0{\cdot}0005818.$$

The depth of water immediately at the dam is $4 + 3 = 7$ ft.; let us now determine the distances from the dam at which the depths are only 6½, 6, 5½, and 5 ft. If, then, we first insert

$$a_0 - a_1 = 0\cdot5,\ F_0 = 80 \times 7 = 560,\ F_1 = 80 \times 6\cdot5 = 520,\ Q = 1400,$$
$$sin.\,\alpha = 0\cdot00058,\ p = 86,$$

and, corresponding to the mean velocity,

$$\frac{2\,Q}{F_0 + F_1} = \frac{2800}{1080} = 2\cdot59 \text{ ft.},$$

$\zeta = 0\cdot0075$, in the formula,

$$l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^{\,2}} - \frac{1}{F_0^{\,2}}\right)\frac{Q^2}{2\,g}}{sin.\,\alpha - \zeta \cdot \frac{p}{F_0 + F_1}\left(\frac{1}{F_0^{\,2}} + \frac{1}{F_1^{\,2}}\right)\frac{Q^2}{2\,g}},$$

we have

$$l = \frac{0\cdot5 - (0\cdot0000036982 - 0\cdot0000031888) \times 30434}{0\cdot00058 - 0\cdot0075\,\frac{86}{1080}\left(0\cdot0000036982 + 0\cdot0000031888\right) \times 30434}$$

$$= \frac{0\cdot5 - 0\cdot0155}{0\cdot00058 - 0\cdot000128} = \frac{0\cdot4845}{0\cdot000452} = 1071 \text{ ft.}$$

In order to find the distance back corresponding to a decrease of 1 ft. in depth, we must again insert $a_0 - a_1 = 0\cdot5$, but now

$$F_0 = 520,\ F_1 = 80 \times 6 = 480,\ p = 85\cdot5,$$

and the mean velocity $\frac{2800}{1000} = 2\cdot80$, gives $\zeta = 0\cdot00749$. Hence by the same formula, we find for the distance in which the depth decreases from 6·5 to 6 ft.,

$$l = \frac{0\cdot5 - 0\cdot0000006421 \times 30434}{0\cdot00058 - 0\cdot00749\,\frac{85\cdot5}{1000} \times 0\cdot0000080385 \times 30434}$$

$$= \frac{0\cdot4805}{0\cdot000424} = 1133 \text{ ft.}$$

Therefore, at a distance, 1071 + 1142 = 2204 ft., the water is only 6 ft. deep, or the height of backwater is but 2 ft.

If, again, we put $a_0 - a_1 = 0\cdot5$, $F_0 = 480$, $F_1 = 80 \times 5\cdot5 = 440$, $p = 85\cdot1$, and $\zeta = 0\cdot00749$, we have for the corresponding distance $l = 1203$ ft., and, again, for a further decrease of 0·5 ft., we find $l = 1413$ ft. Therefore at a distance from the dam of 2213 + 1203 + 1413 = 4829 ft., there is still a swell of 1 ft. For a depth of 4½ ft., we have $l = 1922$ ft.; for 4¼, $l = 1584$ ft.; and for 4·1 ft., $l = 1850$ ft.; so there is still a swell of $\frac{1}{10}$ of a foot at a distance from the dam of 4829 + 1922 + 1584 + 1850 = 10185 ft., which still goes on diminishing up stream. By our formula for a depth of 4 ft. or *no backwater at all*, $l = \infty$.

EXAMPLE 2.—What is the height of swell at a distance of 2500 ft. above the dam in the preceding example?

According to the preceding calculations, there is a rise of 2 ft. at 2122 ft. above the dam, and the question is, how much does this swell diminish in the distance $2500 - 2122 = 378$ ft. Now, the distance back from the 6 ft. depth corresponding to a reduction of 0·5 ft. has been found to be 1205 ft.; therefore, for each foot we have a depression of

$$\frac{0 \cdot 5}{1205} \text{ ft.},$$

and hence for 378 ft. we have a depression of

$$\frac{0 \cdot 5 \times 378}{1205} = 0 \cdot 157 \text{ ft.}$$

The rise of the backwater at the distance of 2500 ft. from the dam is then $2 - 0 \cdot 157 = 1 \cdot 843$ ft., and the depth of water at this point is therefore 5·843 ft.

If we take the formula,

$$a_0 - a_1 = \frac{\left(\sin \alpha - \zeta \cdot \frac{p_0}{a_0 b_0} \cdot \frac{v_0^2}{2g}\right)}{1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}} l,$$

and put $l = 800$, $p_0 = 86$, $a_0 = 7$, $a_0 b_0 = 560$, $v_0 = \frac{1400}{560} = 2 \cdot 5$, and $\zeta = .0075$, we obtain the corresponding diminution of depth 0·399 ft. If we again take $l = 800$, $p_0 = 85 \cdot 8$, $a_0 = 7 - 0 \cdot 399 = 6 \cdot 601$, $a_0 b_0 = 528$, $v_0 = \frac{1400}{528} = 2 \cdot 652$, $\zeta = 0 \cdot 0075$, the corresponding decrease is found to be 0·383 ft. Repeating the operation again, but this time taking $l = 900$ ft., $p_0 = 85 \cdot 5$, $a_0 = 6 \cdot 601 - 0 \cdot 383 = 6 \cdot 218$, $a_0 b_0 = 497 \cdot 44$, $v_0 = \frac{1400}{497 \cdot 44} = 2 \cdot 88$, and $\zeta = 0 \cdot 00749$, we obtain for the decrease of depth $a_0 - a_1 = 0 \cdot 403$ ft. For $800 + 800 + 900 = 2500$ ft. back of the dam therefore, the depth of water is $6 \cdot 218 - 0 \cdot 403 = 5 \cdot 815$ ft., and the height of swell is 1·815 ft. The first method by interpolation gave 1·843, or less than ⅛ of an inch difference.

**§ 154.—Swell of the Backwater.**—If we consider somewhat more closely the equation for the curve of the backwater, viz.,

$$a_0 - a_1 = \left(\frac{\sin. \alpha - \zeta \frac{p}{F} \cdot \frac{v^2}{2g}}{1 - \frac{2}{a} \cdot \frac{v^2}{2g}}\right) l,$$

we shall discover several remarkable relations with reference to it.

In the fraction,

$$\frac{\sin. \alpha - \zeta \cdot \frac{p}{F} \cdot \frac{v^2}{2g}}{1 - \frac{2}{a} \cdot \frac{v^2}{2g}},$$

the numerator and denominator approach zero more nearly, the greater the velocity $v$, and according as one or the other first becomes zero, we have

$$l = \frac{\left(a_0 - a_1\right)\left(1 - \frac{2}{a} \cdot \frac{v^2}{2g}\right)}{0} = \infty$$

or

$$l = \frac{\left(a_0 - a_1\right) . 0}{\sin. \alpha - \zeta \frac{p}{F} \cdot \frac{v^2}{2g}} = 0.$$

We see, therefore, that in the first case the division $l$ and hence the entire length of swell becomes infinite, and in the second case the division $l$ is zero and there is no backwater at all. The numerator becomes zero when

$$\zeta \cdot \frac{p}{F} \cdot \frac{v^2}{2g} = \sin. \alpha,$$

or when the velocity of the backed-up water differs unappreciably from the velocity

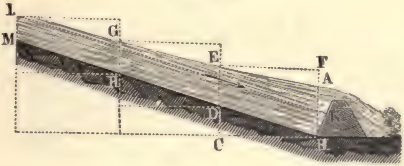
$$v = \sqrt{\frac{2 g F \sin. \alpha}{\zeta p}}$$

of the uniformly flowing water of the undammed stream. The denominator is zero when

$$\frac{2}{a} \cdot \frac{v^2}{2g} = 1 \quad \text{or} \quad \frac{v^2}{2g} = \frac{a}{2},$$

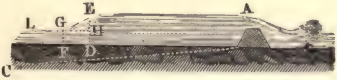
or when the height due to the velocity equals half the depth of the stream. *If, then, the height due to the velocity of the water before it is dammed up is less than half the depth*

FIG. 313.



*of the undammed water, the backwater takes the form shown in Fig.* 313, *and if the height due to the velocity is greater than the half depth, we have the form shown in Fig.* 314, there being a sudden rise or swell at *E G.*

FIG. 314.



If, now, we put $\frac{v^2}{2g} = \frac{a}{2}$ and $F = a\ b$ and $p$ (even if only approximately) equal to $b$, in the equation,

$$sin.\ \alpha = \zeta \cdot \frac{p}{F} \cdot \frac{v^2}{2g},$$

we have $$sin.\ \alpha = \tfrac{1}{2}\ \zeta.$$

Thus a sudden swell is to be expected when the inclination $\alpha$ of the stream is greater than half the coefficient of friction, or putting $\zeta = 0{\cdot}008$, when $\alpha > 0{\cdot}004$ or $\alpha > \frac{1}{250}$. As rivers and canals have generally a less inclination than this, the sudden depression *E G* shown in Fig. 314 does not usually occur.

The height *E H* $= x$ of this depression may be found from the velocity $v$ of the approaching and $v_1$ of the departing water, by putting

$$x = \frac{v^2 - v_1^2}{2g},$$

or since $a\,v = (a + x)\,v_1$, and therefore $v_1 = \left(\frac{a}{a+x}\right) v$

$$x = \left[1 - \left(\frac{a}{a+x}\right)^2\right] \frac{v^2}{2g},$$

or completing the solution for $x$,

$$x = \frac{v^2}{4g} - a + \sqrt{\frac{v^2}{2g}\left(a + \frac{v^2}{8g}\right)}.$$

Accordingly, for $\frac{v^2}{2g} = \frac{a}{2}$

we have, $x = -\frac{3}{4}\,a + \frac{3}{4}\,a = 0,$

as should be, and for $\frac{v^2}{2g} = a,$

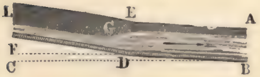
$$x = -\frac{a}{2} + \frac{a}{2}\sqrt{5} = 0{\cdot}618\,a.$$

For $\frac{v^2}{2g} = 2\,a,$

$$x = a\,\sqrt{3} = 1{\cdot}732\,a, \text{ etc.}$$

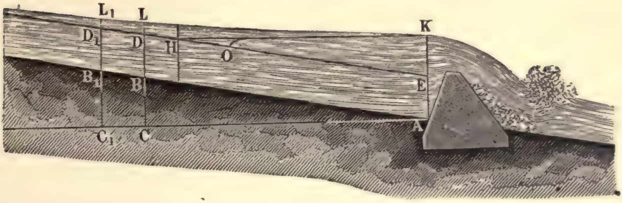
REMARK.—The above noticed sudden depression of the backwater was first noticed by Bidone in a trough twelve inches wide with the mean inclination ratio $\alpha = 0{\cdot}033$. The same happens also not only in the case of water restrained by a weir, but also for a change of inclination of the channel as shown in Fig. 315, as has often been observed by the author. If the inclination ratio of the upper part is greater than $\frac{1}{2}\zeta$ and that of the lower less, there is always at the point of union a sudden elevation, the water in the steeper portion having the least depth.

FIG. 315.



§ **155.—Curve of the Backwater.**—The equation of the curve of the surface of the water may be determined by the aid of the calculus. Let $a$ be the height $A\,E = B\,D$, Fig. 316, of the freely flowing water, $h$ the height of swell

FIG. 316.



$E\,K$ close to the weir, $y$ the height of swell $D\,L$ at a distance $E\,D = x$ from the weir, $\alpha$ the angle of inclination $B\,A\,C$ of the river bed, $c$ the mean velocity of the water for the cross-section $A\,E = B\,D$ before the introduction of the weir, and $v$ the mean velocity at the cross-section $B\,L$. If we put also

$$l = D\,D_1 = d\,x,\ a_0 - a_1 = D\,L - D_1\,L_1 = -\,d\,y,$$

and put in place of $a$, $a + y$ and in place of $\frac{p}{F}$, approximately, $\frac{1}{a+y}$, and put $v = \frac{a\,c}{a+y}$, then the formula,

$$a_0 - a_1 = \left(\frac{\sin.\ \alpha - \zeta\,\frac{p}{F}\cdot\frac{v^2}{2\,g}}{1 - \frac{2}{a}\,\frac{v^2}{2\,g}}\right) l,$$

becomes

$$-\,d\,y = \left(\frac{\sin.\ \alpha - \zeta\,\frac{a^2}{(a+y)^2}\cdot\frac{c^2}{2\,g}}{1 - \frac{2\,a^2}{(a+y)^3}\cdot\frac{c^2}{2\,g}}\right) d\,x.$$

If we also make use of the formula,

$$\sin. \alpha = \alpha = \zeta \frac{1}{a} \frac{c^2}{2g},$$

we have

$$\zeta \frac{c^2}{2g} = \alpha a,$$

and hence

$$\alpha\, d x = -\left(\frac{(a+y)^3 - 2 a^2 \dfrac{c^2}{2g}}{(a+y)^3 - a^3}\right) d y.$$

If, for the sake of abridgment, we put

$$a + y = y_1,\ d y = d y_1 \qquad \text{and} \qquad \frac{c^2}{2g} = k,$$

we have

$$\alpha\, d x = -\left(\frac{y_1^3 - 2 a^2 k}{y_1^3 - a^3}\right) d y_1 = - d y_1 - \left(\frac{a^3 - 2 a^2 k}{y_1^3 - a^3}\right) d y_1,$$

whence

$$\alpha x = - y_1 - a^2 (a - 2 k) \int \frac{d y_1}{y_1^3 - a^3}$$

$$= - y_1 + (a - 2 k) \int \frac{d \left(\frac{y_1}{a}\right)}{1 - \left(\frac{y_1}{a}\right)^3}$$

$$= - y_1 + (a - 2 k) \int \frac{d Z}{1 - Z^3},$$

if we denote $\frac{y_1}{a}$ by $Z$ or put $y_1 = a Z$.

We can put

$$\frac{1}{1 - Z^3} = \frac{1}{(1 - Z)(1 + Z + Z^2)} = \frac{A}{1 - Z} = \frac{B + C Z}{1 + Z + Z^2},$$

therefore

$$1 = A (1 + Z + Z^2) + (B + C Z)(1 - Z),$$

or

$$0 = A + B - 1 + (A - B + C) Z + (A - C) Z^2;$$

hence $A + B = 1,\ A + C = B$ and $A = C,$

or $A = C = \frac{1}{3}$ and $B = \frac{2}{3},$

therefore

$$\frac{1}{1 - Z^3} = \frac{1}{3}\left(\frac{1}{1 - Z} + \frac{2 + Z}{1 + Z + Z^2}\right).$$

Accordingly

$$\int \frac{dZ}{1 - Z^3} = \frac{1}{3}\left(\int \frac{dZ}{1 - Z} + \int \frac{(2 + Z)\,dZ}{1 + Z + Z^2}\right).$$

From III., Art. 22, Vol. I., we have

$$\int \frac{dZ}{1 - Z} = - \int \frac{d\,(1 - Z)}{1 - Z} = - \text{nat. log.}\,(1 - Z).$$

In order to integrate

$$\frac{(2 + Z)\,dZ}{1 + Z + Z^2},$$

we write

$$1 + Z + Z^2 = \tfrac{3}{4} + (\tfrac{1}{2} + Z)^2 = \tfrac{3}{4}\left[1 + \tfrac{4}{3}\,(\tfrac{1}{2} + Z)^2\right] = \tfrac{3}{4}\,(1 + u^2),$$

where

$$u = \sqrt{\tfrac{4}{3}}\,(\tfrac{1}{2} + Z) = \frac{1 + 2Z}{\sqrt{3}},$$

therefore

$$Z = \frac{u\sqrt{3} - 1}{2} \qquad \text{and} \qquad dZ = \frac{du\,\sqrt{3}}{2}.$$

We have then,

$$\frac{(2 + Z)\,dZ}{1 + Z + Z^2} = \frac{\left(2 + \dfrac{u\sqrt{3}}{2} - \dfrac{1}{2}\right)}{\tfrac{3}{4}\,(1 + u^2)} \cdot \frac{du\,\sqrt{3}}{2}$$

$$= \frac{u\,du}{1 + u^2} + \sqrt{3}\,\frac{du}{1 + u^2};$$

but since

$$\int \frac{u\,d\,u}{1+u^2} = \tfrac{1}{2}\int \frac{2\,u\,d\,u}{1+u^2} = \tfrac{1}{2}\int \frac{d\,(1+u^2)}{1+u^2} = \tfrac{1}{2}\,log.\,(1+u^2),$$

and $\int \frac{d\,u}{1+u^2} = tan.^{-1}\,u$ (see VI., Art. 26, Vol. I.),

we have

$$\int \frac{(2+Z)\,d\,Z}{1+Z+Z^2} = \tfrac{1}{2}\,log.\left(1+\frac{(1+2\,Z)^2}{3}\right)$$

$$+\sqrt{3}\ tan.^{-1}\,\frac{1+2\,Z}{\sqrt{3}} + Const.,$$

and

$$\alpha\,x = -\,y_1 + \left(\frac{\alpha-2\,k}{3}\right)\Big[-\,log.\,nat.\,(1-Z)$$

$$+\tfrac{1}{2}\,log.\,nat.\left(1+\frac{(1+2\,Z)^2}{3}\right) + \sqrt{3}\ tan.^{-1}\,\frac{1+2\,Z}{\sqrt{3}}\Big] + Const.$$

$$= -\,y_1 + \frac{a-2\,k}{3}\Big[\tfrac{1}{2}\,log.\,nat.\left(\frac{1+Z+Z^2}{(1-Z^2)}\right)$$

$$+\sqrt{3}\ tan.^{-1}\,\frac{1+2\,Z}{\sqrt{3}}\Big] + Const.$$

$$= -\,y_1 + \frac{a-2\,k}{3}\Big[\tfrac{1}{2}\,log.\,nat.\left(1+\frac{3\,a\,y_1}{(a-y_1)^2}\right)$$

$$+\sqrt{3}\ tan.^{-1}\,\frac{a+2\,y_1}{a\,\sqrt{3}}\Big] + Const.$$

$$= -\,y + \frac{a-2\,k}{3}\Big[\tfrac{1}{2}\,log.\,nat.\left(1+\frac{3\,a\,(a+y)}{y^2}\right)$$

$$+\sqrt{3}\ tan.^{-1}\,\frac{3\,a+2\,y}{a\,\sqrt{3}}\Big] + Const.$$

In the beginning, $x = 0$ and $y = h$, hence

$$0 = -\,h + \frac{a-2\,k}{3}\Big[\tfrac{1}{2}\,log.\,nat.\left(1+\frac{3\,a\,(a+h)}{h^2}\right)$$

$$+ \sqrt{3}\ tan.^{-1} \frac{3\ a + 2\ h}{a\ \sqrt{3}}\Bigg] + Const.,$$

or finally,

$$\alpha\ x = h - y + \frac{a - 2\ k}{3} \left[ \frac{1}{2}\ log.\ nat. \left( \frac{1 + \dfrac{3\ a\ (a + y)}{y^2}}{1 + \dfrac{3\ a\ (a + h)}{h^2}} \right) \right.$$

$$\left. - \sqrt{3} \left( tan.^{-1} \frac{3\ a + 2\ h}{a\ \sqrt{3}} - tan.^{-1} \frac{3\ a + 2\ y}{a\ \sqrt{3}} \right) \right]$$

or,

$$\alpha\ x = h - y + \frac{a - 2\ k}{3} \left[ \frac{1}{2}\ log.\ nat. \left( \frac{y^2 + 3\ a\ (a + y)}{h^2 + 3\ a\ (a + h)} \cdot \frac{h^2}{y^2} \right) \right.$$

$$\left. - \sqrt{3}\ tan.^{-1} \frac{a\ (h - y)\ \sqrt{3}}{6\ a^2 + 3\ a\ (h + y) + 2\ h\ y} \right].$$

By means of this formula, we can find for any distance $x$ from the weir, the depth $y$.

For a small value of $h$ and a very small value of $y$ with reference to $a$, we have simply

$$\alpha\ x = h + \frac{a - 2\ k}{3}\ log.\ nat. \left( \frac{h}{y} \right).$$

If $a = 2\ k = \frac{c^2}{g}$, $\alpha\ x = h - y$, and the surface is a horizontal line $H\ K$. If $a < 2\ k$, $\alpha\ x$ is less than $h - y$, therefore $y$ is less than $h - \alpha\ x$, and we have the swell $O\ K$ as first observed by Bidone.

EXAMPLE.—For $a = 4$, $h = 3$, $c = 4 \cdot 375$ ft., $\alpha = 0 \cdot 000623$ and $y = 0 \cdot 1$ ft., we have for the extent of swell approximately,

$$x = \frac{3 + \frac{1}{3}\ (4 - 0 \cdot 5944)\ log.\ nat.\ 30}{0 \cdot 000623} = \frac{3 + 1 \cdot 1352 \times 3 \cdot 4012}{0 \cdot 000623}$$

$$= \frac{68610}{6 \cdot 23} = 11013 \text{ ft.}$$

In Example (1) of the preceding article, we found $x = 10185$ ft.

REMARK 1.—The quantity of water backed up by the weir is

$$V = \int b\, y\, d x.$$

But approximately

$$\alpha x = h - y + \left(\frac{a - 2\, k}{3}\right) log.\, nat. \left(\frac{h}{y}\right),$$

and hence

$$\alpha\, d x = - d y - \frac{a - 2\, k}{3} \frac{d y}{y};$$

therefore

$$V = - \frac{b}{\alpha} \int \left(y\, d y + \frac{a - 2\, k}{3}\, d y\right) = - \frac{b}{\alpha} \left(\frac{y^2}{2} + \frac{(a - 2\, k)}{3} y\right) + Const.$$

Since for $y = h$, $V = o$, we have

$$V = \frac{b}{\alpha}\left(\frac{h^2 - y^2}{2} + \frac{(a - 2\, k)}{3} (h - y)\right) = \frac{b\,(h - y)}{\alpha} \left(\frac{h + y}{2} + \frac{a - 2\, k}{3}\right);$$

and for $y = o$,

$$V = \frac{b\, h}{\alpha}\left(\frac{h}{2} + \frac{a - 2\, k}{3}\right).$$

If this quantity of water flows over in the time $t$, we have $V = a\, b\, c\, t$, and hence

$$t = \frac{h}{\alpha\, a\, c}\left(\frac{h}{2} + \frac{a - 2\, k}{3}\right).$$

For $a = 2\, k$, we have

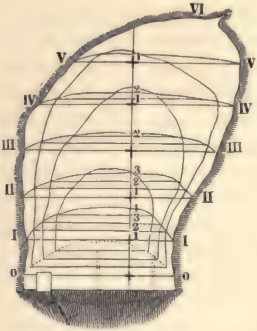
$$V = \frac{b\, h^2}{2\, \alpha} \quad \text{and} \quad t = \frac{h^2}{2\, \alpha\, a\, c}.$$

REMARK 2.—The preceding formulæ were published by the author in the "Allgemeine Maschinencyclopädie," Vol. II., 1844. If we disregard $2\, k = \frac{c^2}{g}$, we obtain the formulæ which Heinemann, in Erbkam's "Ztschr. für Bauwesen," Berlin, 1855 (see also "Polyt. Centralblatt," 1855), attributes to Hagen. The same holds also for the formula given by Gödecker in Vol. VII. of the "Ztschr. des Arch. und Ingenieurvereins für Hannover." Such formulæ naturally give no information as to the swell.

The curve of swell is treated very thoroughly in the second part of the "Cours de Mécanique appliquée," by Bresse, Paris, 1860; also in Rühlmann's "Hydromechanik," Leipzig, 1857. See "Annales des ponts et chauss., 1838," for St. Guilhem's empirical formula for the calculation of the extent of swell, and for Dupuit's formula see his "Etudes théoretiques et practiques sur les mouvement des eaux courantes."

§ **156.—Reservoirs.**—In regions where the supply of water is small and where powerful machines are required, as in mining districts, the construction of reservoirs (Fr. étangs; Ger. Teiche) or of large ponds which are filled during the wet season, and thus supply the demand during the dry season, is of great importance. Such reservoirs are in general situated in ravines and valleys, so as to collect not only the rain and freshet water, but also the brooks and springs found in such situations. In such case, a single dam running square across the valley is sufficient to enclose the reservoir, the rising bottom and the two banks on either side completing the enclosure. The smaller the extent of water surface and the shorter the restraining dam, for the same quantity of water, the better. We should, therefore, select for a site some place where the banks are steep and come close together at the dam. Only in wide valleys we must sometimes enclose the water by two dams or by one main dam and two wing dams. In general, of course, local conditions determine the choice of site, but we may note that the lower the level of the reservoir the greater the supply of water, and the less the fall remaining for the machines; on the other hand, higher reservoirs afford less water, but give an increased fall. That situation is the best for which the product of the water supply into the fall from the reservoir to the machine site, is a maximum. Also by tributary canals and ditches the water supply may be increased. In selecting the site for a reservoir, we must also have regard to the nature of the bottom, and should avoid ground permeable to water, such as creviced rock, quicksand, marsh, bogs, morass, etc. By covering with clay and turf, or ramming with fine sand and clay, we may often make the bottom water-tight. If the banks are not water-tight or not sufficiently impermeable, we must protect them also by layers of clay or turf, masonry, etc.

FIG. 317.



The value of a reservoir depends especially upon its superficial area and cubical contents. To determine both, special measurements are necessary. Thus, by the plane table or with the theodolite and chain, we may locate the points I, II, III, etc. (Fig. 317), of the parallels or cross-sections at these points, and then, with a rod or level or by soundings, may determine the depths at various points of these cross-sections. We can then find the contents required. Thus, if $b$ , $b_1$, $b_2$ .... $b_n$ are the breadths 0 — 0, I — I, II — II, etc., and if the distance between every two parallels is $a$, we have for the surface of the reservoir,

$$G = \left[b_0 + b_n + 4(b_1 + b_3 + \ldots + b_{n-1}) + 2(b_2 + b_4 + \ldots . b_{n-2})\right]\frac{a}{3}.$$

If, in like manner, $F_0$, $F_1$, $F_2$, etc., are the cross-section areas corresponding to the breadths $b_0$, $b_1$, $b_2$, etc., we have for the contents,

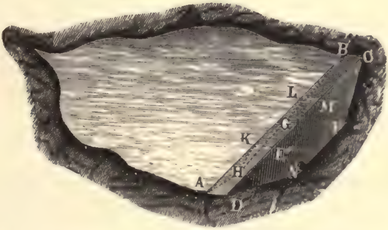
$$V = \left[F_0 + F_n + 4(F_1 + F_3 + \ldots + F_{n-1}) + 2(F_2 + F_4 + \ldots + F_{n-2})\right]\frac{a}{3}.$$

In the same manner we can find the contents for any level of water, by dividing the entire reservoir into layers by horizontal planes.

REMARK.—The measurement and calculation of reservoirs is treated specially in the "Ingenieur," as also in the new "Markscheidekunst," by the author. The subject will also be found treated in "Der Ingenieur," Heft. I., Freiberg, 1846.

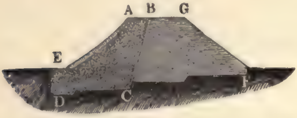
§ **157.—Reservoir Dams.**—Reservoir dams are in general constructed of earth-work and more rarely of stone. They are rendered tight by a thick puddling of clay, which may again be protected by gravel or masonry from the action of waves. Still another foundation layer of clay or turf must be added to prevent the water from penetrating. This layer should be carried down to firm soil, rock, or clay, or, when this is not possible, as in sandy soils, a firm foundation must be made by driving piles. The depth of this foundation layer depends upon the nature of the ground. For stone 5 ft. is often sufficient, while for loose soil 20 ft. may be necessary. Crevices, laminated stone, etc., may be injurious by allowing the water to penetrate beneath the dam. In order to prevent this, the foundations must be carried deep and let far into the banks. The main features of a reservoir dam are as shown in Fig. 318, the cross-section *H K E N* being trapezoidal. The upper surface *A C* is the *crown*, the side exposed to the water *A B G H* is the *face*, and the other side the *back*. *K M N* is the *main portion*, and *A N H* and *B M C* the *wings*. As to the dimensions of the dam, the crown breadth $A D = B C$ is not less than 10 ft., or, when a path is carried on it, 20 ft.; it is also customary to make this breadth at least equal to the height of dam. If, now, we give a batter of 45° to the face and back, the lower breadth will be three times the height or upper breadth. Many dams, however, have a batter of 30° to 40°, so that we may even have a still greater ratio of the

FIG. 318.



lower breadth to the height. The height varies considerably; 15 to 35 ft. is not uncommon. The dam should be 2 to 3 ft. higher than the surface of the water. In Fig. 319 we have the cross-section of a dam. *A B C E* is the clay puddle, well rammed and carried down to firm ground, *B G F C* the earth-work backing, and *A E* the protecting masonry, gravel, or turf, 2 ft. thick above and 4 ft. thick below.

FIG. 319.



REMARK 1.—If $l$ is the upper and $l_1$ the lower length, $b$ the upper and $b_1$ the lower breadth, and $h$ the height of the dam as shown in Fig. 318, the volume is

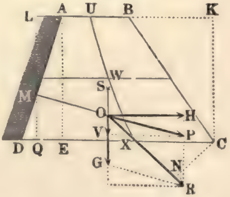
$$V = [l\,b_1 + l_1\,b + 2\,(l\,b + l_1\,b_1)]\,\frac{h}{6} \text{ (Vol. I., Art. 121).}$$

In the application of this formula, we must remember that well-rammed earth occupies not quite half the space of loose earth.

REMARK 2.—One of the largest reservoirs in the Freiberg mining district is the lower "Groszhartsmannsdorfer." It has a superficial area of 6683700 sq. ft. and 69336000 cubic ft. The dam is 2382 ft. long, 56 ft. broad at top, 153 at bottom, and 27 ft. high. In Russia, especially in the Ural, still larger dams are to be found.

**§ 158.—Stability of Reservoir Dams.**—Reservoir dams are exposed to the pressure and sometimes to the shock of water, and it is therefore necessary to give them such proportions that they may neither be overturned nor shoved forward. The conditions of resistance to sliding have already been discussed (Vol. I., Art. 360). It remains only to consider the stability with reference to overturning. The water exerts against the face *A D* of a dam *A B C D*, Fig. 320, a perpendicular pressure *O P* = *P*, whose point of application *M* is at a distance *L M*, or ⅔ the

FIG. 320.



height $CK = \frac{2}{3} h$ from the surface of the water (Vol. I., Art. 358). For a section of the dam whose length is unity, this pressure is

$$P = AD \times 1 \times \gamma \times \frac{h}{2},$$

where $\gamma$ is the weight of one cubic unit of water. The horizontal component of this pressure is

$$H = h \times 1 \times \gamma \times \frac{h}{2} = \tfrac{1}{2} h^2 \gamma,$$

and the vertical component, if $mh$ is the distance $DE$, is

$$V = mh \times 1 \times \gamma \times \frac{h}{2} = \tfrac{1}{2} m h^2 \gamma.$$

The weight of the trapezoidal cross-section for a length of unity, which acts at the centre of gravity $S$, is

$$G = \left(b + \frac{m+n}{2} h\right) h \gamma_1$$

where $\gamma_1$ is the weight of a cubic unit of the dam, $b$ the breadth at crown, and $nh$ the horizontal projection of the rear slope.

From $P$ and $G$, or from $H$, $V$ and $G$, we have a resultant $R = OR$, the statical moment $\overline{CN} \times R$ of which with reference to the edge $C$, expresses the stability of the dam. If we conceive $P$ as well as $H$ and $V$ acting at $M$, we have the statical moment of $P =$ the statical moment of $H$ *minus* the statical moment of $V$; or

$$= \tfrac{1}{2} h^2 \gamma \overline{MQ} - \tfrac{1}{2} m h^2 \gamma \overline{CQ} = \tfrac{1}{2} h^2 \gamma (\overline{MQ} - m \overline{CQ})$$

$$= \tfrac{1}{2} h^2 \gamma \left[\tfrac{1}{3} h - m \left(nh + b + \tfrac{2}{3} mh\right)\right].$$

But we have now the statical moment of the weight $G$ acting in the opposite direction

$$= \tfrac{1}{2} n h^2 \gamma_1 \times \tfrac{2}{3} n h + b h \gamma_1 \left(n h + \frac{b}{2}\right) + \tfrac{1}{2} m h^2 \gamma_1 (n h + b + \tfrac{1}{3} m h)$$

$$= h \gamma_1 (\tfrac{1}{3} n^2 h^2 + n b h + \tfrac{1}{2} b^2 + \tfrac{1}{2} m n h^2 + \tfrac{1}{2} m b h + \tfrac{1}{6} m^2 h^2)$$

$$= h \gamma_1 \left[\left(\frac{m^2 + 2 n^2}{3} + m n\right) \frac{h^2}{2} + \left(n + \frac{m}{2}\right) b h + \tfrac{1}{2} b^2\right];$$

hence the stability of the dam is

$$S = h \left(\left[\left(\frac{m^2 + 2 n^2}{3} + m n\right) \frac{h^2}{2} + \left(n + \frac{m}{2}\right) b h + \tfrac{1}{2} b^2\right] \gamma_1 \right.$$
$$\left. - [\tfrac{1}{3} h - m (n h + b + \tfrac{2}{3} m h)] \frac{h}{2} \gamma\right).$$

In order, now, to find the point $X$ in which the line of resistance $U W X$ intersects the base $C D$, we must determine the distance $C X$ of this point from the edge $C$, by putting the moment $R \times \overline{C N}$ of the resultant $R$, with reference to this point, equal to the moment $(G + V) \times \overline{C X}$ of its vertical component $G + V$.

We have, therefore,

$$\frac{C X}{C N} = \frac{O R}{H R} = \frac{R}{G + V},$$

and hence

$$C X = a = \frac{\overline{C N} \times R}{V + G} = \frac{S}{G + V}$$

$$= \left(\left[\left(\frac{m^2 + 2 n^2}{3} + m n\right) \frac{h^2}{2} + \left(n + \frac{m}{2}\right) b h + \tfrac{1}{2} b^2\right] \gamma_1 \right.$$
$$\left. + \left[\left(\frac{2 m^2 - 1}{3} + m n\right) h + m b\right] \frac{h}{2} \gamma\right)$$

divided by

$$\left(\left[\left(\frac{m + n}{2}\right) h + b\right] \gamma_1 + \tfrac{1}{2} m h \gamma\right);$$

or,

$$a = \frac{[(m^2+2n^2+3mn)h^2+(2n+m)\times 3bh+3b^2]\gamma_1+[(2m^2-1+3mn)h+3mb]h\gamma}{3([(m+n)h+2b]\gamma_1+mh\gamma)}.$$

By means of this formula, we may also determine other points, *W*, etc., in the line of resistance, if for *h* different values be introduced, and may thus determine the stability of any portion of the dam bounded by horizontal planes.

For a dam without batter, $m = n = o$, hence

$$a = \frac{3b^2\gamma_1 - h^2\gamma}{6b\gamma_1} = \tfrac{1}{2}b - \frac{h^2\gamma}{6b\gamma_1}.$$

For a dam with batter of 45° on each side, $m = n = 1$, hence

$$a = \frac{3(2h^2+3bh+b^2)\gamma_1 + (4h+3b)h\gamma}{3[2(b+h)\gamma_1 + h\gamma]}.$$

If, now, $b = h$, we have

$$a = \frac{18\gamma_1 + 7\gamma}{4\gamma_1 + \gamma} \cdot \frac{h}{3}.$$

If, finally, $\gamma_1 = 2\gamma$, we have

$$a = \frac{43}{27}h = \frac{43}{27}b,$$

or, since the lower breadth $b_1 = 3b$, or $b = \frac{1}{3}b_1$, $a = \frac{43}{81}b_1$.

According to Vauban, there is ample security when

$$a = \frac{5}{9} \cdot \frac{b_1}{2} = \frac{5}{18}b_1,$$

so that for the last case there is an excess of stability. It is well in reservoir dams to make *a* at least $= 0{\cdot}4\,b_1$, when the line of resistance will cut the base at $\frac{4}{10}$ of the width of base from the back.

EXAMPLE.—Required the line of resistance for a dam, the face of which has the batter $m = 1$, the back $n = \frac{1}{2}$, the crown breadth $b = 10$ ft., assuming that the specific gravity of the material is 2.

We have here

$$a = \frac{2\,(3\,h^2 + 60\,h + 300) + (\frac{5}{2}\,h + 30)\,h}{3\,(3\,h + 40 + h)} = \frac{1200 + 300\,h + 17\,h^2}{24\,(10 + h)};$$

therefore, for $h = 0$, $a = 5$ ft.;

$$\text{for } h = 5 \text{ ft.}, \; a = \frac{3125}{360} = 8 \cdot 68 \text{ ft.};$$

$$\text{for } h = 10 \text{ ft.}, \; a = \frac{5900}{480} = 12 \cdot 29 \text{ ft.};$$

$$\text{for } h = 15 \text{ ft.}, \; a = \frac{9525}{600} = 15 \cdot 87 \text{ ft.};$$

$$\text{for } h = 20 \text{ ft.}, \; a = \frac{14000}{720} = 19 \cdot 44 \text{ ft., etc.}$$

For a very great height, we may put

$$a = \frac{17\,h}{24} \text{ and } b = \tfrac{3}{2}\,h,$$

therefore

$$\frac{a}{b} = \frac{17}{36}.$$

Since $\frac{17}{36}$ is greater than $0 \cdot 4$, such a dam would be safe against overturning even for an infinite height.

REMARK.—According to the formula $b = \frac{3\,h - a}{2}$ in the example, Vol. I., Art. 280, if we put $a = m\,h$,

$$2\,b = (3 - m)\,h,$$

hence

$$h = \frac{2\,b}{3 - m},$$

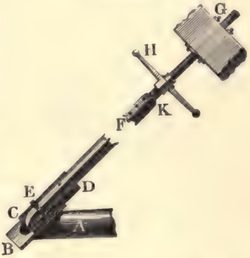
and, therefore, in our last example, in which $m = 1$,

$$h = b = 10 \text{ ft.}$$

§ **159.—Reservoir Sluices.**—For discharging the water from reservoirs we have sluices and waste weirs. The first run through the dam and serve to draw off the water as required. The second are simply notches which allow all excess of water above a certain point to escape. We may have one or more of both of these outlets in a dam. The lowest sluice may serve in general to empty the dam when desirable, the others serve to conduct the water to the machines. In very high dams, it is advantageous to have several sluices at different levels, so that the water may always be drawn off through the highest, and thus as much of the fall as possible utilized. We may also for the same purpose conduct the water to a high receiver or stand pipe, from which the water may be again drawn as required.

The sluice pipes may be either wood, stone, or iron; the latter are the best. Cast-iron pipes from 1 to 2½ ft. in diameter are usual. For regulating the flow, we have sluice gates, of which Fig. 321 gives a representation. Here *A* is the end of the sluice pipe, which is furnished with a flat piece of wood or iron *B*, *C D* is a cast-iron or wooden sluice board fitting in guides, *E F* the sluice rod reaching to the surface, *E* a cross-piece by which, when there are no guides, the sluice is kept in position, *G* is a strong beam through which the screw *G K* passes, so that by a handle at *K* the rod may be moved up and down.

FIG. 321.



The sluice pipe should have a cross-section sufficient to furnish the required discharge for the lowest water level. If $Q$ is the discharge per second, $h$ the least head, $l$ the length, $d$ the diameter $\zeta_0$ the coefficient of resistance at entrance, and $\zeta$ the coefficient of friction, we have from Vol. I., Art. 430,

$$d = \sqrt[5]{\frac{(1+\zeta_0)\,d+\zeta\,l}{2gh} \cdot \left(\frac{4Q}{\pi}\right)^2};$$

or, more simply,

$$d = 0{\cdot}4787\sqrt[5]{[(1+\zeta_0)\,d+\zeta\,l] \cdot \frac{Q^2}{h}}.$$

If, now, we take $\zeta_0$ from the table in Vol. I., Art. 423, and $\zeta$ from the table in Vol. I., Art. 429, we may find by approximation the diameter *d*. For higher levels of the water, a portion of the aperture is closed by the sluice, and therefore, according to Vol. I., Art. 443, we have a greater coefficient of resistance. If the entrance aperture is very small, the water no longer fills the pipe, and the area of the opening is simply

$$F = \frac{Q}{\mu \sqrt{2gh}} = \frac{(1 + \sqrt{\zeta_0})\, Q}{\sqrt{2gh}},$$

where $\zeta_0$ is also given by Art. 443. With the aid of a table of segment areas, such as given on page 152 of the "Ingenieur," the position of the slide may easily be found.

The floodgates or waste weirs are, for the better conduction of the water, placed near the banks. They are at most 5 ft. deep, and 10, 20, and more feet long, and with a stone bottom. They may also be furnished with gates.

EXAMPLE 1.—What is the diameter for a discharge or sluice pipe 100 ft. long, which for 1 ft. head must furnish 10 cubic ft. of water per second?

Assuming the inclination of sluice gate to be 40°, we have $\zeta_0 = 0{\cdot}870$, and for a velocity of 5 ft. we have $\zeta = 0{\cdot}022$. We obtain, therefore,

$$d = 0{\cdot}4787 \sqrt[5]{(1{\cdot}870\, d + 2{\cdot}2) \cdot 100.}$$

The value $d = 1{\cdot}7$ satisfies this equation very nearly. A pipe, then, of $1{\cdot}7 \times 12 = 20{\cdot}4$ inches would fulfil the required conditions.

2. Where must the sluice board be placed in order that the above pipe may furnish still only 10 cubic ft. per second under a head of 16 ft.?

If we assume that the pipe in this case does not run full, we have

$$F = \frac{(1 + \sqrt{\zeta_0})\, Q}{\sqrt{2gh}} = \frac{(1 + \sqrt{0{\cdot}87}) \cdot 10}{8{\cdot}025 \sqrt{16}} = \frac{19{\cdot}327}{8{\cdot}025 \times 4} = 0{\cdot}602 \text{ sq. ft.}$$

This segment for radius $\frac{1{\cdot}7}{2}$, reduced to a radius 1, $= 0{\cdot}431 \frac{4}{2{\cdot}89} = 0{\cdot}598$, and from a table of segments we find the height for such a segment to be 5 inches.

§ **160.—Watercourses.**—The water is led from the reservoir to the machine by watercourses, races, canals, or mill leads. These channels are generally dug out of the natural earth surface, sometimes raised upon embankments, carried over valleys by aqueducts, and through elevations by means of tunnels. The bed may be formed of the natural earth, sand, gravel, stones, or cement or masonry, and it may often consist of wood, stone, or iron. The cross-section of the canal may consist of straight lines or a slightly curved trapezium; the cross-section of a race or *trough* is, in general, rectangular. The best form of cross-section has already been discussed in Vol. I., Art. 472. The cross-section of watercourses is generally, on an average, from 1½ to 3 times as wide as it is deep; in canals for transport, however, 5 to 10 times. In the case of masonry set in cement, little or no slope is given to the sides; for dry stone banks an inclination of 2 to 1 is given; in the case of compact earth 1 to 1, and in sand or loose earth 1 to 2. Fig. 322 gives the construction of a watercourse in loose ground not water-tight. The sides and bottom are rammed 1 to 2 ft. thick with clay, and then covered by 1½ to 2 ft. thick side walls with slight batter. If the watercourse is situated on a side hill, as shown in Fig. 323, a portion only is cut, and the earth thus obtained is used for the other portion. In order to protect the

FIG. 322.

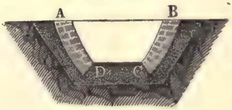
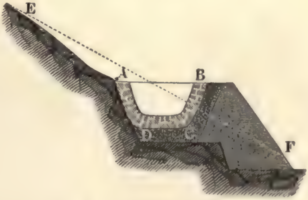


FIG. 323.



bottom *C D*, it is also laid in masonry as well as the sides. High embankments must sometimes be supported by retaining walls as shown by *A B* and *C D*, Fig. 324. Subterraneous watercourses are either cut out of solid rock or built of masonry, as shown in Fig. 325. In Fig. 326 we have a watercourse of masonry, rounded at top, protected within by cement and without by a clay packing. Fig. 327 shows a wooden race or trough. The bottom and sides are formed of planks, and held together as shown at *D E F G*. Fig. 328 shows a cast-iron race. The sides have flanges, as shown at *A B*, *B C*, etc., and are fastened together by bolts. To underground watercourses belong

FIG. 324.

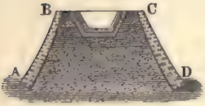


FIG. 325.

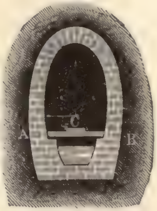


FIG. 326.

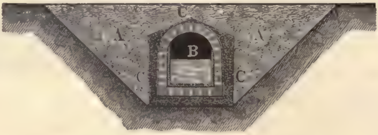


FIG. 327.

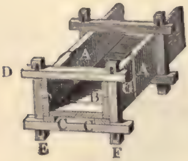
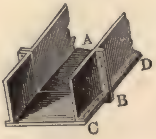


FIG. 328.



also *sewers* (Fr. égouts; Ger. Schleusen). They differ from ordinary underground water-courses only in that the water conducted by them is very impure and full of foreign substances, and within considerable limits very variable in quantity. For this reason, they receive considerable slope, at least $\frac{1}{50}$ of the length. In order to resist the earth pressure, and to increase the velocity, the cross-section *A B*, Fig. 329, is egg-shaped, and to make them water-tight the bottom is protected by a layer *B* of beton. Vent holes as at *C* are also provided, closed by means of an iron cap *D D*.

FIG. 329.

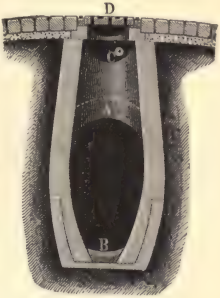
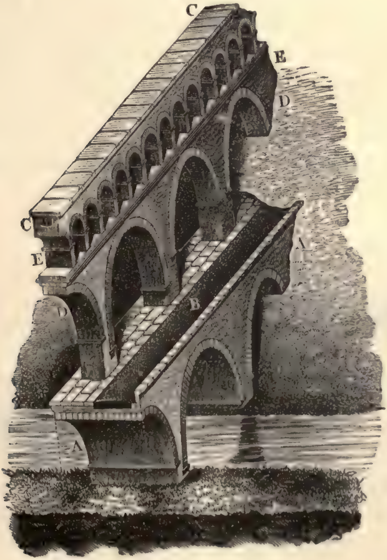


FIG. 330.



REMARK.—Fig. 330 shows the ancient aqueduct du Gard at Nismes. It is 160 ft. high. The canal *C C*, in which the water flows, is 4½ ft. wide and 5 ft. high. It rests upon three tiers of arches, and is covered by stone plates. The lowest arch tier *A A* consists of six semicircular arches from 55 to 77 ft. span, and carries also a road *B*. The second tier *D D* consists of ten arches, and the top tier *E E* of a very large number of small arches.

**§ 161.**—The junction of a watercourse *A A*, Fig. 331, with a river *B* should be effected by a gradual widening and rounding off, and the banks should be protected at the head *D* from the action of the flowing water by masonry. Sluice gates may be inserted, as at *C D*, to regulate the supply of water. To guard against sudden freshets, floodgates and races may be arranged as at *E F*. Self-acting floodgates are often used, as, for instance, floats which, being raised by the water, open a valve and thus discharge the surplus water, or a vessel fills and sinks as the water rises, and thus raises a valve. The simplest contrivance is a syphon, as shown in Fig. 332, *A B C*. As soon as the water rises to the level *B*, the pipe fills and runs with full mouth under the head *C H*. If, however, the water sinks, air enters and the discharge ends. If the water fills only a portion of the

FIG. 331.

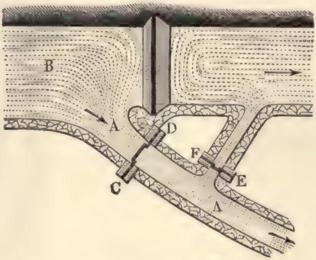
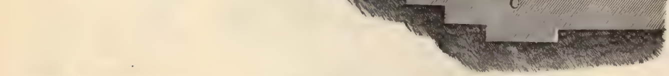


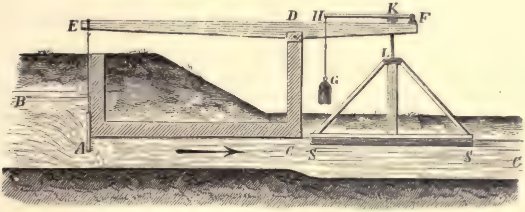
FIG. 332.



cross-section *B D*, we have of course only an overflow, as shown in the figure.

A self-acting sluice is shown in Fig. 333. The sluice board *A* is suspended from a lever *E F*, which turns about

FIG 333.



*D*, and is connected with the float *S S*. As the water level *C C* rises, the float *S S* rises also, and the board *A* falls. If the level *C C* falls, *A* is raised. In the first case, the discharge at *A* is diminished, and in the second increased. In order not to obstruct the rise of the float, when the sluice *A* is shut and the level *C C* is raised by floods, the float acts by means of a rod *K L* upon a lever *F H*, which carries a counterpoise *G*.

§ **162.**—**Fall of Watercourses.**—The velocity of the water in a watercourse should be an average one; not too little, lest the channel become obstructed by sand or sediment, and not too great, because the bottom is liable to be injured, and also because too great a fall deprives the machine of that which it might have. In order to prevent the deposit of sediment, the velocity should at least exceed 7 to 8 inches, and to prevent deposit of sand, it should not be less than $1\frac{1}{4}$ ft. As to the maximum velocity of the water, this depends upon the quality of the bottom. In order that the bottom may not be affected, the velocity should not exceed:

for mud bottom, ¼ ft.;
" clay bottom, ½ ft.;
" sand bottom, 1 ft.;
" gravel bottom, 2 ft.;
" loose stone bottom, 4 ft.;
" conglomerate or slate bottom, 5 ft.;
" laminated stone bottom, 6 ft.;
" hard and unlaminated bottom, 10 ft.

If the velocity at bottom is less than the mean velocity for the entire cross-section, it will be advisable, for the sake of security, not to allow this last to exceed the above limits.

From the assumed mean velocity $c$ and the quantity of water $Q$, we may find the area of the cross-section $F$, and from this again the *wetted perimeter* $p$, or water profile. If we put this value in the formula,

$$\delta = \frac{h}{l} = \zeta \cdot \frac{p}{F} \cdot \frac{c^2}{2g} \text{ (see Vol. I., Art. 475, \textit{et al.})},$$

we obtain the fall $\delta$ for the watercourse, from which we may easily obtain the fall $h = \delta\, l$ for the entire length $l$.

We have, therefore, for different relations very different falls; but since $\zeta$ on the average $= 0{\cdot}007565$, $c$ in general lies between 1 and 5 ft., and for conduit canals $\frac{p}{F}$ is between ⅕ and 2, the limits for fall of such canals are

$$0{\cdot}007565 \times \tfrac{1}{5} \times 1 \times 0{\cdot}016 = 0{\cdot}000024,$$

and

$$0{\cdot}007565 \times 2 \times 25 \times 0{\cdot}016 = 0{\cdot}00605.$$

To canals which conduct the water *away* from the machine, a greater fall is generally given, so as to obtain a greater velocity and to conduct the water away as quickly as possible.

Since, according to Vol. I., Art. 474, for canals with similar profiles,

$$\frac{p}{F} = \frac{m}{\sqrt{F}},$$

the inclination of the bed is

$$\delta = \zeta \frac{m}{\sqrt{F}} \cdot \frac{c^2}{2g};$$

and the fall is therefore greater the less the profile of the canal.

For the same reason, large streams have for the same velocity a less fall than small brooks and canals. If $p$, $F$, $l$ and $c$ apply to small channels, and $p_1$, $F_1$, $l_1$ and $c_1$ to the stream near which the channel runs; if, therefore,

$$h = \zeta \cdot \frac{p\,l}{F} \cdot \frac{c^2}{2g}$$

is the fall of the first, and

$$h_1 = \zeta \frac{p_1 l_1}{F_1} \cdot \frac{c_1^2}{2g}$$

the fall of the second, we have for the effective fall gained by the channel

$$h_2 = h_1 - h = \zeta \frac{p_1 l_1}{F_1} \frac{c_1^2}{2g} - \zeta \frac{p\,l}{F} \frac{c^2}{2g}.$$

Since in general $\frac{p_1}{F_1} < \frac{p}{F}$, we must have $l c^2 < l_1 c_1^2$; *i. e.*, the length of the channel less than that of the stream, and the velocity of the first less than that of the last.

REMARK 1.—In the Freiberg mining districts, the conduits have inclinations varying from 0·00025 to 0·0005, or from 15 inches to 30 inches per mile, the tail races having generally an inclination of from 0·001 to 0·002. The Roman aqueduct at Arcueil, near Paris, has an inclination $\delta = 0{\cdot}000416$, or 2 ft. per mile. The New River, which in part supplies London, has an inclination $\delta = 0{\cdot}00004735$. [The Croton aqueduct has $\delta = 0{\cdot}000208$, or 1·1 ft., and the Boston aqueduct 0·00004735, or 3 inches per mile, the same as New River.]

REMARK 2.—Sudden changes of direction and of sectional area should be avoided, because these occasion not only a loss of head, but also have an injurious effect upon the bottom. When curves cannot be avoided, the radius should be taken as large as possible, or at least the sectional area increased.

REMARK 3.—Depositions of mud, sand, etc., and the growth of plants, cause a contraction of sectional area and consequent loss of head. The watercourse should therefore from time to time be cleaned out. Water is also lost by evaporation and leakage which may again be supplied by rains and springs.

REMARK 4.—If in the formula

$$h = \zeta \frac{m\,l}{\sqrt{F}} \frac{c^2}{2\,g} = \zeta \frac{m\,l\,Q^2}{2\,g\,F^{\frac{5}{2}}},$$

$F$ increases by $\Delta F$, $h$ decreases by

$$\Delta h = \tfrac{5}{2}\,\zeta \frac{m\,l\,Q^2\;\Delta F}{2\,g\;F^{\frac{7}{2}}},$$

and

$$\frac{\Delta h}{h} = -\tfrac{5}{2}\,\frac{\Delta F}{F},$$

or

$$\frac{\Delta F}{F} = -\tfrac{2}{5}\,\frac{\Delta h}{h}$$

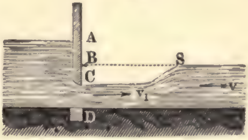
The relative increase of head is, therefore, $\frac{5}{2}$ times the relative diminution of sectional area, or the latter is $\frac{2}{5}$ of the former. The quantity of water remains the same, for example, whether we make the cross-section 2 per cent greater or smaller, or the fall 5 per cent smaller or greater.

§ **163.—Sluices.**—Water may either enter a canal freely or the flow may be regulated by a sluice. If the water flows freely out of the reservoir, where it may be assumed as still, there is a depression of the surface where the flow commences, which depression is due to the velocity $v$ of the flow, and therefore equal to

$$\frac{v^2}{2\,g}.$$

This height must be deducted from the total fall of the watercourse. For moderate velocities of 3 to 4 ft., this depression amounts, however, only to 1½ to 3 inches. If the water entering the canal is regulated by a sluice, we may distinguish two cases. Either the water flows freely through the sluice, or it flows out under the water in the canal below the sluice. In general, the depth of water below the sluice is greater than the height of the sluice opening, and there is therefore at a certain distance from the sluice *A C*, Fig. 334, a sudden rise *S*. The height $BC = x$ of this rise may be found from the velocity $v$ of the issuing water and the velocity $v_1$ of the approaching water, by means of the formula

FIG. 334.



$$x = \frac{v_1^2}{2g} - \frac{v^2}{2g},$$

and if we deduct this height from the head corresponding to the velocity $v_1$,

$$AC = h = \frac{v_1^2}{2g},$$

we have the head corresponding to the velocity $v$,

$$AB = h_1 = h - x = \frac{v_1^2}{2g} - \left(\frac{v_1^2}{2g} - \frac{v^2}{2g}\right) = \frac{v^2}{2g},$$

or precisely the same as for free discharge. Since the opening is never perfectly smooth, there is always a certain resistance which may increase the head required for a given velocity by 10 per cent or more.

If we put the area of cross-section of the departing water $= G$, and that of the opening $CD = F$, and represent the coefficient of contraction by $\alpha$, we have

$$Gv = \alpha F v_1,$$

and hence the height of swell

$$x = a - a_1 = \left[1 - \left(\frac{\alpha F}{G}\right)^2\right] \frac{v_1^2}{2g}.$$

If for $\frac{v_1^2}{2g}$, we put the head $A\,C = h$, and introduce the coefficient of resistance $\zeta_0$, so that

$$h = (1 + \zeta_0)\frac{v_1^2}{2g},$$

we have

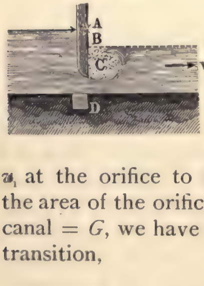
$$x = \left[1 - \left(\frac{\alpha F}{G}\right)^2\right]\frac{h}{1 + \zeta_0}.$$

If the difference $a - a_1$ of the heads $a$ and $a_1$ is less than

$$\left[1 - \left(\frac{\alpha F}{G}\right)^2\right]\frac{v_1^2}{2g},$$

the rise is found further down the stream. If, on the other hand, it is greater, the rise is nearer the sluice, until finally the discharge takes place under backwater, as shown in Fig. 335. In this case, the head $A\,B = h$ not only produces the velocity $v$ of the issuing water, but must also overcome the resistance due to the change from the velocity $v_1$ at the orifice to the velocity $v$ in the canal. If we put the area of the orifice $C\,D = F$ and the cross-section of the canal $= G$, we have for the loss of head occasioned by this transition,

FIG. 335.



$$h_1 = \frac{(v_1 - v)^2}{2g} = \left(\frac{G}{\alpha F} - 1\right)^2\frac{v^2}{2g},$$

and hence the whole head

$$A\,B = h = \frac{v^2}{2g} + \left(\frac{G}{\alpha F} - 1\right)^2\frac{v^2}{2g};$$

or
$$h = \left[1 + \left(\frac{G}{\alpha F} - 1\right)^2\right]\frac{v^2}{2g}.$$

We see that this head, or the difference of level of the water before and behind the sluice, is greater, the less the area of the orifice $F$ in comparison to the cross-section $G$ of the canal.

Example.—A canal of 5 ft. mean width and 3 ft. depth furnishes 45 cubic ft. of water per second. If the sluice opening is 4 ft. wide and 1 ft. high, how much higher is the level of the water in front of the sluice than behind?

Here we have $G = 5 \times 3 = 15$ sq. ft., and $F = 4 \times 1 = 4$ sq. ft.; also $v = \frac{45}{15} = 3$ ft., and $v_1 = \frac{45}{4} = 11\frac{1}{4}$ ft.

Since, now,

$$\left[1 - \left(\frac{F}{G}\right)^2\right]\frac{v_1^2}{2g} = \left[1 - \left(\frac{4}{15}\right)^2\right] 1 \cdot 96 = 1 \cdot 82 \text{ ft.}$$

is less than $a - a_1 = 3 - 1 = 2$ ft., it is evident there will not be a free discharge. The formula

$$h = \left[1 + \left(\frac{G}{F} - 1\right)^2\right]\frac{v^2}{2g}$$

gives then the difference of level required. Hence

$$h = (1 + 2 \cdot 75^2)\, 0 \cdot 139 = 8 \cdot 56 \times 0 \cdot 139 = 1 \cdot 18 \text{ ft.}$$

This, however, owing to the resistance at the orifice, should be increased at least 10 per cent.

§ **164.—Conduit Pipes.**—Conduit pipes serve in general to conduct smaller quantities of water, such as are required by water-pressure engines or turbines, of high fall. As they are enclosed on all sides, they can be led up as well as down hill. The inclination may also be varied at pleasure so long as the exit is below, and the highest point of the conducting pipe is not over one atmosphere (34 ft.) above, or, better still, also below the entrance end. Valleys and elevations may also be crossed by conduit pipes without bridges and tunnels. They may be made of wood, pottery, stone, glass, iron, lead, etc. Wood, iron, and stone pipes are those most often employed.

Wooden conduit pipes are made from large trees which furnish straight sections of 12 to 20 ft. in length. The bore is generally from 1½ to 8 inches, but should not exceed one third of the pipe diameter. The methods of uniting the sections are shown in Figs. 336 and 337. Fig. 336 shows a conical mortice with an iron binding ring *A B*, and a packing of tarred hemp or linen. Fig. 337 shows a connection by means of an iron spigot sunk from 1 to 2 inches into the ends.

FIG. 336.

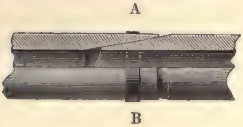
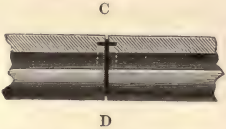


FIG. 337.

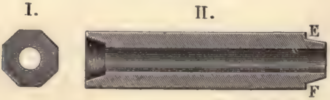


Stone pipes are from 5 to 6 ft. long. They are butted together and united by cement or hydraulic mortar, and an iron binding ring around the joint.

In many cases, asphalt pipes may be used to advantage. So also lead pipes and soldered zinc pipes.

Fig. 338 shows a cross and longitudinal section of a stone pipe with conical mortice.

FIG. 338



Iron pipes surpass all others in strength and durability. They are cast of any diameter, at the least ½ inch thick, and in sections from 5 to 10 ft. long. They should be subjected to a hydrostatic test before use. In order to prevent internal oxidation, they are sometimes tarred or varnished, or coated with hydraulic mortar. The thickness depends upon the diameter and upon the pressure, as laid down in Vol. I., Art. 363. The jointing is effected either by flanges

FIG. 339.

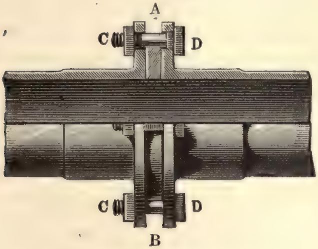
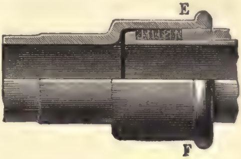


FIG. 340.



*A B* and bolts *C D*, as shown in Fig. 339; or by sockets *E F*, as shown in Fig. 340; or by rings *G H*, as in Fig 341.

The packing is either leather, felt, rubber, lead, cement, or wood, the latter being driven into the joint in wedges. Sometimes thin iron or copper rings are applied internally, so as to cover the joint. Wooden and stone pipes may also be joined to iron ones by means of flanges.

FIG. 341.

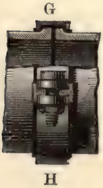
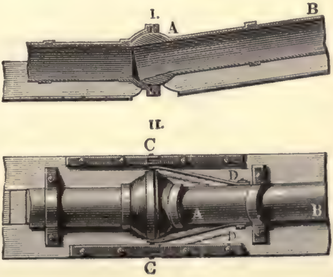


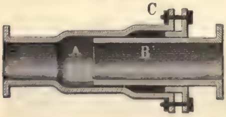
FIG. 342.



In Fig. 342, I. and II., we have also a connection by a ball and socket joint, by means of which the pipes may be accommodated to any angle. This connection has also an axle *C C* and two arms *C D*, *C D*, which turn about the axle and are fastened to the tube *A B*.

If the cast-iron pipes are not laid deep enough beneath the surface, they are subject to changes of temperature, causing expansion or contraction of the tubes. In order to avoid the evil effects of this, compensation or expansion joints, as shown in Fig. 343, may be used. The expansion of cast-iron is 0·0000111 of its length for each degree centigrade, and therefore the expansion for 50° change of temperature, or from winter to summer, is $50 \times 0{\cdot}0000111 = 0{\cdot}000555$ For every 1800 ft. of length then, there is a change of length of about one foot for this variation of temperature. This is compensated by the joint at *A*, which allows the tube *B* to slide in the stuffing-box *C*. In general, there should be such a joint for every 300 ft. of length.

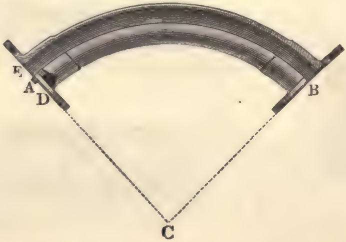
FIG. 343.



In order to prevent freezing, the pipes should be laid not less than 3 ft. below the surface of the ground, by which, of course, the contraction is also diminished.

§ **165.—Arrangement of Conduit Pipes.**—Pipes cannot, of course, always be laid in a straight line, but must rise and fall, and turn to the right and left. The rule should always be observed, to avoid all sudden changes of direction or knee joints, and to give curved pipes a large radius or a large diameter. Such a curved cast-iron section is shown in Fig. 344. Here the angle $ACB = 90°$ and the ratio of the diameter $DE$ to the radius $CA$ is $\frac{1}{6}$. Sudden changes of cross-section should also be avoided, and every transition effected by gradual increment or decrement of

FIG. 344.



diameter. Vertical curves or bends, Fig. 345, have the disadvantage of collecting the air at the summit *L*, thus causing a contraction of section which may even prevent the flow entirely. In order to prevent this accumulation, vertical pipes called *wind pipes*, or *ventilators* (Fr. ventouses; Ger. Luftständer), are inserted, by which the air may escape. The ventilator may be furnished with a cock, which must be turned from time to time by hand. In order to make the ventilator self-acting, we may use a float, as shown in Fig. 346. Here the discharge valve *V* is attached to a hollow float *S* of tinned iron, which, as long as water fills the tube, is pressed upward, and keeps the valve shut, but which falls and opens the valve as soon as air collects.

FIG. 345.

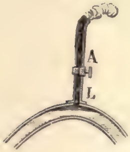


FIG. 346.

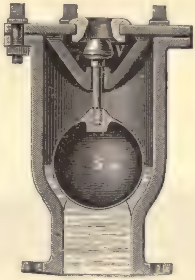
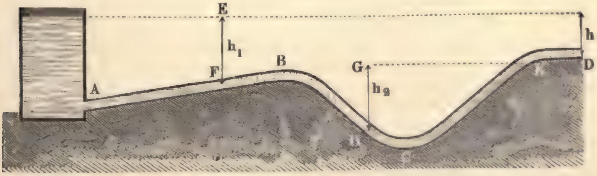


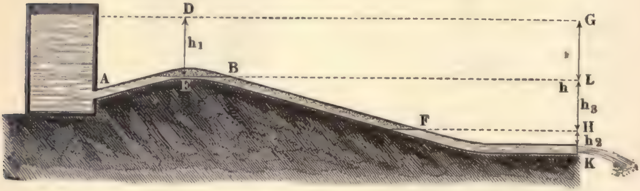
FIG. 347.



If a pipe system, *A B C D*, Fig. 347, has at the point *B* no ventilator, the accumulated air is pressed upon on one side by the head of water $E F = h_1$, and upon the other by $G H = h_2$. If, then, $h_2 = h_1$ and the surface of the water at *K* does not reach the orifice *D*, the air pressure in *F B H* is in equilibrium with these two heads, and there is no discharge.

The lack of a ventilator may also sometimes simply diminish the discharge without wholly preventing it. Such a case is shown in Fig. 348, where the height $z$ of the water column which measures the pressure of the air in

FIG. 348.

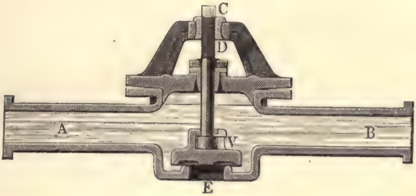


$E\,B\,F$ is only a little less than the head $D\,E = h_1$ of the approaching water, and hence the velocity of this last is very small. From $E$ the water then flows down to $F$ upon the bottom of the pipe without any change of pressure, and from $F$ on, it flows to $K$ with full cross-section. The head at the orifice $K$ is therefore not $G\,K = h$, but $H\,K = h_2 + z$, or nearly $h_1 + h_2$, and hence the head $H\,L = h_3$, between $E$ and $F$, is entirely lost.

Not only does air collect at the highest points of a system of pipes, but in like manner sand and mud are deposited at the lowest points. In order to remove this deposit from time to time, we use at these points *flushing pipes* or *mud boxes*. The flush pipes open sideways into the pipe, and are closed by cocks or valves. The mud boxes are vessels into which both portions of the pipe are inserted, and through which, therefore, the water must flow with diminished velocity. The deposit of matter is not only aided by the diminished velocity, but sieves or partitions may also be made use of for the same purpose. By opening a valve in the bottom, the box may from time to time be cleaned out. It is also necessary to have traps at intervals of every 100 ft. or so, in order to facilitate inspection and cleaning of the pipes. The pipes may be cleaned by the discharge of the water, by wood or iron rods, and by dissolving the lime incrustations by hydrochloric acid. The piezometer (Vol. I., Art. 435) is also very useful for ascertaining obstructions, etc.

For regulating the flow, we have cocks, slides, and valves. A simple stop valve is shown in Fig. 349. This valve *V* sits on a screw bolt *C D V*, and closes a side opening *E* of the pipe *A B*. When it is desired to discharge water through *E*, the stem *C D* is turned by a wrench. The action of this apparatus has already been considered in Vol. I., Art. 443. In order, finally, to diminish the effect of shock by the sudden closure of the aperture, it is well to make use of loaded valves, which open and shut gradually as the pressure changes.

FIG. 349.



REMARK.—The subject of water conduction will be found treated in detail in Geniey's "Essai sur les moyens de conduire, d'élever et de distribuer les eaux," also in "Traité théoretique et pratique de la conduite et de la distribution des eaux," etc., par Dupuit, Paris, 1854. Also "Les Fontaines publiques de la ville de Dijon," par Henry Darcy, Paris, 1856. See also upon water conduction more especially Hagen's "Wasserbaukunst," Theil. I., Gerstner's "Mechanik," Theil. II., and Eytelwein's "Hydraulik." Also Bornemann's "Hydrometrie," Freiberg, 1849.

**§ 166.—Motion of Water in Conduit Pipes.**—The conditions of motion in conduit pipes have already been discussed. If $h$ is the fall, $l$ the length, $d$ the diameter, $\zeta_0$ the coefficient of resistance at entrance, $\zeta$ the coefficient of friction; if, further, $\zeta_1$, $\zeta_2$, etc., are the other coefficients of resistance for bends, cocks, etc., and if $v$ is the velocity of discharge, we have

$$h = \left(1 + \zeta_0 + \zeta\frac{l}{d} + \zeta_1 + \ldots\right)\frac{v^2}{2g},$$

or, if $Q$ is the quantity of water,

$$h = \left(1 + \zeta_0 + \zeta\frac{l}{d} + \zeta_1 + \ldots\right)\left(\frac{4Q}{\pi}\right)^2\frac{1}{2gd^4}.$$

We see, therefore, that for the conduction of a certain quantity $Q$, less fall is necessary, the greater the diameter. If, instead of one pipe, two are used, which together have the same cross-section as the single one, and if we let each carry half the whole amount of water, the necessary fall is

$$h_1 = \left(1 + \zeta_0 + \zeta\frac{l}{d\sqrt{\frac{1}{2}}} + \zeta_1 + \ldots\right)\left(\frac{2Q}{\pi}\right)^2\frac{1}{2g(d\sqrt{\frac{1}{2}})^4}$$

$$= \left(1 + \zeta_0 + \zeta\frac{l\sqrt{2}}{d} + \zeta_1 + \ldots\right)\left(\frac{4Q}{\pi}\right)^2\frac{1}{2gd^4}.$$

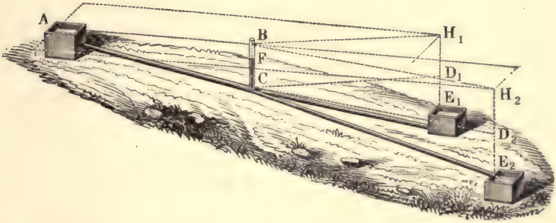
The head required is therefore greater than in the first case, and it is mechanically better to use only one pipe whose cross-section is equal to the sum of the others.

Calculations for entire systems of pipes which have numerous ramifications, become extremely complicated. The same holds good when many branches unite. The general method of such calculations is as follows: If the division of the water takes place in a reservoir of much greater section than the main pipe, the water comes again to rest and the entire *vis viva* is therefore lost and must be again acquired in the branch pipes. The same loss occurs when many branches discharge into a reservoir from which a single main leads out. In this case, the calculation may be made for each branch pipe separately, and requires no further notice. In order that the division or collection of the water in such intermediate reservoirs may entail but little loss of head, they should be placed so high as to ensure the same mean velocity in all the pipes.

In the case of a simple branch or *fork*, it is desirable to so arrange that the water may move with the same velocity in all the pipes. If, then, the branches be properly curved so that there is no sudden change of direction in passing from the main into the branches, it may be assumed that there is no noticeable loss of head or *vis viva*.

In the case represented in Fig. 350, let $h$ be the head $BC = H_1 D_1 = H_2 D_2$, $l$ the length and $d$ the diameter of the main pipe $AC$; also, $h_1$ the head $D_1 E_1$, $l_1$ the length and $d_1$ the diameter of one of the branch pipes, and $h_2$ the head $D_2 E_2$, $l_2$ the length and $d_2$ the diameter of the other. Let, also, $c$, $c_1$, $c_2$, be the velocities of the water in these three pipes, and $\zeta_0$ the coefficient of resistance at entrance, $\zeta$ the coefficient of friction.

FIG. 350.



If, now, $z$ is the height of the piezometer or the pressure head at the junction of the branches, we may put

$$BF = CB - CF = h - z = \left(1 + \zeta_0 + \zeta \frac{l}{d}\right) \frac{c^2}{2g} \quad . \quad . \quad \text{I.}$$

Further,

$$CF + D_1 E_1 = z + h_1 = \frac{c_1^2}{2g} - \frac{c^2}{2g} + \zeta \frac{l_1}{d_1} \cdot \frac{c_1^2}{2g} \quad . \quad . \quad . \quad \text{II.}$$

$$CF + D_2 E_2 = z + h_2 = \frac{c_2^2}{2g} - \frac{c^2}{2g} + \zeta \frac{l_2}{d_2} \cdot \frac{c_2^2}{2g} \quad . \quad . \quad . \quad \text{III.}$$

Since the quantity of water

$$Q = \frac{\pi d^2}{4} c,$$

passed by the main, is equal to the sum of the quantities

$$Q_1 = \frac{\pi d_1^2}{4} c_1 \quad \text{and} \quad Q_2 = \frac{\pi d_2^2}{4} c_2$$

passed by the two branches, we have also

$$d^2 c = d_1^2 c_1 + d_2^2 c_2 \; . \; . \; . \; . \; . \; . \; . \quad \text{IV.}$$

By the aid of these four equations, we may determine four quantities. In general, the fall, length of pipe, and quantity of water are given, and it is required to find the necessary diameter of pipe. If we assume the velocity $c$ of the water in the main as known, we have for its diameter

$$d = \sqrt{\frac{4Q}{\pi c}} = 1 \cdot 1284 \sqrt{\frac{Q}{c}} \; . \; . \; . \; . \; . \quad (1),$$

and can then from I. find the height of the piezometer at the point $C$,

$$z = h - \left(1 + \zeta_0 + \zeta \frac{l}{d}\right) \frac{c^2}{2g} \; . \; . \; . \; . \; . \quad (2).$$

If we insert this value of $z$ in equations II. and III., we have, after reduction, the following values for the diameters of the branch pipes:

$$d_1 = \sqrt[5]{\frac{\zeta l_1 + d_1}{2g(z + h_1) + c^2} \left(\frac{4Q_1}{\pi}\right)^2} \; . \; . \; . \; . \quad (3)$$

$$d_2 = \sqrt[5]{\frac{\zeta l_2 + d_2}{2g(z + h_2) + c_2} \left(\frac{4Q_2}{\pi}\right)^2} \; . \; . \; . \; . \quad (4).$$

In order to obtain a first approximation, we may omit $d_1$ and $d_2$ under the radical. If $c_1$ and $c_2$ are found to be very different from $c$, the coefficient of friction $\zeta$ should be regarded as variable, a special value taken for each of the three pipes, and the determination of $d_1$ and $d_2$ repeated.

EXAMPLE.—A system of pipes consisting of one main and two branches is required to discharge by one branch 15 and by another 24 cubic ft. of water per minute. The levels show the main pipe to have a fall of 4 ft. in 1000, the first branch 3 ft. in 600, and the other branch 1 foot in 200. What diameter should the pipes have?

If we wish a velocity of 2½ ft. in the main, we must give it a diameter,

$$d = \sqrt{\frac{4\,Q}{\pi\,c}} = \sqrt{\frac{4 \times 39}{\frac{5}{2} \times 60\,\pi}} = \sqrt{\frac{26}{25\,\pi}} = 0\cdot5754 \text{ ft.} = 6\cdot9 \text{ inches.}$$

If, now, we take (according to Vol. I., page 859) the coefficient of resistance for entrance $\zeta_0 = 0\cdot505$, and the coefficient of friction (Vol. I., page 869) for the velocity $c = 2\cdot5$, $\zeta = 0\cdot0253$, then since $2g = 64\cdot4$ and $\left(\frac{4}{\pi}\right)^2 = 1\cdot621$, we have for the height of piezometer at junction of branches,

$$z = h - \left(1 + \zeta_0 + \zeta\frac{l}{d}\right)\frac{c^2}{2g} = 4 - \left(1 + 0\cdot505 + 0\cdot0253\,\frac{12000}{6\cdot9}\right)\frac{6\cdot25}{64\cdot4}$$

$$= 4 - (1\cdot505 + 44)\,\frac{1}{10\cdot3} = 4 - 4\cdot418 = -\,0\cdot418 \text{ ft.}$$

If, now, we assume at first for the branch pipes $\zeta = 0\cdot0253$, and neglect at first the terms $d_1$ and $d_2$, we have

$$z + h_1 = -\,0\cdot418 + 3 = 2\cdot582,$$
$$z + h_2 = -\,0\cdot418 + 1 = 0\cdot582,$$

$$\left(\frac{4\,Q_1}{\pi}\right)^2 = 1\cdot621\left(\frac{15}{60}\right)^2 = \frac{1\cdot621}{16} = 0\cdot10131$$

and

$$\left(\frac{4\,Q_2}{\pi}\right)^2 = 1\cdot621\left(\frac{24}{60}\right)^2 = 1\cdot621\,(0\cdot4)^2 = 0\cdot25936.$$

Also

$$d_1 = \sqrt[5]{\frac{0\cdot0253 \times 600 \times 0\cdot10131}{64\cdot4 \times 2\cdot582 + 6\cdot25}} = \sqrt[5]{\frac{15\cdot18 \times 0\cdot10131}{172\cdot5308}} = 0\cdot389 \text{ ft.}$$

and

$$d_1 = \sqrt{\frac{0\cdot0253 \times 200 \times 0\cdot25936}{64\cdot4 \times 0\cdot582 + 6\cdot25}} = \sqrt[5]{\frac{5\cdot06 \times 0\cdot25936}{43\cdot7308}} = 0\cdot496.$$

For these diameters we have the corresponding velocities,

$$c_1 = \frac{4\,Q}{\pi\,d_1^2} = \frac{4 \times 15}{60\,\pi\,(0 \cdot 389)^2} = 2 \cdot 12 \text{ ft.}$$

and

$$c_2 = \frac{4\,Q}{\pi\,d_2^2} = \frac{4 \times 24}{60\,\pi\,(0 \cdot 496)^2} = 2 \cdot 07 \text{ ft.},$$

which (Vol. I., page 859) correspond to the coefficients of resistance,

$$\zeta = 0 \cdot 0262 \quad \text{and} \quad \zeta = 0 \cdot 0268.$$

We have, therefore, more accurately for the first branch pipe,

$$d_1 = \sqrt[5]{\frac{0 \cdot 0262 \times 600 + 0 \cdot 389}{172 \cdot 5308}} \cdot 0 \cdot 10131 = 0 \cdot 4 \text{ ft. (nearly)} = 4 \cdot 8 \text{ inches.}$$

$$d_2 = \sqrt[5]{\frac{0 \cdot 0268 \times 200 + 0 \cdot 496}{43 \cdot 7308}} \cdot 0 \cdot 25963 = 0 \cdot 5 \text{ ft. (nearly)} = 6 \text{ inches.}$$

§ **167.—Compound Systems.**—If the division of the main into two branches takes place in a vessel in which the water has a free surface, the preceding equations I., II. and III. become as follows:

$$\text{I. } h = \left(1 + \zeta_0 + \zeta\frac{l}{d}\right)\frac{c^2}{2g},$$

$$\text{II. } h_1 = \left(1 + \zeta_0 + \zeta\frac{l_1}{d_1}\right)\frac{c_1^2}{2g},$$

and

$$\text{III. } h_2 = \left(1 + \zeta_0 + \zeta\frac{l_2}{d_2}\right)\frac{c_2^2}{2g},$$

where $h$ is the distance of the water surface $A$ in the upper reservoir above that in the centre, and $h_1$, $h_2$ the distance of this last above the water surface $E_1$ in the lower vessel or to the orifice of discharge $E_2$ of the branch pipe $C E_2$, according as the water discharges under water or free. If here also we have $c$ given or $d = \sqrt{\frac{4\,Q}{\pi\,c}}$, we can from the first equation determine the height $h$, and if we subtract this from the whole fall between $A$ and $E_1$ and between $A$ and $E_2$, we have the fall $h_1$ and $h_2$ of the branch pipes

$CE_1$ and $CE_2$, whose diameters may then be found by the formulæ,

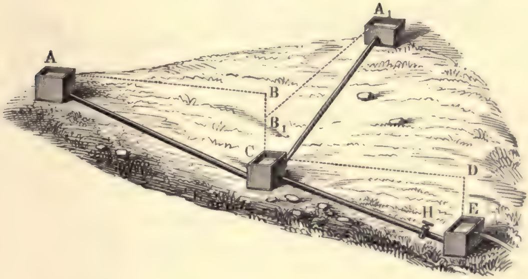
$$d_1 = \sqrt[5]{\frac{(1+\zeta_0)\, d_1 + \zeta\, l_1}{2\, g\, h_1} \left(\frac{4\, Q_1}{\pi}\right)^2}$$

and

$$d_2 = \sqrt[5]{\frac{(1+\zeta_0)\, d_2 + \zeta\, l_2}{2\, g\, h_2} \left(\frac{4\, Q_2}{\pi}\right)^2}.$$

The above formulæ apply also when, as in Fig. 351, two pipes $A\,C$ and $A_1\,C$ empty into a reservoir $C$, and their united discharge is conveyed by the main $C\,E$. In this case, $h$ is the fall $D\,E$, $l$ the length, $d$ the diameter of the main $C\,E$; $h_1$ the fall $B\,C$, $l_1$ the length, $d_1$ the diameter of the branch $A\,C$; $h_2$ the fall $B_1\,C$, $l_2$ the length and $d_2$ the diameter of the other branch $A_1\,C$. The above formulæ also apply to such a confluence, when, in place of the reservoir $C$, there is a simple fork as in Fig. 350.

FIG. 351.



If curves, bends, and elbows occur, their resistance must be taken into account. The same holds good for valves and cocks, such as $H$, which may be used. If $\zeta_2$ is the coefficient of resistance for a certain position of the valve (Vol. I., Art. 443), we have in such of the preceding formulæ as apply to the pipe thus affected merely to increase the coefficient of resistance $\zeta_0$ for entrance by $\zeta_2$, and hence to insert $\zeta_0 + \zeta_2$ in place of $\zeta_0$.

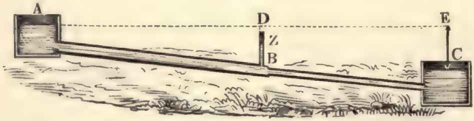
If in a conduit pipe there occurs an angular change of cross-section which causes a sudden change of velocity, we have also an additional resistance, which is measured by

$$h_1 = \frac{(c_1 - c)^2}{2g}$$

if $c_1$ and $c$ are the two velocities of the water.

If the pipe is composed of a wide and narrow portion, as shown in Fig. 352, the resistance encountered by the water is of course different from that due to a pipe of constant diameter.

FIG. 352.



If $l$ is the length, $d$ the diameter, and $h$ the head of the lower pipe $BC$, $c$ being the velocity of the water in it, and $l_1$, $d_1$, $h_1$ and $c_1$ the same quantities for the upper pipe $AB$, $h$ being the entire head $CE$, and $z$ the height of piezometer at the point $B$ where the cross-section changes, then,

$$h_1 - z = \left(1 + \zeta_0 + \zeta_1 \frac{l_1}{d_1}\right) \frac{c_1^2}{2g},$$

$$z + h - h_1 = \frac{c^2}{2g} - \frac{c_1^2}{2g} + \frac{(c - c_1)^2}{2g} + \zeta \frac{l}{d} \cdot \frac{c^2}{2g}.$$

Since

$$\frac{c_1}{c} = \frac{d^2}{d_1^2},$$

we have

$$c_1 = \left(\frac{d}{d_1}\right)^2 c,$$

and if, according to Vol. I., Art. 436, we put

$$\frac{(c - c_1)^2}{2g} = \left(1 - \frac{c_1}{c}\right)^2 \frac{c^2}{2g} = \left[1 - \left(\frac{d}{d_1}\right)^2\right]^2 \frac{c^2}{2g} = \zeta_2 \frac{c^2}{2g},$$

we obtain

$$2gh = \left[1 + \zeta_2 + \zeta\frac{l}{d} + \left(\zeta_0 + \zeta_1\frac{l_1}{d_1}\right)\left(\frac{d}{d_1}\right)^4\right]c^2.$$

If the entire fall is given, we have, therefore, the velocity of discharge,

$$c = \sqrt{\frac{2gh}{1 + \zeta_2 + \zeta\frac{l}{d} + \left(\zeta_0 + \zeta_1\frac{l_1}{d_1}\right)\left(\frac{d}{d_1}\right)^4}},$$

from which we may determine the quantity discharged,

$$Q = \frac{\pi d^2}{4}c.$$

If this last is given, we have, on the other hand, for the diameter of pipe,

$$d = \sqrt[5]{\frac{\zeta l + (1 + \zeta_2)d}{2gh\left(\frac{\pi}{4Q}\right)^2 - \left(\zeta_0 + \zeta_1\frac{l_1}{d_1}\right)\frac{1}{d_1^4}}}.$$

EXAMPLE.—Suppose a system of pipes such as represented in Fig. 352, consisting of a pipe *B C*, 200 ft. long and 3 inches diameter, and of a pipe *A B*, 300 ft. long and 5 inches diameter, the total fall being 5 ft. We can assume

$$\zeta_0 = 0\cdot505 \text{ and } \zeta_2 = \left[1 - \left(\frac{d}{d_1}\right)^2\right]^2 = (1 - 0\cdot36)^2 = 0\cdot410,$$

and preliminarily, $\zeta = 0\cdot024$ and $\zeta_1 = 0\cdot030$. The velocity in the narrower pipe is then

$$c = \sqrt{\frac{64\cdot4 \times 5}{1\cdot410 + 0\cdot024 \times 800 + (0\cdot505 + 0\cdot03 \times 300 \times \frac{12}{5})\left(\frac{3}{5}\right)^4}} = \sqrt{\frac{322}{23\cdot475}} = 3\cdot76 \text{ ft.}$$

The velocity in the wider pipe is then

$$c_1 = \left(\frac{d}{d_1}\right)^2 c = 0\cdot36\,c = 1\cdot35 \text{ ft.}$$

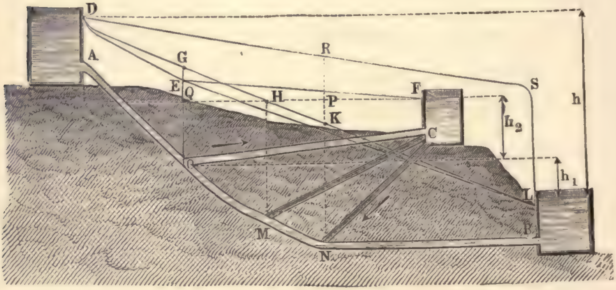
If we take accordingly $\zeta = 0{\cdot}0233$ and $\zeta_1 = 0{\cdot}0291$, we have more accurately the velocity $c = 3{\cdot}80$, and the corresponding quantity of water discharged,

$$Q = \frac{\pi d^2}{4} c = \frac{\pi}{4} \cdot \frac{3{\cdot}80}{16} = 0{\cdot}1865 \text{ cubic ft.}$$

### § 168.—Pressure Line of a System of Conduit Pipes.—

The pressure line *D G H K L* of a system of pipes, *A M B*, Fig. 353, determined by the piezometer height, gives complete information as to the pressure of the water at every point of the system. For example, at *O* the pressure is measured by the piezometer height *O G*, at *M* by the piezometer height *M H*, etc. In systems which have changes of cross-section and direction, the pressure line is curved. Where the pipe is narrow and the water flows rapidly, it is drawn down, and where the cross-section is larger and the water flows more slowly, it rises. If the system *A M B* which unites two reservoirs *A* and *B*, communicates by a second pipe with a third reservoir *C*, the question first arises, whether the water will flow from *C* to *A B*, or from *A B* to *C*. If the plane of the water surface in *C* cuts the pressure line *D G H K L* in *H*, the point *M* of the pipe system vertically below *H* is that point where a branch pipe *C M* will neither lead water to nor away from *C*. If the junction is at *N*, the distance *N P* below the water surface at *C*

FIG. 353.



being greater than the piezometer height $NK$ at this point, then the water will flow from $C$ to $N$ and from there on to $B$. If, on the other hand, the branch pipe makes junction at $O$, the depth $O\,Q$ below water surface at $C$ is less than the piezometer height or hydraulic head $O\,G$ at $O$, the water from $A$ will flow not only to $B$, but also a portion will flow towards $C$.

If, again, as in Art. 166, $l$, $l_1$ and $l_2$ are the lengths; $d$, $d_1$ and $d_2$ the diameters, and $h$, $h_1$ and $h_2$ the heads of the portions $A\,O$, $O\,B$ and $O\,C$, and if we put the piezometer height at $O = z$, and consider only the frictional resistance of the pipes, we have simply

$$h - z = \zeta \frac{l}{d} \frac{c^2}{2g},$$

$$z + h_1 = \zeta \frac{l_1}{d_1} \frac{c_1^2}{2g},$$

$$z - h_2 = \zeta \frac{l_2}{d_2} \frac{c_2^2}{2g},$$

or, putting the quantities of water

$$Q = \frac{\pi d^2}{4} c, \quad Q_1 = \frac{\pi d_1^2}{4} c_1, \quad \text{and} \quad Q_2 = \frac{\pi d_2^2}{4} c_2,$$

and putting for simplicity,

$$\left(\frac{4}{\pi}\right)^2 \frac{\zeta}{2g} = \psi,$$

$$h - z = \left(\frac{4}{\pi}\right)^2 \frac{\zeta}{2g} \frac{l\,Q^2}{d^5} = \frac{\psi\,l\,Q^2}{d^5},$$

$$z + h_1 = \frac{\psi\,l_1\,Q_1^2}{d_1^5}, \qquad \text{and} \qquad z - h_2 = \frac{\psi\,l_2\,Q_2^2}{d_2^5}.$$

But, $Q = Q_1 + Q_2$, hence

$$\sqrt{\frac{(h - z)\,d^5}{l}} = \sqrt{\frac{(z + h_1)\,d_1^5}{l_1}} + \sqrt{\frac{(z - h_2)\,d_2^5}{l_2}},$$

or if the pipe system has the same diameter,

$$\sqrt{\frac{h-z}{l}} = \sqrt{\frac{z+h_1}{l_1}} + \sqrt{\frac{z-h_2}{l_2}}.$$

In this last case, therefore, the piezometer height $z$ at the point of junction $O$ is not dependent upon the pipe diameter $d$ nor the quantity of water $Q$.

If the reservoir $C$ is shut off from $A\ B$, the discharge towards $C$ is

$$Q_0 = \sqrt{\frac{(h+h_1)\,d^5}{\psi\,(l+l_1)}},$$

and we have

$$Q_0^2\,(l+l_1) = Q^2\,l + Q_1^2\,l_1 = (Q_1+Q_2)^2\,l + Q_1^2\,l_1,$$

hence

$$Q_1^2 + \frac{2\,Q_2\,l}{l+l_1}\,Q_1 = Q_0^2 - \frac{Q_2^2\,l}{l+l_1}.$$

The solution of this quadratic equation gives the quantity of water passing through $O\ B$ into the reservoir $B$,

$$Q_1 = -\frac{Q_2\,l}{l+l_1} + \sqrt{Q_0^2 - Q_2^2\,\frac{l\,l_1}{(l+l_1)^2}},$$

or, approximately, when $Q_2$ is not large with respect to $Q_0$,

$$Q_1 = Q_0 - \frac{l}{l+l_1}\,Q_2 \quad \ldots\ldots\ (1)$$

$$Q = Q_1 + Q_2 = Q_0 + \frac{l_1}{l+l_1}\,Q_2 \quad \ldots\ (2).$$

By the introduction of the pipe $O\,C$, the pressure line $D\,G\,H\,K\,L$ becomes $D\,E\,L$, and the pressure line $D\,E\,F$ also enters. The piezometer height at $O$, $O\,E$ is greater than $O\,Q$, but less than $O\,G$, or the head differences for $A\ O$ are less than $h - h_2$, while the head differences of $O\ B$ are greater than $h_1 + h_2$.

We have, then,

$$Q < \sqrt{\frac{(h - h_2)\, d^5}{\psi\, l}},$$

and

$$Q_1 > \sqrt{\frac{(h_1 + h_2)\, d^5}{\psi\, l_1}},$$

also

$$Q_2 = Q - Q_1 < \sqrt{\frac{(h - h_2)\, d^5}{\psi\, l}} - \sqrt{\frac{(h_1 + h_2)\, d^5}{\psi\, l_1}}.$$

If we assume provisionally,

$$Q_2 = \sqrt{\frac{(h - h_2)\, d^5}{\psi\, l}} - \sqrt{\frac{(h_1 + h_2)\, d^5}{\psi\, l_1}},$$

we can find by means of the above formulæ (1) and (2), $Q$ and $Q_1$, approximately, and then more exactly $Q_2 = Q - Q_1$. By repeated application of the formulæ, we can determine $Q$, $Q_1$, and hence $Q_2$, more and more exactly.

If by means of a cock or other regulator at $B$ the pressure in the pipes $AMB$ is increased, so that the pressure line becomes $DRSL$, the piezometer height $NR$ at the point $N$ is greater than the distance to water surface in $C$, and water flows also through the pipe $NC$, from $A$ to $C$. In order to conduct more or less water to $B$, according to desire, it is only necessary, therefore, to have a greater or less opening of the regulator at $B$.

# HYDRAULIC MOTORS.

## CHAPTER I.

### VERTICAL WATER-WHEELS.

§ **169.—Water-Power.**—Water acts as a motor, or generates motion in machines, either by its weight or by its inertia or impulse. When acting by its weight, it sinks gradually and slowly in the machine through a certain height or *fall* (Fr. chute; Ger. Gefälle), while, when acting by impulse, it passes through the machine with constantly *decreasing velocity*, and performs a work equivalent to this loss of velocity upon the surfaces with which, during its progress through the machine, it comes in contact.

If $Q$ is the whole quantity of water entering in one second, $Q \gamma$ its weight, $\gamma$ being the weight of one unit of volume, and if $h$ is the fall through which the water sinks while acting by its weight; then the mechanical effect or work which during this fall it performs is

$$L = Q \gamma h.$$

If, however, $c$ is the velocity with which it enters the machine, then the work which, by reason of its inertia or living force, it can perform before coming to rest is,

$$L = Q \gamma \frac{c^2}{2 g}.$$

But in order that the water starting from rest may attain the velocity $c$, it must fall through the distance

$$\frac{c^2}{2g};$$

hence in this second case also we can put

$$L = Q \gamma h.$$

Therefore, the work inherent in the water is always *the product of its weight into the height through which it falls, or should fall in order to attain its given velocity.*

Sometimes the water acts at the same time both by weight and impulse together, while falling through the height $h$ and entering with a velocity $c$. In such case, its mechanical effect is of course

$$L = Q \gamma h + Q \gamma \frac{c^2}{2g} = \left(h + \frac{c^2}{2g}\right) Q \gamma.$$

The effective delivery $Pv$ of an hydraulic motor is always less than the above total available work $Q h \gamma$, because many causes of loss exist which prevent all of this work being utilized. Thus, all the water may not enter; a portion of the fall itself may not be used; the water may leave the machine with a certain velocity, in which case it is capable still of performing a certain amount of work which has not been given up to the machine; and, lastly, we have work expended in overcoming friction, resistance of the air, etc. Accordingly we can place the *efficiency* of an hydraulic motor,

$$\eta = \frac{P v}{Q h \gamma},$$

and the degree of perfection of such a machine is greater the nearer this fraction approaches unity—*i. e.*, the nearer the useful work approaches to equality with the total available work.

From the general formula $L = Q h \gamma$, we see that both the fall and quantity of water have equal influence upon the work received by the machine. Thus a double fall doubles this work, as does also a double quantity of water. Therefore the same effect is to be expected from two machines of which one receives three times as much water as the other, but the other has three times as much fall.

EXAMPLE.—A machine has at disposal 12 cubic ft. of water per second, and a fall of 10 ft., but it actually uses only 8·5 ft., and the water leaves it with a velocity of 9 ft. Finally, it also loses 750 ft. lbs. of work, which go to the overcoming of friction, etc. What is its degree of efficiency?

The whole available work is $L = 12 \times 10 \times 62 \cdot 5 = 7500$ ft. lbs. per second. If the machine actually used all this, it would be perfect, and its efficiency would be 1. But it only actually uses 8·5 ft. of the whole fall, and hence the work which this fall would give it is $12 \times 8 \cdot 5 \times 62 \cdot 5 = 6375$ ft. lbs. per second. But even this work is not all utilized. The departing water carries away with it a work of $0 \cdot 016 \times 9^2 \times 12 \times 62 \cdot 5 = 972$ ft. lbs.; and, moreover, the friction of the machine consumes 750 ft. lbs more. Thus the useful work of the machine is $P v = 6375 - (972 + 750) = 4653$ ft. lbs. The efficiency of the machine is then

$$\frac{4653 \cdot 0}{7500} = 0 \cdot 62 = \eta.$$

**§ 170.—Water-Wheels.**—Hydraulic motors may be divided into water-wheels and water-pressure engines. Water-wheels (Fr. roues hydrauliques; Ger. Wasserräder) consist of a wheel and axle set in motion by water-power. Water-pressure engines (Fr. machines à colonne d'eau; Ger. Wassersäulenmaschinen) consist essentially of a column of water and a piston, which is set in motion by the pressure of the water column upon its surface.

We may distinguish *vertical* water-wheels—*i. e.*, wheels with a horizontal axis—and *horizontal* water-wheels, or wheels with a vertical axis. Vertical water-wheels, which we shall first discuss, are either *overshot*, or *middleshot*, or *undershot*. In wheels of the first kind, the water enters near the top; in those of the second kind, it enters near the middle; while in undershot-wheels, it flows under, tangent to the circumference. We may also distinguish the *back-pitch* water-wheel, in which the water enters between the top and the middle, and which therefore occupies an intermediate position between the overshot and the middleshot water-wheel. In the overshot-wheel, the water acts principally by its weight, but in the undershot principally by its inertia or *vis viva*, while in the middleshot-wheel it acts commonly both by reason of its weight and inertia combined. Undershot-wheels may either hang freely in an unconfined current, or they may revolve in a confined race. To the first class belong ship mill-wheels. The second class may revolve either in a straight race, or may be partially enclosed by a circular channel or trough. The term *breast-wheel* generally implies such a circular channel or apron, and to such the name will be here restricted. When this breast or apron becomes a slight circular depression only at the bottom, the breast-wheel becomes the undershot. The overshot-wheel then takes its water at or near the top. If on the further side, it is overshot proper; if on the near side, it is back-pitch. The breast-wheel admits the water at or near the centre. If without an apron, we may call it, for convenience, "middleshot;" if with an apron fitting closely to the wheel, the object of which is to retain the water as long as possible in the buckets, it is called the breast-wheel proper. If the water runs under the wheel tangentially, or at most there is but a slight depression in which the wheel plays, it is undershot.

Finally, we have *Poncelet's wheel*, in which the water acts solely by its *vis viva* while running up and down a curved surface.

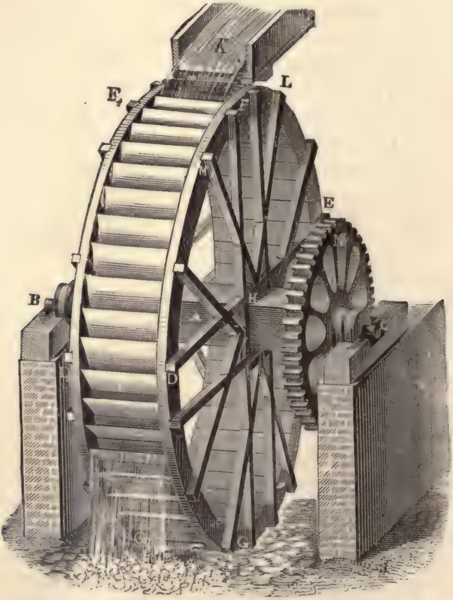
§ 171.—**Bucket-Wheels.**—An ordinary vertical water-wheel consists of a wooden or iron *axle* with two *journals*, two (more rarely one, three or more) circular *crowns*, and a number of radial *arms*, which connect the crowns with the axle. Between the crowns are the *floats*, and, finally, we may or may not have the *sole*, which forms a cylindrical surface upon the interior circumference of the crowns. The floats divide the space enclosed by the crowns and the sole into a number of partitions; and when the floats are more tangential than radial, they enclose spaces which can contain and retain water, called *buckets*. We have, therefore, two kinds of water-wheels—viz., *float* or *paddle wheels* and *bucket-wheels*. The latter occur in all cases where the water acts by its weight, and therefore in overshot, back-pitch, and, if desired, breast wheels. The former may also occur in breast-wheels, as the apron enables them to retain water, but otherwise only in such wheels as act by impulse of the water only and not by its weight.

We have first to consider *overshot water-wheels*. The water is brought to the wheel by a head race or canal, and its discharge is regulated by a sluice or "*pen stock*" at the end. It then enters the first, second, or third bucket from the top. As now the wheel moves, each bucket fills as it passes by the inlet orifice, and empties itself as it approaches the bottom, so that we always have upon one side of the wheel a certain number of buckets filled with water, which by their weight cause the constant revolution of the wheel. Overshot water-wheels occur in falls of from 8 to 50 ft., and of from 3 to 25 cubic ft. of entrance water per second. The smallest fall and least quantity of water give a delivery of from 3 to 5 horse-power; for large falls and feed, we may have as high as 130 horse-power. In this latter case, however, it is more advantageous to use two wheels, as water-wheels of more than 80 horse-power become inconveniently heavy.

The fall of a water-wheel is reckoned from the surface of the water in the head race, or just in front of the sluice, to the surface of the water in the tail race, the depth of which depends upon the quantity of water, the breadth, and the gradient. In order to lose as little of the fall as possible, the bottom of the wheel should no more than just clear the lower water surface, in which case the fall may also be taken as the distance from the upper surface to the bottom of the wheel. Only when a backing up of the escaping water and consequent "*drowning*" of the wheel is to be feared, is it desirable to hang the wheel somewhat higher, so that at its lowest point it may have the desirable amount of clearance, in which case it "hangs free."

§ **172.—Construction of Wheel.**—We may construct the water-wheel of wood or iron, or partly of wood and partly of iron. The methods of fastening the arms or spokes to the axle are various. In wheels all of wood, we have often two pairs of arms at right angles embracing the square axle (see Fig. 354); less often the arms are let into the axle itself. This last method is employed only for light wheels. For very high or heavy wheels, the first method is not sufficient, and auxiliary arms are introduced between the principal pairs. This last construction is shown in Fig. 354. In this figure, *A* is the axle, *B* and *C* the journals, *D E* and *F G* the principal arms, *H M*, *H L*, etc.,

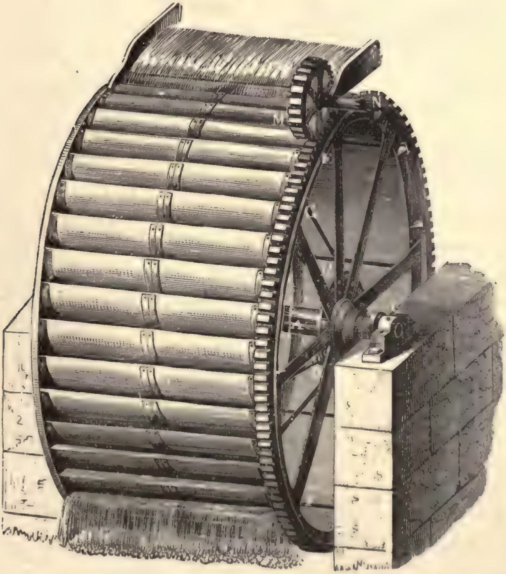
FIG. 354.



the auxiliary arms. Also *E L M D G* and $E_1$ $D_1$ $G_1$ are the crowns, and *K* is the head race. The crowns are formed of wooden segments. The arms are fastened to them by screws. Long screw bolts fasten the crowns to each other, passing through both crowns and two opposite arms at once. In order to insert the floats, grooves are made on the inner surface of the crowns. The toothed wheel *N* serves to transmit the motion of the wheel. The *sole* forms a cylindrical ring composed of planks screwed on to the inner edges of the crowns.

The following Fig. 355 shows an iron wheel of newer construction. Here the arms *B E*, *D F*, are fastened by screws to a disk or rosette *B D* which is attached to the axle *A C*. These wheels are ordinarily made very wide, and often have, therefore, besides the two outer crowns, a

FIG. 355.



third in the centre. This third crown plate is braced by diagonal arms, as $B\,G$, etc. Upon the outer edge of one of the crowns a toothed wheel $E\,L\,F$ engages with a smaller one $M$ by which the axle $M\,N$ is set in motion. The floats consist here of iron plate fastened by means of screws to flanges cast on the inner sides of the crowns, or bolted to them.

§ **173.—Radius of Wheel.**—The object of our whole discussion of water-wheels is to determine those mechanical principles and results in accordance with which the wheel must be constructed and set up, in order to obtain the maximum efficiency; or from the known dimensions and data to determine what the efficiency is.

The first thing which claims our attention is the *velocity* $v$, *at the circumference*, or the number of revolutions $u$ per minute. From one or the other of these, the *radius* of the wheel may next be determined. We shall see hereafter that overshot water-wheels should have a small velocity. For high wheels, it may rise as high as 10 ft., while wheels of medium height have about 5 ft., and even for the lowest the velocity should not sink below 2½ ft. The velocity $c$ of the entering water depends upon the velocity $v$ of the wheel, and is greater than this last in a certain proportion. For the generation of this velocity $c$, a certain fall is necessary, as in the Fig. 356. Thus the velocity of entrance $c$ is that due to the height

$$A\,B = h_1 = \frac{c^2}{2g},$$

which thus leaves remaining out of the total fall $A\,F = h$ only the fall occupied by the wheel proper—viz.,

$$B\,F = h_2 = h - h_1 = h - \frac{c^2}{2g}.$$

Since even for the most perfect discharge, 5 per cent of living force is still lost (see Vol. I., § 405), we take as the ordinary loss in practice 10 per cent, and hence the fall actually required to produce the velocity $c$ is

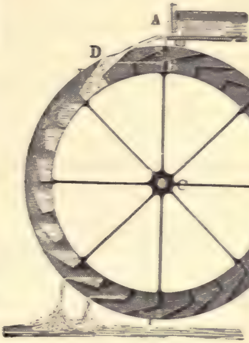
$$h_1 = 1 \cdot 1 \frac{c^2}{2g},$$

and therefore

$$h_2 = h - 1 \cdot 1 \frac{c^2}{2g}.$$

From the wheel fall $h_2$ we can now find the radius $CF = CD = r$. Denote the angle $SCD$ between the crown and where the water enters by $\theta$, and consider this angle known. Then

FIG. 356.



$$h_2 = CF + CB = r + r \cos.\theta = (1 + \cos.\theta)\, r,$$

and hence

$$r = \frac{h - h_1}{1 + \cos.\theta}.$$

From the radius $r$ and velocity $v$ of the wheel, we can now find the number of revolutions per minute,

$$u = \frac{60\, v}{2\pi r} = \frac{30\, v}{\pi r}.$$

In practice, we have generally $u$ known, and wish to determine $r$ and $v$. We have, therefore,

$$v = \frac{\pi u r}{30} \qquad \text{and} \qquad c = \kappa \frac{\pi u r}{30}$$

where $\kappa$ is a certain ratio, the so-called coefficient of velocity or ratio of $\frac{c}{v}$. Then

$$(1 + \cos.\theta)\, r = h - \frac{1 \cdot 1}{2g} \left(\frac{\kappa \pi u r}{30}\right)^2,$$

and from this we obtain, by putting

$$g = 32 \cdot 2$$

and

$$\pi = 3 \cdot 1416,$$

for the value of $r$,

$$r = \frac{h - 0 \cdot 000187\,(\kappa\, u\, r)^2}{1 + \cos.\,\theta}.$$

Solving this quadratic equation, we have for the value of $r$ in terms of $h$, $\kappa$, $u$, and $\theta$, all known quantities,

$$r = \frac{\sqrt{0 \cdot 000748\,(\kappa\, u)^2\, h + (1 + \cos.\,\theta)^2} - (1 + \cos.\,\theta)}{0 \cdot 000374\,(\kappa\, u)^2} \quad . \quad . \quad (1),$$

or approximately,

$$r = \frac{h\,[1 - 0 \cdot 000047\,(\kappa\, u)^2\, h]}{1 + \cos.\,\theta} \text{ ft.}$$

Accordingly, we have the velocity at circumference

$$v = \frac{\pi\, u\, r}{30} = 0 \cdot 1047\, u\, r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

EXAMPLE.—1. For a fall of 30 ft., we wish to construct a wheel which shall have 8 ft. velocity, when the water enters with a velocity of 16 ft. at 12° from the top. What is the necessary radius, and what is the number of revolutions per minute?

We have here $c = 16$ ft. and $h_1 = 1 \cdot 1 \times 0 \cdot 0155 \times 16^2 = 4 \cdot 36$ ft.

Hence

$$r = \frac{h - h_1}{1 + \cos.\,\theta} = \frac{30 - 4 \cdot 36}{1 + \cos.\,12^\circ} = \frac{25 \cdot 64}{1 \cdot 978} = 12 \cdot 96 \text{ ft.},$$

and

$$u = \frac{30 \times 8}{\pi \times 12 \cdot 9} = 5 \cdot 92.$$

2. If the number of revolutions is given, $u = 5$, then for the same fall and as above $\kappa = 2$, we have for the radius

$$r = \frac{\sqrt{2 \cdot 244 + 3 \cdot 9125} - 1 \cdot 978}{0 \cdot 0374} = \frac{0 \cdot 503}{0 \cdot 0374} = 13 \cdot 45 \text{ ft.}$$

Further, the velocity $v = 0 \cdot 1047 \times 5 \times 13 \cdot 45 = 7 \cdot 04$ ft., the entrance velocity $c = \kappa\, v = 14 \cdot 08$ ft., and the fall for the generation of this last velocity $h_1 = 1 \cdot 1 \times 0 \cdot 0155 \times 14 \cdot 08^2 = 3 \cdot 36$ ft.

**§ 174.—Depth of Crown and Width of Wheel.**—The breadth of the crown and width of the wheel are also of importance. In overshot-wheels, the depth of crown is usually made from 10 to 12 inches, more rarely 14 to 15 inches, principally for the reason that the water acts with a greater lever arm in a wheel of small crown depth than in a wheel of the same diameter and greater crown depth. As to the width of the wheel, this depends upon the water space to be given to it. If $d$ is the depth of crown and $e$ the width of wheel, the cross-section of the cylindrical space formed by the sole and crowns is $d\, e$, and if $v_1$ is the velocity of the wheel at the centre of the crown depth, we have $d\, e\, v_1$ for the space presented to the entrance water in one second. This space, however, should not be equal to the quantity $Q$ of the entrance water per second, because the contents of a bucket are not so great as the entire space between two successive floats, and, moreover, it is not advantageous to fill the buckets entirely on account of the too early exit of the water. We have, therefore, $\varepsilon\, d\, e\, v_1 = Q$ and $\varepsilon < 1$. As a rule, this coefficient, called the *coefficient of fill*, is taken at $\varepsilon = \frac{1}{4}$ to $\frac{1}{3}$. In any case, we have for the width of the wheel

$$e = \frac{Q}{\varepsilon\, d\, v_1},$$

or if we put approximately

$$v_1 = v = \frac{\pi\, r\, u}{30}, \quad e = \frac{30\, Q}{\varepsilon\, \pi\, u\, r\, d},$$

or

$$e = 9 \cdot 55 \frac{Q}{\varepsilon\, u\, r\, d}.$$

Putting for $\varepsilon$ the value $\frac{1}{4}$, we have

$$e = 38 \cdot 2 \frac{Q}{u\, r\, d}.$$

For very high wheels, we may even take $\varepsilon = \frac{1}{5}$.

§ **175.—Number of Floats.**—The number of floats, $n$, is another important element of the wheel. It is easy to see that a smaller quantity of water remains longer in the space between the floats than a larger, and since, under similar circumstances, this quantity is smaller the greater the number of floats, it follows that, in general, the more numerous the buckets the greater the efficiency. This number must, however, have a limit, not only because the floats, in consequence of their thickness, occupy a certain amount of space, whereby wheels with thin iron plate floats may have a greater number of buckets than those with thick wooden floats, but also because the floats may be so numerous as to impede the ready entrance of the water. The form of the floats has also an essential influence, as has also the method of letting on the water, since a sufficient cross-section should be presented to the entering water.

If the distance between two floats is given, then the number $n$ of floats is proportional to the perimeter or radius $r$, and increases with $r$. In general, for ordinary depths of crown from 10 to 12 inches, $n = 5\,r$ to $6\,r$, $r$ being taken in feet.

Wheels of greater crown depth have a smaller proportional number of buckets than those of less crown depth.

From the number $n$ of floats, we have for the angle between any two, or the *pitch angle*, $\phi = \frac{360^\circ}{n}$.

EXAMPLE.—If an overshot-wheel of 15 ft. radius, 1 ft. crown depth, and 10 cubic ft. feed per second, makes 5 revolutions per minute, we have for the width of wheel $e = 38 \cdot 2 \frac{10}{5 \times 15 \times 1} = 5 \cdot 1$ ft., and the number of buckets is $n = 5\,r = 5 \times 15 = 75$, or, in order to facilitate the division, 72; finally, the pitch angle is $\phi = \frac{360^\circ}{72} = 5^\circ$.

§ **176.—Form of Buckets.**—The form given to the buckets is of great influence upon the action of the wheel. The buckets must have such shape and be so disposed as not only to admit the water without hindrance, but also to retain it to the lowest point possible upon the wheel. Many of the different methods in use answer these ends but very imperfectly. For the same number and same quantity of feed water, the exit and entrance of the water depend upon the position of the outer bucket end $A\ B$, Fig. 357. This makes with the circumference a certain

FIG. 357.

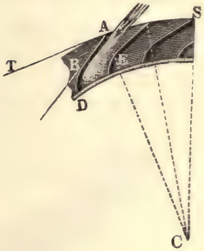
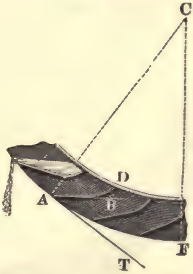


FIG. 358.



angle $B\ A\ T = \beta$, which we shall call the angle of entrance of the water. This angle is the complement of the angle $B\ A\ C$ which the end of the float makes with the radius $C\ A$. The outer end $A\ B$ forms the enclosing side of a bucket whose capacity depends upon its form and position. Thus, when, in revolving this element, $A\ B$ takes a horizontal position, as shown in Fig. 358, the capacity of the bucket is zero and no water can remain in it. At this moment, the angle $A\ C\ F = B\ A\ T = \beta$. In order, therefore, that the water may be retained as long as possible in the descending bucket, this angle must be as small as possible. Since, however, for the admission of water to the wheel, a certain cross-section $A\ E$ is necessary, which depends upon the amount of the entrance angle and becomes zero simultaneously with it, it is necessary for the best result *that the entrance angle should be indeed small, but yet should not be less than a certain fixed limit.*

The capacity of the bucket also depends upon the form and extent of the float, and it is easy to see that it will be greater, the greater the surface of the float and the further it extends out from the interior circumference. Although now it is required, in order to retain the water as long as possible, to increase as much as may be the capacity of the bucket; still we must not overpass the limit within which the buckets encroach upon one another, or assume dimensions which prevent the entrance and proper exit of the water. For these reasons, the simple plain floats *A B*, Fig. 359, are either inapplicable or are at least disadvantageous, and we must replace them by compound or curved floats which must indeed meet the outer circumference under the given angle $\beta$, but are nearly or quite at right angles with the inner circumference.

FIG. 359.

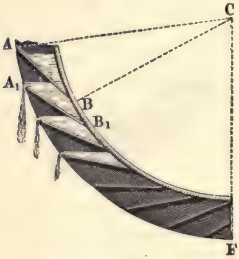
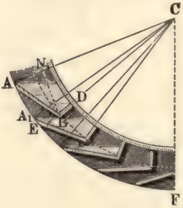
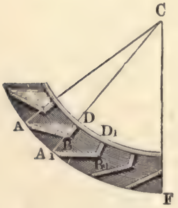


FIG. 360.



In general, then, we make wooden floats of two portions *A B* and *B D*, Fig. 360, which meet at an obtuse angle. The first portion meets the outer circumference at an angle $\beta$, and the second is radial, or sometimes even at right angles with the first. The circle which passes through the junction of these two portions may be in the centre between the outer and inner circumferences or at a distance of $\frac{3}{8}$ the crown depth, for a large entrance angle.

FIG. 361.



A very usual and simple method of float construction is to make the portion $A B$, Fig. 361, enclose the angle $A C B = \phi$, to make $B D$ radial, and the intersection $B$ always on the circle of division.

In order to obtain a smaller entrance angle, we may make the angle $A C B = \psi$, Fig. 360, greater than the angle between the ends of two floats $A C A_1 = \phi$, say $\psi = \frac{5}{4} \phi$.

If $r$ is the outer radius $C A$, and $r_1 = C B$ that of the dividing circle, we have for the entrance angle $E A B = A B N = \beta$ corresponding to the angle $\psi$,

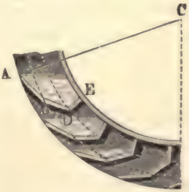
$$tan. \beta = \frac{A N}{B N} = \frac{r - r_1 \cos. \psi}{r_1 \sin. \psi},$$

in which we have simply to put $\phi$ in place of $\psi$ for the previously described more usual method of construction. If, now, the depth of crown $D E = d$, we have, when the dividing circle is in the centre or at $\frac{2}{3}$ from outer circumference, $r_1 = r - \frac{1}{2} d$ or $r_1 = r - \frac{2}{3} d$, to be substituted in the above formula.

Floats of cast-iron or iron plate consist of but one piece, and are usually curved (Fig. 358). Since the contraction of the bucket space by angles is thus avoided, such floats afford a better entrance to the water than the wooden ones consisting only of two straight portions.

For this reason, we may construct the wooden float of three straight portions instead of two (Fig. 362). We may also increase the bucket space without prejudice to the introduction of the water, by making the second portion $B D$, Fig. 363, of the float meet the inner circumference at an angle more or less sharp, instead of having it radial. For instance, we may make it form an angle of 45° instead of 90°. In order to obtain such an angle with iron floats, we can make them entirely or in part of circular arcs which make at the ends the

FIG. 362.



desired angles. The centre $M$ of such a circular arc $A\,D$, Fig. 364, is to be found in the line $A\,M$, which forms with

FIG. 363.

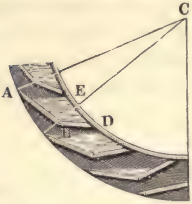
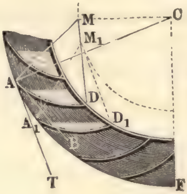


FIG. 364.



the radius $C\,A$ the angle of entrance $C\,A\,M = B\,A\,T = \beta$. The centres for the other buckets $M_1\,M_2$, etc., lie in the circle described with $C\,M$ as a radius.

EXAMPLE.—In a wheel whose height is $2\,r = 30$ ft., and crown depth $d = 10$ inches. and which has accordingly, from Art. 30, for the number of buckets $5\,r = 5 \times 15 = 75$, or, for convenience of division, 72; we find for the angle of division

$$\phi = \frac{360^\circ}{n} = \frac{360}{75} = \frac{24}{5} = 4\tfrac{4}{5} \text{ degrees.}$$

If we make the angle between the float ends $\psi = \frac{5}{4}\,\phi$, and take the dividing circle through the centre, we have

$$\psi = \frac{24}{5} \times \frac{5}{4} = 6^\circ\,;$$

further, for the dividing circle itself,

$$r_1 = r - \frac{d}{2} = 15 - \frac{5}{12} = 14 \cdot 5833 \text{ ft.},$$

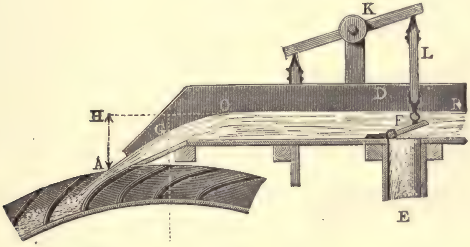
and for the entrance angle $\beta$,

$$\tan.\,\beta = \frac{r - r_1 \cos.\,\psi}{r_1 \sin.\,\psi} = \frac{15 - 14\cdot 5833 \cos.\,6^\circ}{14\cdot 5833 \sin.\,6^\circ} = \frac{0\cdot 496}{1\cdot 527} = 0\cdot 325,$$

and hence $\beta = 18^\circ\,1'$, and the angle which the float end makes with the radius is $90 - \beta = 71^\circ\,59'$ or $72^\circ$.

§ **177.—Sluices.**—The method in which the water is brought into the wheel is of considerable importance. We may either allow the water to fall freely from the race into the wheel, or we may shut it off and regulate the flow by a sluice. In the first case, the velocity of entrance depends almost entirely upon the height of fall; in the second, it may be regulated by the height of water. For this reason, a sluice is to be preferred to free entrance through an unobstructed head race. In Fig. 365, we have the water brought on without a gate or sluice. The water is conducted through a race *D O* to a spout *G* which gives it the proper direction. In order merely to regulate the flow, we have a waste pipe *E*, by which surplus water may be discharged, closed by a flap *F*, worked by a lever *K*. If the velocity in the race is $c_0$, and the fall before reaching the wheel is $A\,H = h_1$ from the surface of the water in the race to the point of entrance *A*, we have for the velocity of the entrance water nearly

Fig. 365.



$$c = \sqrt{2\,g\,h_1 + c_0^2} = \sqrt{2\,g\,h_1 + \left(\frac{Q}{F}\right)^2}$$

where *Q* is the quantity of water and *F* is the cross-section of the current.

Sluices (Fr. vannes; Ger. Spannschützen), or, as they are also called, hatches, pen stocks, or shuttles, are either horizontal, vertical, or inclined. The arrangement of a horizontal sluice is shown in Fig. 366, Fig. 367 shows a vertical sluice, and Fig. 368 an inclined sluice.

FIG. 366.

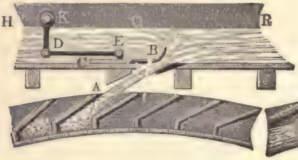
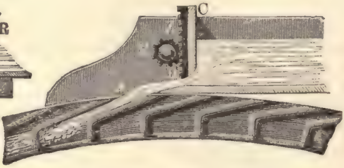
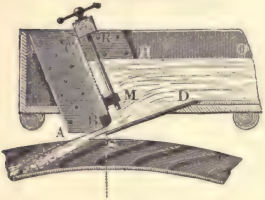


FIG. 367.



In all such constructions, the interior of the opening ought to be well rounded and as smooth as possible, or formed to the contracted stream, in order to prevent contraction, and offer as little resistance to the flow as possible.

FIG. 368.



If the water falls freely after passing out, and if the opening can be put at right angles to the stream, it is advantageous to have it an orifice in a thin plate, but care must be exercised lest partial contraction of the stream ensue. (See Weisbach, Vol. I., Art. 414.)

For discharge with sluice, we have due to the height of water back of sluice $h_0$, the velocity

$$c_0 = \mu \sqrt{2 g h_0}.$$

If, now, $z$ is the free fall from orifice to point of entrance in the wheel, we have for the velocity of entrance

$$c = \sqrt{c_0^2 + 2 g z} = \sqrt{2 g (\mu^2 h_0 + z)}.$$

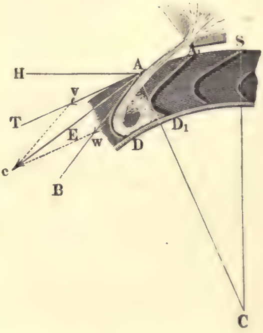
If we take the coefficient of discharge $\mu = 0{\cdot}95$, we have

$$c = \sqrt{2 g (0{\cdot}9 h_0 + z)}.$$

We see from this that for the same height of fall $z$, the velocity of entrance $c$ is nearly the same, whether the water runs on freely or is regulated by a sluice.

**§ 178.—Entrance of the Water.**—In order that the water may enter without hindrance into the buckets, it should not strike the buckets at the outer circumference, but the shock should take place later and nearer the bottom. For this reason, not only the outer edge of the bucket should be sharpened, but also the stream of water $A\,c$, Fig. 369, should be so directed that its velocity can be decomposed into two components, of which one coincides with the velocity $v$ at the circumference, and the other has the direction $A\,B$ of the float, or the outer float end. Since we can regard this latter direction as known, as also the velocity $v$ at right angles to $C\,A$, and since the entrance velocity $c$ is given, we can find the direction of this last by drawing through $v$ a parallel to $A\,B$, then with $A$ as centre and $A\,c$ as radius describe an arc, and then draw $A\,c$ to point of intersection of arc with $v\,c$. If, finally, we draw through $c$ a parallel to $A\,v$, we find $A\,w$ the relative velocity of the water with reference to the wheel. By calculation, we have the following: If $\alpha$ is the angle of approach $c\,A\,T$ under which the approaching stream strikes the circumference, and $\beta$ is the given entrance angle under which the float meets the circumference, we have

FIG. 369.



$$\frac{sin.\ \alpha}{sin.\ \beta} = \frac{w}{c} \quad \text{and} \quad \frac{sin.\ (\beta - \alpha)}{sin.\ \beta} = \frac{v}{c}.$$

From the last equation, we have

$$sin.\ (\beta - \alpha) = \frac{v}{c}\ sin.\ \beta = \frac{sin.\ \beta}{\kappa}$$

where $\beta$ is the entrance angle and $\kappa$ is the velocity ratio $\frac{c}{v}$, and from these two the angle $c\ A\ B$ of the stream with the float may be found, as also the entrance angle

$$\alpha = \beta - (\beta - \alpha).$$

We have also from the first equation

$$w = \frac{c\ sin.\ \alpha}{sin.\ \beta}$$

for the relative velocity. We can also find this velocity by the formula

$$w = \sqrt{c^2 + v^2 - 2\ c\ v\ cos.\ \alpha} = v \sqrt{1 - 2\ \kappa\ cos.\ \alpha + \kappa^2}.$$

Since the angle of approach is small, $cos.\ \alpha$ is nearly 1, and hence, approximately,

$$w = c - v = (\kappa - 1)\ v$$

and

$$sin.\ \alpha = \frac{c - v}{c}\ sin.\ \beta = \left(\frac{\kappa - 1}{\kappa}\right)\ sin.\ \beta,$$

or more simply,

$$\alpha = \left(\frac{\kappa - 1}{\kappa}\right) \beta.$$

Therefore, by means of the given velocity ratio

$$\kappa = \frac{c}{v},$$

we can find from the entrance angle $\beta$ what the angle of approach is. We see also that $\kappa > 1$ and $c > v$.

Since the water entering a bucket and striking against the lower portion takes an opposite motion, it would escape in part at least, if the relative velocity of the water were very great, and if the water did not strike against the following float and thus take a different direction. In this respect, therefore, the method of float construction consisting of two straight portions, one of which is radial, as shown above, is preferable to that method of construction where the second portion is not radial, but inclined to the inner circumference at an acute angle, as illustrated in the preceding article, and the form of bucket shown in the figure to this article is best for a great relative velocity of the water.

Since the relative velocity of entrance,

$$w = c - v = (\kappa - 1)\, v,$$

increases not only with $v$ but also with $\kappa$, this ratio $\kappa$ should never be great, at most only from $\frac{3}{2}$ to 2.

If we know now the angle $SCA = \theta$ by which the point of entrance $A$ is distant from the crown, we can easily find the angle $EAH = \nu$ of the entering stream with the horizon. Thus

$$\nu = TAH + cAT = \theta + \alpha.$$

Example.—If a water-wheel 36 ft. high makes four revolutions per minute, we have for the velocity at the circumference

$$v = \frac{2\,\pi\, r\, u}{60} = \frac{\pi\, r\, u}{30} = 0 \cdot 1047 \times 18 \times 4 = 7 \cdot 54 \text{ ft. per second,}$$

and hence if the ratio

$$\kappa = \frac{c}{v} = \alpha,$$

the requisite absolute velocity of the water is $c = \kappa\, v = 2\, v = 15 \cdot 08$ ft. If we make the entrance angle $\beta = 20°$, then for the angle of approach $\alpha$ we have

$$sin.\, (\beta - \alpha) = \frac{sin.\, \beta}{\kappa} = \tfrac{1}{2}\, sin.\, \beta = \tfrac{1}{2} \times 0 \cdot 3420 = 0 \cdot 1710,$$

hence $\beta - \alpha = 9°\, 51'$, and therefore $\alpha = 20° - 9°\, 51' = 10°\, 9'$, and the relative velocity of the entering water is

$$w = \frac{sin.\, \alpha}{sin.\, \beta}\, c = \frac{0 \cdot 1762}{0 \cdot 3420} \times 15 \cdot 08 = 7 \cdot 78 \text{ ft.}$$

According to the approximate formulæ $w = c - v = (\kappa - 1)\, v = v = 7{\cdot}54$ ft., and

$$\alpha = \left(\frac{\kappa - 1}{\kappa}\right) \beta = \tfrac{1}{2}\, \beta = 10^\circ.$$

If the angle between the top and point of entrance is $\theta = 12^\circ$, then for the necessary inclination of the stream of water to the horizon we have

$$r = \theta + \alpha = 12^\circ + 10^\circ\, 9' = 22^\circ\, 9'.$$

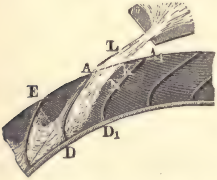


FIG. 370.

**§ 179.—Number of Floats—Exit of Air.**—If $b$ is the length $A\,K$, Fig. 370, of the arc upon the outer circumference occupied by the entering stream, then the thickness of this stream just before entrance is

$$K\,L = A\,K \textit{ sin. } K\,A\,L,$$

or $b$ *sin.* $\alpha$, $\alpha$ being the angle of approach or the angle made by the stream with the outer circumference. On the other hand, after the entrance of the stream, we have $A\,N = A\,K$ *sin.* $A\,K\,N = b$ *sin.* $\beta$, $\beta$ being the angle of entrance, or the angle made by the float with the outer circumference. If, now, $e$ is the width of the wheel, and hence the width of the entering stream, we have for the cross-section of this stream $e \times A\,N = e\,b$ *sin.* $\beta$ and $e \times K\,L = e\,b$ *sin.* $\alpha$, and therefore for the quantity of water $Q = e\,b$ *sin.* $\alpha \times c = e\,b$ *sin.* $\beta \times w$, $c$ being the absolute and $w$ the relative velocity.

But we have also $Q = \varepsilon\, d\, e\, v$, where $\varepsilon$ is the coefficient of fill, $d$ the depth of crown, and $v$ the velocity of circumference, hence

$$\textit{sin. } \alpha = \frac{Q}{e\,c\,b} = \frac{\varepsilon\, d\, e\, v}{e\,c\,b} = \frac{\varepsilon\, d\, v}{c\,b}$$

and

$$\textit{sin. } \beta = \frac{Q}{e\,b\,w} = \frac{\varepsilon\, d\, v}{e\,b\,w} = \frac{\varepsilon\, d\, v}{b\,w}.$$

Inversely, the length of the arc occupied by the stream is

$$b = \frac{v}{c}\, \frac{\varepsilon\, d}{\textit{sin. } \alpha} = \frac{v}{w}\, \frac{\varepsilon\, d}{\textit{sin. } \beta}.$$

Approximately, $w = c - v = (\kappa - 1)\, v$ and

$$b = \frac{\varepsilon\, d}{(\kappa - 1)\, sin.\, \beta},$$

where $\kappa$ is the ratio $\frac{c}{v}$.

Since overshot water-wheels cannot be *ventilated—i. e.*, cannot have openings in the sole to allow of the escape of the air in the buckets as the water enters—the mouth of the bucket must not be completely filled by the entering stream, but room must be left for the escape of the compressed air. If, now, the breadth of the entering stream is but little less than that of the wheel $e$, the air must escape along the whole breadth, and hence it is necessary that the above value for $b$ should be less than the length of arc from end to end of float $A\, A_1$.

If $n$ is the number of floats and $r$ the outer radius, then the distance from $A$ to $A_1$ is

$$b_1 = \frac{2\, \pi\, r}{n},$$

and if we put this equal to $b$, as found above, we have for the *greatest possible* number of floats when $b = b_1$

$$n = \frac{2\, \pi\, r}{b} = (\kappa - 1)\, \frac{2\, \pi\, r\, sin.\, \beta}{\varepsilon\, d}.$$

For perfect action, we take in practice a much smaller number, even as low as only the half of this number, or

$$n = (\kappa - 1)\, \frac{\pi\, r\, sin.\, \beta}{\varepsilon\, d}.$$

We see from this formula that the number of floats of a wheel is greater the greater the radius $r$ and the angle of entrance $\beta$ and the ratio $\kappa = \frac{c}{v}$, and the smaller the coefficient of fill $\varepsilon$ and the smaller the breadth $d$.

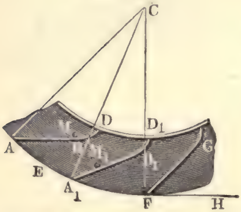
EXAMPLE.—For an overshot water-wheel of 24 ft. height and 1 ft. depth of crown, taking $\kappa = \frac{c}{v} = 2$ and $\varepsilon = \frac{1}{4}$ and the entrance angle $\beta = 20^\circ$, we have for the greatest possible number of floats

$$n = (\kappa - 1)\frac{2\,\pi\,r\,sin.\,\beta}{\varepsilon\,d} = \frac{2\,\pi \times 12\,sin.\,20^\circ}{\frac{1}{4}}$$

$$= 96\,\pi\,sin.\,20^\circ = 96 \times 3\cdot 14 \times 0\cdot 342 = 103\cdot 1.$$

If we take ¾ of this for practice, we have about 72 floats, allowing for exit of air.

FIG. 371.



**§ 180.—Entrance Angle.**—We have assumed in the preceding that the water completely leaves a bucket when the portion $A\,B$, Fig. 371, or at least the outer end is horizontal. This is, however, only approximately true, for the last particle of water as $M$, having no pressure behind it, will roll along $A\,B$ while the float goes round and takes up a greater and greater inclination. The time necessary can be determined as follows: When the originally horizontal float $A\,B$ has turned through the angle $A\,C\,A_1 = \psi$, and when its inclination to the horizon is therefore $= \psi$, the acceleration of the particle at $M_1$ is $p = g\,sin.\,\psi$. But, now, for the velocity corresponding to this acceleration we have (Vol. I., Art. 20) $d\,w = p\,d\,t$ or $d\,w = g\,sin.\,\psi\,d\,t$. If, now, the wheel turns with the velocity $v$, we have $r\,\psi = v\,t$, or $r\,d\,\psi = v\,d\,t$, hence

$$d\,w = g\,sin.\,\psi\,\frac{r\,d\,\psi}{v} = \frac{g\,r}{v}\,sin.\,\psi\,d\,\psi.$$

Therefore the relative velocity of the particle is

$$w = \frac{g\,r}{v}\int sin.\,\psi\,d\,\psi = \frac{g\,r}{v}(1 - cos.\,\psi).$$

We have also for the space over which the particle passes in the time $t$,

$$ds = w\,dt = \frac{w\,r\,d\psi}{v} = \frac{g r^2}{v^2}(1 - cos.\,\psi)\,d\psi,$$

and hence for the distance,

$$s = \frac{g r^2}{v^2}\int(1 - cos.\,\psi)\,d\psi = \frac{g r^2}{v^2}(\psi - sin.\,\psi).$$

If the wheel revolves quickly, the force of gravity will be reinforced by the centrifugal force, and we have therefore instead of $g$,

$$g + \frac{v^2}{r},$$

and hence

$$s = \frac{r^2}{v^2}\left(g + \frac{v^2}{r}\right)(\psi - sin.\,\psi),$$

or

$$\psi - sin.\,\psi = \frac{v^2 s}{(g r + v^2) r}.$$

Since

$$\frac{\psi - sin.\,\psi}{2}$$

are the contents of a circular segment of radius 1, we can regard $\psi$ as the central angle of a circular segment whose contents are

$$\frac{\frac{1}{2} v^2 s}{(g r + v^2) r}.$$

In order, then, that all the water may have left the bucket when the outer end $A$ passes to $F$, the above formula must hold good when we put for $s$ the length $A B = F G$ and for $\psi$ the entrance and exit angle—*i. e.*, the angle $B A E = G F H = \beta$. Thus from the formula

$$\beta - sin.\,\beta = \frac{v^2 s}{(g r + v^2) r},$$

or approximately,

$$\sin.\beta = \sqrt[3]{\frac{6v^2s}{(gr+v^2)r}},$$

we may determine the allowable entrance angle $\beta$, which we have thus far assumed as given. We see also that the entrance angle $\beta$ is smaller the greater the radius $r$ and the less the velocity $v$ and the distance $AB = s$.

EXAMPLE.—1. If $s = 1$ ft. and $v = 5$ ft. and $r = 10$ ft., we have

$$\beta - \sin.\beta = \frac{25}{(32\cdot2 \times 10 + 25)\,10} = \frac{2\cdot5}{347} = \frac{5}{694} = \frac{1}{139} = 0\cdot0071942,$$

hence

$$\frac{\beta - \sin.\beta}{2} = 0\cdot0035971$$

Hence we have $\beta = 20\frac{1}{8}$ degrees. The approximate formula gives

$$\sin.\beta = \sqrt[3]{6 \times 0\cdot0071942} = \sqrt[3]{0\cdot04316} = 0\cdot354 \text{ or } \beta = 20\tfrac{7}{8} \text{ degrees.}$$

2. For a high wheel of 20 ft. radius and 10 ft. velocity, if as before $s = 1$, we have

$$\frac{\beta - \sin.\beta}{2} = \frac{50}{(32\cdot2 \times 20 + 100)\,20} = \frac{2\cdot5}{744} = \frac{1}{297\cdot6} = 0\cdot0033602,$$

and accordingly $\beta$ nearly $= 20°$.

3. For a very swift revolving low wheel of 5 ft. radius and 8 ft. velocity,

$$\frac{\beta - \sin.\beta}{2} = \frac{32}{(32\cdot2 \times 5 + 64)\,5} = \frac{6\cdot4}{225} = 0\cdot0284,$$

or $\beta =$ nearly $40°$.

We see from these examples that the floats must make an angle of from 20° to 40° with the circumference; the first for high and slow, the second for low and swift wheels.

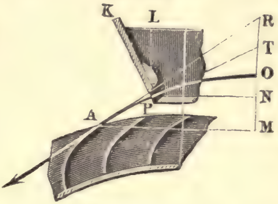
§ 181.—**Introduction of the Water.**—In order that the water may enter the wheel in the given direction, we must either place the sluice opening very near the point of entrance and have the sluice board at right angles to the direction of the entering stream; or we must bring the water on in the required direction by a race; or we must so place the sluice board that the water in falling freely describes a parabola and enters in the required direction.

In order to find the direction of the sluice board in the case when the water falls freely, upon the wheel, we may make use of the principles of projectiles given in Vol. I., Art. 39. For the velocity

$$A\,c = c,$$

Fig. 372, and the angle of inclination $R\,A\,M = v$ of the stream to the horizon at the point $A$, we have for the vertical co-ordinate of the crown of the parabola

FIG. 372.



$$M\,O = x = \frac{c^2\,sin.^2\,v}{2\,g},$$

and for the horizontal co-ordinate,

$$A\,M = y = \frac{c^2\,sin.\,2\,v}{2\,g}.$$

If we wish to place the sluice opening at any point $P$ of the parabola, and if the height $M\,N = z$ of this opening above $A$ is known, we have for the co-ordinates $O\,N = x_0$ and $N\,P = y_0$ of this point, $x_0 = x - z$ and

$$v_0 = y\sqrt{\frac{x-z}{x}} = y\sqrt{1 - \frac{z}{x}},$$

and for the angle of inclination $T\,P\,N = v_0$, which the parabola at this point makes with the horizon,

$$tan.\,v_0 = \frac{T\,N}{P\,N} = \frac{2\,O\,N}{P\,N} = \frac{2\,x_0}{y_0} = \frac{2\sqrt{x\,(x-z)}}{y}.$$

The plane $P\,K$ of the sluice board must now be perpendicular to $P\,T$. We can therefore find the required position of the sluice board by laying off $O\,T = N\,O$, drawing $PT$ and then a perpendicular $PK$ to $PT$. If the sluice opening is at the vertex of the parabola, the sluice board of course becomes vertical.

The velocity of efflux at $P$ is

$$c_0 = \sqrt{c^2 - 2gz},$$

and the corresponding theoretical height is

$$h_0 = \frac{c^2}{2g} - z,$$

or for the effective height,

$$h_0 = 1 \cdot 1 \left(\frac{c^2}{2g} - z\right)$$

for rounded and smooth orifice. The breadth of the sluice opening should be made but little less than that of the wheel.

EXAMPLE.—For the velocity $c = 15$ ft., and the angle of inclination $\nu = 20\frac{1}{4}°$, we have for the co-ordinates of the vertex of the parabola,

$$x = 0 \cdot 015 \times 15^2 (sin.\ 20\tfrac{1}{4}°)^2 = 0 \cdot 4312 \text{ ft.},$$

and

$$y = 0 \cdot 015 \times 15^2 (sin.\ 40\tfrac{1}{2}°) = 2 \cdot 338 \text{ ft.}$$

If, now, we wish the centre of the sluice opening at a distance of

$$z = 4 \text{ inches} = 0 \cdot 333 \text{ ft.}$$

above the point of entrance, we have for the co-ordinates of the opening,

$$x_0 = 0 \cdot 4312 - 0 \cdot 333 = 0 \cdot 0979$$

and

$$y_0 = 2 \cdot 338 \sqrt{\frac{979}{4312}} = 1 \cdot 114 \text{ ft.}$$

For the inclination of the stream to the horizon,

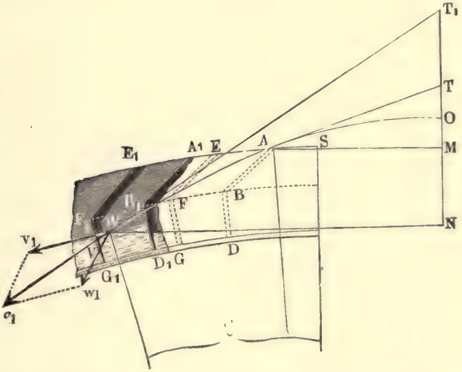
$$tan.\ \nu_0 = \frac{1958}{11140}$$

or $\nu_0 = 9° 58'$, and, therefore, for the inclination of the sluice itself,

$$90 - \nu_0 = 80° 2'.$$

§ **182.**—In the manner of bringing on the water shown in Art. 178, the parabolic course of the stream suffers no change until the water strikes the float or the water already in the bucket. We may determine the point $W$, Fig. 373, where the water thus strikes by the preceding principles. If $z_1$ is the perpendicular distance $MN$ of the point of entrance $A$ from the surface $W$ of the water at the moment the influx ceases, we have for the abscissa of $W$,

FIG. 373.



$$ON = OM + MN$$

or $x_1 = x + z_1$, and for the ordinate

$$NW = y_1 = y\sqrt{\frac{x_1}{x}} = y\sqrt{1 + \frac{z_1}{x}},$$

and finally for the angle of inclination $T_1 WN$,

$$tan.\, v_1 = \frac{2\, x_1}{y_1} = \frac{2\sqrt{x\,(x + z_1)}}{y}.$$

For the angle $WCS = \theta_1$ between $W$ and the top,

$$sin.\,\theta_1 = \frac{WN - AM + AS}{CW} = \frac{y_1 - y + r\,sin.\,\theta}{r_1}$$

where $r_1$ is the radius of $W$; and hence for the angle between the velocity $c_1$ of the point $W$ and the velocity of revolution $v_1$ at this point, we have $\alpha_1 = \nu_1 - \theta_1$. The velocity $c_1$ finally with which the water strikes at $W$ is given by the formula

$$\frac{c_1^2}{2g} = \frac{c^2}{2g} + z_1,$$

or

$$c_1 = \sqrt{c^2 + 2gz_1},$$

or, according to Art. 177,

$$c_1 = \sqrt{2g(0 \cdot 9h_0 + z + z_1)}.$$

EXAMPLE.—If, in the last example, we make $z_1 = 9$ in. $= 0 \cdot 75$ ft., we have for $W$, $ON = x_1 = x + z_1 = 0 \cdot 4312 + 0 \cdot 75 = 1 \cdot 1812$, and

$$NW = y_1 = y\sqrt{1 + \frac{z_1}{x}} = 2 \cdot 338\sqrt{1 + \frac{0 \cdot 75}{0 \cdot 4321}} = 2 \cdot 338\sqrt{2 \cdot 7356} = 3 \cdot 867 \text{ ft.}$$

Further, for the angle of inclination of the stream at $W$,

$$tan.\,\nu_1 = \frac{2x_1}{y_1} = \frac{2 \cdot 3624}{3 \cdot 867},$$

or $\nu_1 = 31° 25'$.

On the other hand, for the centre angle at $W$, when $r = 18$ ft. and

$$ACS = \theta = 12°,$$

we have

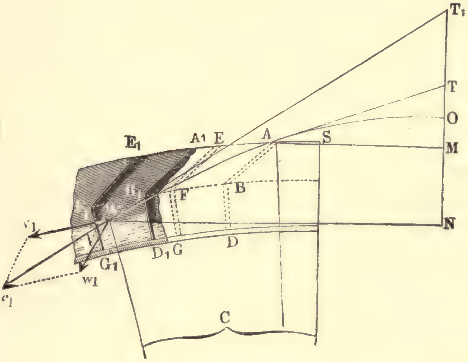
$$sin.\,\theta_1 = \frac{y_1 - y + r\,sin.\,\theta}{r_1} = \frac{3 \cdot 867 - 2 \cdot 338 + 18\,sin.\,12°}{17 \cdot 25} = \frac{1 \cdot 529 + 3 \cdot 742}{17 \cdot 25} = \frac{5 \cdot 271}{17 \cdot 25}$$

or $\theta_1 = 17° 48'$; and for the angle of inclination of the stream at $W$ to the tangent to the wheel, $\alpha_1 = \nu_1 - \theta_1 = 13° 37'$. Finally, the velocity of the water at $W$ is

$$c_1 = \sqrt{c^2 + 2gz_1} = \sqrt{15^2 + 64 \cdot 4 \times 0 \cdot 75} = \sqrt{225 + 48 \cdot 304} = \sqrt{273 \cdot 304} = 16 \cdot 54 \text{ ft.}$$

§ **183.—Motion of the Water within the Wheel.**—The manner in which the water proceeds within a bucket is as follows: Let *A F W*, Fig. 374, be the axis of the stream, and *A B D* a float, and *E F G* the next preceding, so that *A D G E* is the bucket which receives the body of water whose axis is *A F W*.

FIG. 374.



In the construction of floats shown in Art. 178, where the end *F E* meets the circumference at the proper angle, the water enters the bucket almost without impulse, or at least only the foremost particles strike at *F* upon *E F G*, and the principal impact occurs while the bucket passes gradually from the position *A D G E* to $A_1\ D_1\ G_1\ E_1$, during which time the foremost float is struck in succession by all the particles of the body of water *A F*. The impact within the bucket ends as soon as the last element *A* strikes the foremost float $E_1 F_1 G_1$ in *V*, or the water in the partially filled bucket at *W*. For the corresponding position of the bucket $A_1\ G_1 E_1$, therefore, we may assume the filling ended, and the impact of the water as having ceased, while its full action by pressure is just commencing. In order to find this position of the bucket, we have to observe that the motion of the float $E F G$ to the position $E_1 F_1 G_1$ must take place in the same time as the last particle of water passes from $A$ to $V$ or $W$.

If we denote the distance $A A_1 = E E_1$ by $s$, then, since the velocity of $A$ is $v$, we can put the time

$$t = \frac{s}{v}.$$

If, on the other hand, we denote the length of the parabolic arc $A F V$ by $s_1$, and assume that the last particle of water starting from $A$ with a velocity $c$, and arriving at $V$ with the velocity $c_1$, moves with the mean velocity

$$\frac{c + c_1}{2},$$

we can put the time

$$t = \frac{2 s_1}{c + c_1}.$$

Now, since these two expressions for the time must be equal, we have

$$\frac{s}{v} = \frac{2 s_1}{c + c_1}.$$

By reason of the slight deviation of the stream $A F V$ from the circumference $A E_1$, we can put approximately $s_1 = A F V = A E + E F + E E_1$. Now, $A E$ is known, and equal to the distance between two floats, viz.,

$$b = \frac{2 \pi r}{n},$$

and $E F$ is given by the proportion

$$\frac{E F}{E A} = \frac{w}{v} = \frac{c - v}{v} = \kappa - 1,$$

or

$$E F = (\kappa - 1) E A = (\kappa - 1) b.$$

Hence

$$s_1 = b + (\kappa - 1) b + s = \kappa b + s.$$

We have, then,

$$\frac{s}{v} = \frac{2}{c + c_1} (\kappa b + s),$$

or

$$(c + c_1 - 2 v) s = 2 \kappa v b,$$

and therefore, for the distance $s$ passed through by the float during impact,

$$s = \kappa \frac{2 v}{c + c_1 - 2 v} b = \frac{2 \kappa b}{\left(1 + \frac{c_1}{c}\right) \kappa - 2}.$$

By the aid of $s = A A_1 = E E_1$, we can now locate the corresponding position of the float. Since, from the given quantity of feed water $Q$ per second, we can determine the number of revolutions of the wheel per minute, and also, from the number of floats $n$, can determine the amount of water in each bucket, viz.,

$$V = \frac{60 Q}{n u}$$

where $u$ is the number of revolutions per minute, and from this, again, can determine the cross-section of the bucket,

$$F = \frac{V}{e} = \frac{60 Q}{n u e},$$

we can now also determine the position of the water surface $W$ in the bucket $A_1 G_1 E_1$, and the height $M N = z_1$, which in the preceding article we have assumed.

Since

$$c_1 = \sqrt{c^2 + 2 g z_1},$$

the entire determination of $s$, as given above, depends upon $z_1$. In general, we can take for $z_1$ a mean value, and replace it in the last expression by an approximate value.

EXAMPLE.—An overshot water-wheel has a height of 36 ft., 96 floats, and revolves with a velocity of $7\frac{1}{2}$ ft. The water enters with a velocity $c = 2v = 15$ ft., and acquires in the wheel a velocity of $c_1 = 16 \cdot 49$ ft. (see the example of the preceding article). Then the distance between the ends of floats measured on the circumference is

$$b = \frac{2\pi r}{n} = \frac{36 \times 3 \cdot 1416}{96} = 1 \cdot 18 \text{ ft.},$$

and the motion of the float during impact is

$$s = \frac{2\kappa b}{\left(1 + \frac{c_1}{c}\right)\kappa - 2} = \frac{2b}{1 + \frac{16 \cdot 49}{15} - 1} = \frac{2 \times 1 \cdot 18}{1 \cdot 099} = 2 \cdot 15 \text{ ft.}$$

§ **184.—Effect of Impulse.**—The water in an overshot-wheel acts principally by its weight, and only in a small degree by impact. The effect due to impact we can find by subtracting from the entire work due to the *vis viva* of the entering water, the work inherent in the water when it leaves the wheel, together with that lost by agitation and disturbance in the bucket. If we denote the velocity of the departing water by $v_1$, we have for the work inherent in it

$$\frac{v_1^2}{2g} Q \gamma,$$

$Q$ being the quantity and $\gamma$ the weight of one unit of volume. The loss of work due to the agitation in the bucket is

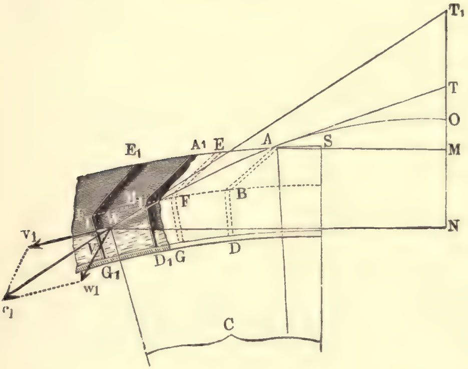
$$\frac{w_1^2}{2g} Q \gamma,$$

as in impact, if $w_1$ is that velocity which the water suddenly loses upon entrance. If, then, $c_1$ is the velocity $Wc_1$, Fig. 375, of the entering water, we have for the remaining work performed by it,

$$L_1 = \left(\frac{c_1^2 - v_1^2 - w_1^2}{2g}\right) Q \gamma.$$

Now, we may decompose $c_1$ into the velocity $W v_1 = v_1$, and $W w_1 = w_1$, where $v_1$ is the velocity of the water when it leaves the wheel. Hence the other component $w_1$ is the velocity lost. If we put the angle $c_1 \, W v_1$ which the water, on entering, makes with the tangent to the wheel $= \alpha_1$, we have

FIG. 375.



$$w_1^2 = c_1^2 + v_1^2 - 2 \, c_1 \, v_1 \cos. \alpha_1,$$

and hence the work

$$L_1 = \left( \frac{c_1^2 - v_1^2 - c_1^2 - v_1^2 + 2 \, c_1 \, v_1 \cos. \alpha_1}{2 g} \right) Q \gamma,$$

or

$$L_1 = \frac{(c_1 \cos. \alpha_1 - v_1) \, v_1}{g} \, Q \gamma.$$

Since

$$\frac{1}{g} = 0 \cdot 031 \quad \text{and} \quad \gamma = 62 \cdot 5 \text{ lbs.},$$

we have

$$L_1 = 2 \cdot 0193 \, (c_1 \cos. \alpha_1 - v_1) \, v_1 \, Q \text{ ft. lbs.}$$

We see at once that the work of impact is greater the less $\alpha_1$ and the greater $c_1$. But $(c_1 \cos. \alpha_1 - v_1) v_1$ will be a maximum for

$$v_1 = \tfrac{1}{2} c_1 \cos. \alpha_1.$$

The corresponding maximum work will then be

$$L_1 = \tfrac{1}{2} \frac{c_1^2 \cos. \alpha_1^2}{2g} Q \gamma,$$

or, making $\alpha_1 = 0$, and therefore $\cos. \alpha_1 = 1$,

$$L_1 = \tfrac{1}{2} \frac{c_1^2}{2g} Q \gamma.$$

Since

$$\frac{c_1^2}{2g}$$

is the height corresponding to the velocity, it follows that the effect of impact in the most favorable case is only half as great as the whole disposable work, or *by impact half the power is lost.* For this reason, it is desirable to make use of as little of the fall as possible in impact, and to use as much as possible in pressure. If we could make

$$c_1 \cos. \alpha_1 = v_1 \qquad \text{or} \qquad c_1 = \frac{v_1}{\cos. \alpha_1},$$

the fall necessary to bring the water on to the wheel would be

$$\frac{v_1^2}{2g \cos. \alpha_1^2},$$

and there would be no impact. If, on the other hand, we make

$$c_1 = \frac{2 v_1}{\cos. \alpha_1},$$

that is, if we use the fall

$$4 \frac{v_1^2}{2g \cos. \alpha_1^2},$$

four times as great as before, we have for the work of impact

$$\frac{1}{2}\,\frac{4\,v_1^{\,2}}{2g}\,Q\gamma = 2\,\frac{v_1^{\,2}}{2g}\,Q\gamma,$$

and lose entirely, therefore, the fall

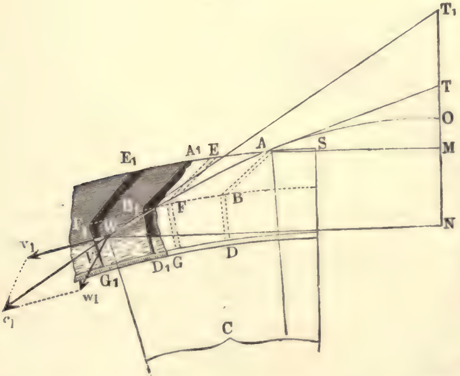
$$\left(\frac{4}{cos.\ \alpha_1^{\,2}} - 2\right)\frac{v_1^{\,2}}{2g},$$

or when $\alpha_1$ is very small,

$$2\,\frac{v_1^{\,2}}{2g},$$

which might have been used in pressure—*i. e.*, double as much as when we have no impact or let the water in just as fast as the wheel revolves. We see also that a greater performance is to be expected the smaller $v_1$, or the slower the wheel revolves. But the breadth $e$ of the wheel increases as the velocity diminishes, for the same amount of water. Hence the weight also increases as the velocity diminishes. Since, now, the axle is of greater diameter as the weight is greater, and the moment of the axle friction

FIG. 376.



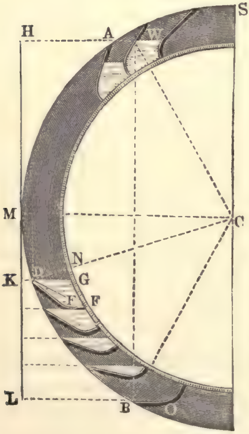
increases with its diameter, more work will be consumed by friction by the slow wheel, and we may therefore easily conclude that the greatest delivery of a water-wheel is by no means when its velocity is very slow. Since (Art. 178) the entrance velocity $c$ is always greater than $v$, much more is $c_1$ greater than $v_1$. The loss of work due to impact then always exceeds the quantity

$$\frac{v_1^2}{2g} Q \gamma.$$

§ 185.—**Effect of Pressure.**—The filled or partially filled buckets of a water-wheel form a cylindrical space, $A\,B$, Fig. 377, which we may call the water arc or loaded portion. Since the water enters at the upper end of this arc and departs from the lower, the height $A\,B = h_2$ is the real effective fall of the water, and hence the mechanical work or delivery of the wheel by means of pressure is $h_2\, Q\, \gamma$.

The height of the water arc $A\,B = H\,L$ can be divided into three portions. The first $H\,M$ lies above the centre, and depends upon the angle $S\,C\,W = \theta_1$ of the point of entrance from the summit. If we put $C\,W = r_1$, we have for the height of this portion $H\,M = r_1 \cos. \theta_1$.

FIG. 377.



The second portion lies below the centre and depends upon the position of the bucket end $D$, where the water first begins to flow out. If we put the angle $M\,C\,D = \lambda$, we have for the height of this second portion $M\,K = r \sin. \lambda$. The third portion, finally, corresponds to the arc $D\,B$ in which the buckets are losing water—*i. e.*, between $D$, where exit commences, and $B$, where it ends, the portion $B\,O$ of the float being here horizontal. If we put the angle $M\,C\,B = \lambda_1$, we have for the height $K\,L$ of this third portion $K\,L = r\,(sin.\,\lambda_1 - sin.\,\lambda)$. While, now, in the two first portions we obtain the full effect of all the water, in the last we have but a part of the full effect, because in this last portion the water is escaping from the wheel. If, therefore, we put for $Q$ the entire amount of water entering per second, and $Q_1$ a portion only of $Q_1$—*i. e.*, the mean amount acting in $D\,B$—we have for the entire effect or work of the water by its weight,

$$(r_1\,cos.\,\theta_1 + r\,sin.\,\lambda)\,Q\,\gamma + r\,(sin.\,\lambda_1 - sin.\,\lambda)\,Q_1\,\gamma.$$

If to this we add the work due to impact, just found, we have for the *entire work of an overshot-wheel,*

$$L = \left[\frac{(c_1\,cos.\,\alpha_1 - v_1)\,v_1}{g} + r_1\,cos.\,\theta_1 + r\,sin.\,\lambda\right] Q\,\gamma$$
$$+ r\,(\,sin.\,\lambda_1 - sin.\,\lambda)\,Q_1\,\gamma.$$

Or, denoting the height $(r_1\,cos.\,\theta_1 + r\,sin.\,\lambda)$ of that part of the water arc which entirely retains its water, by $h_3$, the remaining part $r\,(sin.\,\lambda_1 - sin.\,\lambda)$ by $h_4$, and the ratio $\frac{Q_1}{Q}$ by $\xi$, we have

$$L = P\,v = \left[\frac{(c_1\,cos.\,\alpha_1 - v_1)\,v_1}{g} + h_3 + \xi\,h_4\right] Q\,\gamma,$$

and for the force at the extremity of the wheel,

$$P = \left[\frac{(c_1\,cos.\,\alpha_1 - v_1)\,v_1}{g} + h_3 + \xi\,h_4\right] \frac{Q}{v}\,\gamma.$$

EXAMPLE.—In an overshot water-wheel 30 ft. high, the entrance velocity is $c_1 = 16$ ft.; the velocity at the surface of the water $v_1 = 7$ ft.; the angle $\alpha_1$ between the direction of the stream at the surface and the tangent to the wheel is 12°, and the radius $C\,W$ or $r_1 = 14$ ft. Further, the angle

$S C W = \theta_1 = 18°$, the angle $M C D$ to the point of commencement of exit is $\lambda = 58\frac{1}{2}°$, and $M C B = \lambda_1 = 70\frac{1}{2}°$. Finally, $Q = 5$ cubic ft. and

$$\xi = \frac{Q_1}{Q} = \frac{1}{2}.$$

Required the mechanical work imparted to the wheel.

We have the fall during impact

$$= 0·031\,(16\,cos.\ 12° - 7) \times 7 = 0.217\,(15·65 - 7) = 1·877 \text{ ft.,}$$

and the fall during pressure

$$= 14\,cos.\ 18° + 15\,[sin.\ 58\tfrac{1}{2} + \tfrac{1}{2}\,(sin.\ 70\tfrac{1}{2} - sin.\ 58\tfrac{1}{2})]$$

$$= 13·315 + 15\,(0·8526 + \tfrac{1}{2} \times 0·09) = 13·315 + 13·464 = 26·779 \text{ ft.}$$

Hence, for the entire work

$$L = (1·877 + 26·779) \times 5 \times 62·5 = 28·656 \times 312·5 = 8955 \text{ ft. lbs.}$$

The force at the circumference of the wheel, where the velocity is $v = 7\frac{1}{2}$ ft., is therefore

$$P = \frac{L}{v} = \frac{8955}{7·5} = 1194 \text{ lbs.}$$

**§ 186.—Exit of Water from the Wheel.**—We see, therefore, at once that for an exact determination of the work due to the pressure of the water in an overshot-wheel, it is quite important to determine the first and last points of exit, or the two limits of the discharging water arc, as well as the ratio

$$\xi = \frac{Q_1}{Q}$$

of the mean contents of a bucket in the discharging arc to the original quantity of water in the bucket before discharge commenced. We have therefore to determine the rules necessary for this purpose.

If the wheel has $n$ floats or buckets and makes $u$ revolutions per minute, then in each second we have

$$\frac{n\,u}{60}$$

buckets presented for the reception of the water $Q$, and hence in each bucket we have a volume of water

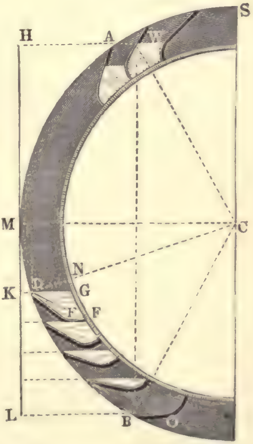
$$V = Q : \frac{n\,u}{60} = \frac{60\,Q}{n\,u}.$$

If, as before, $e$ is the breadth of the wheel, then the cross-section of the water in a bucket is

$$F_0 = \frac{V}{e} = \frac{60\,Q}{n\,u\,e}$$

(see Art. 183). If, now, $DEFG$, Fig. 378, is that bucket in which discharge first commences, we have $F_0$ = the segment $DEF$ + the triangle $DFG$; or since the triangle $DFG$ = triangle $DFN - DGN$, $F_0 = seg.\ DEF + tri.\ DFN - tri.\ DGN$.

FIG. 378.



If, now, we put the area of the segment $DEF = S$ and that of the triangle $DFN = D$, we have

$$DGN = S + D - F_0.$$

But since

$$DGN = \frac{DN \times NG}{2} = \tfrac{1}{2} d^2\, tan.\, \lambda,$$

we have, finally, approximately and more correct, the greater the number of buckets,

$$tan.\, \lambda = \frac{S + D - F_0}{\tfrac{1}{2} d^2}.$$

Accordingly, the angle $\lambda$ is determined in terms of measurable quantities. This angle gives the point $D$ at which the water commences to flow out. Further, the water will completely leave the wheel when the outer float end is horizontal. The angle $CBO$ then is equal to $MCB$ or $\lambda_1$. This angle gives the position of the point $B$ at which the bucket is entirely empty. In order, now, to find the ratio

$$\xi = \frac{Q_1}{Q} = \frac{F_m}{F_0} =$$

the ratio of the mean cross-section of all the buckets in the arc $DB$ to

$$F_0 = \frac{V}{e} = \frac{60\,Q}{n\,u\,e};$$

we divide the height $KL = r(\sin.\lambda_1 - \sin\lambda)$ into an even number of equal parts, and let a bucket end correspond to each point of division, and determine the areas $F_1, F_2, F_3, \cdots F_n$, of their cross-sections.

Now, the mean value $F_m$ can be found by Simpson's rule,

$$F_m = \frac{F_0 + F_n + 4\,(F_1 + F_3 + \cdots + F_{n-1}) + 2\,(F_2 + F_4 + \cdots + F_{n-2})}{3\,n},$$

and hence we have for the ratio

$$\xi = \frac{Q_1}{Q} = \frac{F_m}{F_0} = \frac{F_0 + F_n + 4\,(F_1 + F_3 + \cdots + F_{n-1}) + 2\,(F_2 + F_4 + \cdots F_{n-2})}{3\,n\,F_0}.$$

We can now, then, from easily measurable or given data, determine the work imparted by the water to an overshot water-wheel.

EXAMPLE.—A 40 ft. high water-wheel receives per minute 300 cubic ft. of water, and makes per minute 4 revolutions. What is the work performed? If we take the depth of crown at 1 ft., we have for the breadth of wheel

$$e = \frac{4 \times 300}{\pi \times 40 \times 1 \times 4} = \frac{30}{4\,\pi} = 2 \cdot 4 \text{ ft.}$$

If the wheel has 136 floats, we have for the quantity of water in a bucket

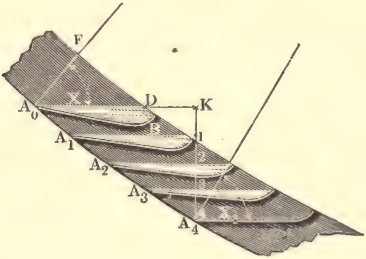
$$V = \frac{300}{4 \times 136} = \frac{75}{136} = 0{\cdot}5515 \text{ cubic ft.},$$

and hence for the cross-section,

$$F_0 = \frac{0{\cdot}5515}{2{\cdot}4} \text{ sq. ft.} = \frac{144 \times 0{\cdot}5515}{2{\cdot}4} = 33{\cdot}09 \text{ sq. in.}$$

For the form of float shown in the figure, we have by careful measurement

FIG. 379.



for the area of the segment $A_0 B D$, $S = 24{\cdot}5$ sq. in., and for the triangle $A_0 F D$, 102 sq. in. We have, therefore, for the beginning of exit,

$$tan.\ \lambda = \frac{24{\cdot}5 + 102 - 33{\cdot}09}{\frac{1}{2} \times 144} = \frac{93{\cdot}41}{72} = 1{\cdot}2973,$$

or

$$\lambda = 52''\ 22\tfrac{1}{2}'.$$

The angle $\lambda_1$ is 62° 30′, hence the height $K A_4$ of the discharging portion is

$$r(sin.\ \lambda_1 - sin.\ \lambda) = 20(0{\cdot}8870 - 0{\cdot}7920) = 1{\cdot}9 \text{ ft.}$$

If, now, we inscribe in this space three more floats, we find, by measurement and calculation of the cross-sections of the water in these positions, $F_1 = 24{\cdot}50$, $F_2 = 14{\cdot}48$, $F_3 = 6{\cdot}60$, $F_4 = 0$. We have, therefore,

$$\xi = \frac{F_m}{F_0} = \frac{33{\cdot}09 + 4(24{\cdot}5 + 6{\cdot}60) + 2 \times 14{\cdot}48}{12 \times 33{\cdot}09} = \frac{15{\cdot}5375}{33{\cdot}09} = 0{\cdot}469.$$

If, now, the height of the water surface in the highest bucket above the centre is $r_1\, cos.\, \theta_1 = 18$ ft., then we have for the delivery of the wheel due to the weight of the water alone, without taking into account impact and neglecting friction,

$$L = [r_1 \cos. \theta_1 + r\,[\sin. \lambda + 0\cdot469(\sin. \lambda_1 - \sin. \lambda)]\,]\,Q\gamma$$

$$= [18 + 20\,(0\cdot7920 + 0\cdot469 \times 0\cdot0950)]\,5 \times 62\cdot5$$

$$= (18 + 16\cdot73)\,308\cdot75 = 10723 \text{ ft. lbs.}$$

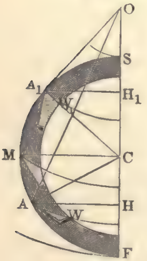
The amount to be added to this for impact may easily be found as in the preceding article, when the velocities $c_1$ and $v_1$ are known, as also the angle $\alpha_1$.

**§ 187.—Influence of the Centrifugal Force.**—For the same velocity at the circumference, small wheels make a greater number of revolutions than high ones; moreover, the necessity of uniform motion, and the object of the machine, as in saw mills, etc., demands more often a high than low rate of speed for small wheels. For these reasons, small wheels often make a large number of revolutions (25) per minute. For such large values of $u$, the centrifugal force of the water in the buckets becomes of importance, acting as it does to give a decided inclination of the surface of the water with the horizon, and thus causing exit to begin earlier than for slowly moving wheels. We must not, then, omit to discuss the influence of the centrifugal force in such cases. We have already seen (Vol. I., Art. 354) that the surface of the water in the buckets takes under the action of the centrifugal force a cylindrical shape, the axis $O$, Fig. 380, being parallel to that of the wheel at a height

$$CO = k = \frac{g}{\omega^2} = g\left(\frac{30}{\pi u}\right)^2 = \frac{2936}{u^2} \text{ ft.}$$

above the centre. This distance, therefore, increases inversely as the *square* of the number of revolutions, and for a great number of revolutions, becomes quite small. Only, then, at the summit $S$ and at the foot $F$ would the water surface be

FIG. 380.



horizontal; at any position above $M$ it will make an angle with the horizon. Let the deviation

$$H A W = A O C = \chi$$

for any point $A$ whose angle $A C M = \lambda$ is below $M$. Then

$$tan.\, \chi = \frac{A H}{O H} = \frac{r\, cos.\, \lambda}{k + r\, sin.\, \lambda}.$$

For any point $A_1$ above $M$, $\lambda$ is negative and

$$tan.\, \chi = \frac{r\, cos.\, \lambda}{k - r\, sin.\, \lambda}.$$

If we draw from $O$ a tangent to the circumference, we have at the point of tangency $A_1$ that position for which the water surface makes the greatest angle with the horizontal, and where, therefore, $\chi$ is a maximum and equals $\lambda$. For this point, we have

$$sin.\, \chi = \frac{r}{k} = \frac{\pi^2 r u^2}{900 g} = \frac{r u^2}{2936}.$$

The inclination $\chi$ of the water surface increases directly as $r$ and as the square of the number of revolutions.

EXAMPLE.—1. For a wheel which makes 5 revolutions per minute, we have

$$k = \frac{2936}{25} = 117 \text{ ft.}$$

If, now, $r = 16$ ft. and the exit angle $\lambda = 50°$, we have at the place of exit

$$tan.\, \chi = \frac{16\, cos.\, 50°}{117 + 16\, sin.\, 50°} = \frac{10 \cdot 285}{129 \cdot 256},$$

hence

$$\chi = 4° \; 33'.$$

Therefore, at this point the water is inclined to the horizontal at an angle of $4\frac{1}{2}°$.

2. For a wheel making 20 revolutions, we have

$$k = \frac{2936}{400} = 7 \cdot 34.$$

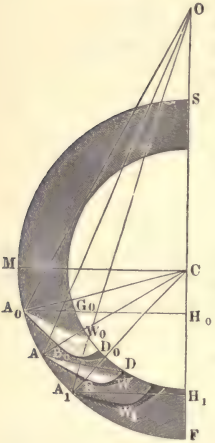
If, now, $r = 5$ ft. and $\lambda = 0$, we have

$$tan.\, \chi = \frac{5}{7 \cdot 34}, \qquad \text{or} \qquad \chi = 34° \; 16'.$$

Finally, 44° 34′ above the centre, we have $\chi = 44° \; 34'$.

**§ 188.**—If, now, we take into account the influence of the centrifugal force, which for swift wheels we must do, the formula found above for the arc of discharge must be replaced by others. Let $A_0$, Fig. 381, be the beginning of exit,

FIG. 381.



$$M\,C\,A_0 = H_0\,A_0\,C = \lambda$$

the exit angle,

$$H_0\,A_0\,W_0 = A_0\,O\,C = \chi$$

the deviation of the water surface from the horizon, therefore

$$\angle\, G_0\,A_0\,W_0 = \lambda + \chi,$$

and

$$\varDelta\, A_0\,G_0\,W_0 = \tfrac{1}{2}\,d \times d\; tan.\,(\lambda + \chi)$$

$$= \tfrac{1}{2}\,d^2\; tan.\,(\lambda + \chi).$$

If we represent as before by $S$ the area of the segment $A_0\,B_0\,D_0$, by $D$ that of the triangle $A_0\,G_0\,D_0$, and the cross-section of the water $A_0\,W_0\,D_0$ by $F_0$, we have

$$F_0 + \tfrac{1}{2}\,d^2\; tan.\,(\lambda + \chi) = S + D,$$

and hence

$$tan.\,(\lambda + \chi) = \frac{S + D - F_0}{\tfrac{1}{2}\,d^2} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But we have also

$$\frac{sin.\,A_0\,O\,C}{sin.\,O\,A_0\,C} = \frac{C\,A_0}{C\,O},$$

or

$$\frac{sin.\,\chi}{sin.\,[90^\circ - (\lambda + \chi)]} = \frac{r}{k},$$

hence

$$sin.\;\chi = \frac{r\,cos.\,(\lambda + \chi)}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From the first formula, we can find $(\lambda + \chi)$, and then by substitution of this in the second can find $\chi$. We can then find $\lambda$ by subtraction of these two values, thus,

$$\lambda = (\lambda + \chi) - \chi.$$

At the end of the arc of discharge $A_1$ the outer float end coincides with the water surface $A_1 W_1$, and therefore

$$C A_1 W_1 = \lambda_1 + \chi_1 =$$

the known angle $\delta = 90^\circ - \beta$, hence

$$sin.\, \chi_1 = \frac{r\, cos.\, \delta}{k} = \frac{r\, sin.\, \beta}{k}$$

and $\lambda_1 = \delta - \chi_1$. Hence we have the angle of $A_1$ below $M$.

We can now, as before, determine the height

$$H_0 H_1 = h_4 = r\,(sin.\, \lambda_1 - sin.\, \lambda),$$

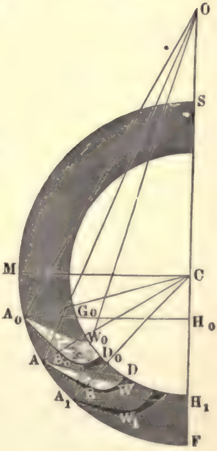


FIG. 382.

Fig. 382, divide it into an even number of parts (4 or 6), and determine the contents of the buckets for each position, and thus find

$$\xi = \frac{Q_1}{Q} = \frac{F_m}{F_0},$$

as in Art. 186, and then find the action of the water in the arc of discharge. For this purpose, we may use the above formulæ reversely. Thus $\lambda$ is now known, and accordingly

$$tan.\, \chi = \frac{r\, cos.\, \lambda}{k + r\, sin.\, \lambda}$$

and

$$F = S + D - \tfrac{1}{2} d^2\, tan.\, (\lambda + \chi).$$

When the water no longer fills the entire segment, when, therefore,

$$F < S \quad \text{or} \quad \tfrac{1}{2} d^2\, tan.\, (\lambda + \chi) > D,$$

we have

$$F = seg.\ A\,B\,D - \triangle\,A\,D\,W,$$

or for straight floats,

$$F = S - \tfrac{1}{2}\,s^2\,\frac{sin.\ (\lambda + \chi - \delta)\ sin.\ \delta_1}{sin.\ (\lambda + \chi)},$$

where $s$ is the diagonal $A\,D$ and $\delta_1$ the angle $D\,A\,C$, $\delta$ being $B\,A\,C$.

EXAMPLE.—A small wooden water-wheel, 12 ft. high, 1 ft. depth of crown, and 4 ft. wide, revolves 17 times per minute, and uses 1080 cubic ft. of water. What is the mechanical effect? Here we have

$$r = 6,\ d = 1,\ e = 4,\ r_1 = 5\cdot 5,\ Q = \frac{1080}{60} = 18, \text{ and } u = 17.$$

If, now, the wheel has 24 floats, then

$$\phi^0 = \frac{360}{24} = 15^\circ,$$

and

$$F_0 = \frac{1080}{24 \times 17 \times 4} = \frac{45}{68} = 0\cdot 662 \text{ sq. ft.}$$

If, further, $D = 0\cdot 652$ and $S = 0\cdot 373$, we have

$$tan.\ (\lambda + \chi) = \frac{0\cdot 373 + 0\cdot 652 - 0\cdot 662}{\frac{1}{2}} = 0\cdot 363 \times 2 = 0\cdot 726,$$

hence $\lambda + \chi = 35^\circ\ 59'$. Now,

$$C\,O = k = \frac{2936}{17^2} = 10\cdot 16 \text{ ft.},$$

hence

$$sin.\ \chi = \frac{6\ cos.\ 35^\circ\ 59'}{10\cdot 16} = 0\cdot 4779$$

and accordingly $\chi = 28^\circ\ 33'$ and $\lambda = 7^\circ\ 26'$. At $6\frac{1}{2}^\circ$ below the middle the overflow commences. In order to find the place where the exit of water ends and the bucket is empty, we put for $r$, $r_1$ and for $\beta$ the entrance angle, which is here $= 10^\circ\ 46'$. Accordingly,

$$sin.\ \chi_1 = \frac{r\ sin.\ \beta}{k} = \frac{5\cdot 5\ sin.\ 10^\circ\ 46'}{10\cdot 16},$$

or

$$\chi_1 = 5^\circ\ 59',$$

and

$$\lambda_1 = 79^\circ\ 14' - 5^\circ\ 59' = 73^\circ\ 15'.$$

Therefore we have for the height of the arc of discharge,

$$h_4 = r_1 \sin. \lambda_1 - r \sin. \lambda = 5 \cdot 5 \sin. 73^\circ\ 15' - 6 . \sin. 7^\circ\ 26'$$

$$= 5 \cdot 2666 - 0 \cdot 7764 = 4 \cdot 49 \text{ ft.}$$

Now, this height we can divide into four equal parts, and determine, by exact drawing and careful measurement and estimation, the three other values of $F$. We thus find, for instance, $F_1 = 0 \cdot 501$, $F_2 = 0 \cdot 409$, $F_3 = 0 \cdot 195$, and, as above, $F_0 = 0 \cdot 662$. Therefore,

$$\xi = \frac{F_m}{F_0} = \frac{0 \cdot 662 + 4 (0 \cdot 501 + 0 \cdot 195) + 2 \times 0 \cdot 409}{12 \times 0 \cdot 662} = 0 \cdot 537,$$

and the mechanical work of the water in the arc of discharge is then

$$L_4 = \xi\, h_4\, Q \gamma = 0 \cdot 537 \times 4 \cdot 49 \times 18 \times 62 \cdot 5 = 2712 \text{ ft. lbs.}$$

If the water enters with a velocity of 20 ft. 20° below the crown, so that the stream makes an angle of 25°, with the tangent at the point of entrance, then we have for the remaining action of the water,

$$L_3 = (5 \cdot 5 \cos. 20^\circ + 6 \sin. 7^\circ\ 26')\, 18 \times 62 \cdot 5 = 5 \cdot 944 \times 1125 = 6687 \text{ ft. lbs.}$$

For the work of impact, since

$$v_1 = \frac{11 \times \pi \times 17}{60} = 9 \cdot 791,$$

$$L_1 = 0 \cdot 032 (20 \cos. 25^\circ - 9 \cdot 791)\, 9 \cdot 791 \times 18 \times 62 \cdot 5 = 2 \cdot 611 \times 18 \times 62 \cdot 5$$

$$= 2937 \text{ ft. lbs.}$$

The whole work is then

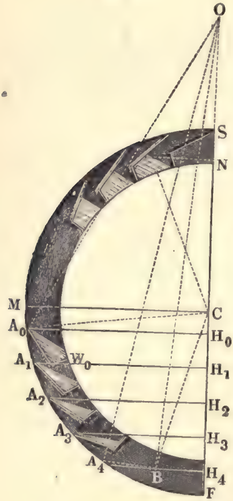
$$L = L_1 + L_3 + L_4 = 12336 \text{ ft. lbs.}$$

§ **189.—Size of the Arms.**—The necessary dimensions to be given to the sections of the arms of a water-wheel, and also the size of the shaft and that of the journals, depend upon the size of the wheel and the way in which it works. To ascertain these dimensions, we must refer principally to Vol. I., Section IV.

The power of the wheel is generally transmitted by a cog-wheel, which is placed as follows:

1. On the shaft, or,
2. On one of the sets of arms, or,
3. Is fastened to one of the crowns.

FIG. 383.



In the first case, the power is communicated to the shaft by the arms and by the shaft to the driving-wheel; in the second case, on the other hand, it passes to the driving-wheel by means of the arms only; and in the third case, the transmission of the power is effected almost directly. The first case is by far the most common, especially as this also includes those cases in which the transmission of power is effected by drums, cranks, etc.

If $m$ denotes the number of arms, $b_1$ the width, and $h_1$ the thickness of an arm, the former being measured parallel with the axis of the wheel and the latter with its circumference, then, according to the formula given in Vol. I., § 236, viz.,

$$Pl = b_1 h_1^2 \cdot \frac{T}{6},$$

in which the force $P$ is to be replaced by $\frac{P}{m}$, and $l$ by $r$, the length of an arm in inches, we have

$$\frac{Pr}{m} = 9.549 \frac{L}{mu} = b_1 h_1^2 \frac{T}{6},$$

and if we also put tne ratio $\frac{b_1}{h_1} = \mu$, a constant quantity, for wood, for example, $= \frac{5}{7}$, and for cast-iron $= \frac{1}{5}$, we obtain for the required thickness of the arms

$$h_1 = \sqrt[3]{\frac{6}{\mu T}\,\frac{Pr}{m}} = \sqrt[3]{\frac{6}{\mu T} \times 9 \cdot 549\,\frac{L}{m u}} = 3 \cdot 86 \sqrt[3]{\frac{L}{\mu T m u}}.$$

If, as is usual, we express $r$ in feet and $L$ in horse-powers (1 horse-power being 33,000 ft. lbs. per minute), we obtain

$$h_1 = 4 \cdot 16 \sqrt[3]{\frac{Pr}{\mu T m}} = 72 \cdot 31 \sqrt[3]{\frac{L}{\mu T m u}} \text{ inches.}$$

Assuming now for wood $\mu = \frac{5}{7}$ and $T = 1000$ pounds (vid. Vol. I., § 240), we have for wooden arms

$$h_1 = 0 \cdot 465 \sqrt[3]{\frac{Pr}{m}} = 8 \cdot 09 \sqrt[3]{\frac{L}{m u}} \text{ inches.}$$

For safety and for the reason that the arms have to carry the weight of the wheel also, in practice, their dimensions are taken fully twice as great, and we therefore put

$$\text{I.} \quad h_1 = 0 \cdot 930 \sqrt[3]{\frac{Pr}{m}} = 16 \cdot 2 \sqrt[3]{\frac{L}{m u}} \text{ inches.}$$

If, on the other hand, for cast-iron, we take $\mu = \frac{1}{5}$ and $T = 7000$, we have for cast-iron arms

$$h_1 = 0 \cdot 372 \sqrt[3]{\frac{Pr}{m}} = 6 \cdot 46 \sqrt[3]{\frac{L}{m u}}.$$

In practice, we double this dimension, that is to say,

$$\text{II.} \quad h_1 = 0 \cdot 7 \sqrt[3]{\frac{Pr}{m}} = 13 \sqrt[3]{\frac{L}{m u}} \text{ inches.}$$

EXAMPLE.—If an overshot water-wheel of wood, with 16 arms, is to make 5 revolutions per minute, and is to give a power amounting to 20 horse-powers, and transmit this power by its shaft, its arms must have the following dimensions for their cross-sections :

$$h_1 = 16 \cdot 2 \sqrt[3]{\frac{20}{16 \times 5}} = 16 \cdot 2 \sqrt[3]{0 \cdot 25} = 10 \cdot 2 \text{ inches,}$$

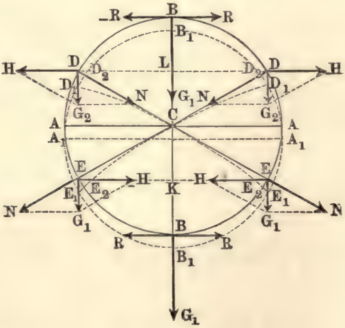
and

$$b_1 = \mu \, h_1 = \tfrac{5}{7} \times 10 \cdot 2 = 7 \cdot 29.$$

These dimensions can, of course, be somewhat decreased towards the outer ends.

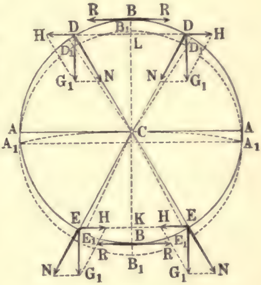
**§ 190.**—When the power of an overshot-wheel is transmitted by means of a cog-wheel on the circumference of the main wheel, the arms have to carry chiefly the weight of the wheel alone, and accordingly, in this case, the size of the arms depends upon this weight and almost nothing else. As, during a revolution of the wheel, its arms gradually come into all possible positions, with respect to the direction of the weight, the force which an arm has thereby to sustain is variable, and accordingly, in the determination of the cross-section of an arm, its different positions have to be taken into consideration. Let us first assume only one set of arms, six in number, *CB*, *CD*, *CE*, Fig. 384 and Fig. 385, and also perfect rigidity in the crown of the wheel. With the position in Fig. 384 there are two arms, *CB*, *CB* vertical, and four arms, *CD*, *CE*, etc., inclined to the horizontal

FIG. 384.



zon at an angle of 30°. The arm pointing vertically upwards is under compression, the one directed downwards is under tension, while the others undergo compound strains; *CD*, *CD*, in fact, undergoing compression and bending, and *CE* and *CF* being subjected to tension and bending. Since the resistance to compression and tension allow the wheel only a slight vertical movement, the deflections of the arms are also very small, and we can, therefore, leave wholly out of consideration the force which measures this deflection.

FIG. 385.



Let *G* be that part of the wheel's weight which the set of arms under consideration has to transmit to the shaft *C*, and let $G_1$ be the part of the weight which each of the two vertical arms receives, and $G_2$ that received by each of the inclined arms. This last force is decomposed into a horizontal force,

$$H = G_2 \, tan. 60° = \sqrt{3} \cdot G_2,$$

and into a force in the direction of the arm,

$$N = \frac{G_2}{cos. 60°} = 2\, G_2.$$

Since the horizontal forces *H*, *H*.... virtually balance each other in the wheel, of course the latter, in consequence of the elasticity of the arms, can only sink perpendicularly and to the amount $\overline{BB_1} = \overline{DD_1} = \overline{EE_1} = \sigma$. But if the ends *D*, *E*....., have the vertical movements $\overline{DD_1} = \overline{EE_1}$...., the arms themselves must be shortened or extended to a corresponding amount, viz.,

$$\overline{DD_2} = \overline{EE_2} = \overline{DD_1} \, cos. \, D_1 DD_2 = \sigma \, cos. \, 60° = \tfrac{1}{2}\sigma,$$

and therefore the force $N$ in the direction of the arms $CD$, $CE$.... is also one half of the force $G_1$, borne by each of the arms $CB$, $CB$, which are respectively shortened and lengthened by the amount $\sigma$, and consequently we must put

$$G_2 = \tfrac{1}{2} N = \tfrac{1}{4} G_1.$$

If we introduce this value in the equation

$$2 G_1 + 4 G_2 = G,$$

we obtain the following expressions for $G_1$ and $G_2$:

$$G_1 = \tfrac{1}{3} G \qquad \text{and} \qquad G_2 = \tfrac{1}{12} G.$$

Finally, denoting the section of an arm by $F$ and its modulus of proof strength by $T$, we obtain these formulæ:

$$F = \frac{G_1}{T} = \frac{G}{3\,T},$$

and

$$F = \frac{N}{T} = \frac{2\,G_2}{T} = \frac{G}{6\,T}.$$

The first equation is, of course, to be used in determining the section.

In the position of the arms in Fig. 385, where two arms, $CA$, $CA$, are horizontal, only the four arms, $CD$, $CD$ and $CE$, $CE$ are exposed to compressive and tensile strains, and the force of compression or tension is

$$N = \frac{G_1}{\cos. 30^\circ} = G_1 \sqrt{\tfrac{4}{3}} = \frac{G}{4} \sqrt{\tfrac{4}{3}},$$

and therefore the corresponding section of an arm is

$$F = \frac{N}{T} = \frac{G_1}{4\,T} \sqrt{\tfrac{4}{3}} = \frac{G}{3{\cdot}464\,T},$$

which is, therefore, smaller than for the position in Fig. 384, so that the section to be used in practice is

$$F = \frac{G}{3\,T}.$$

When using only four arms, we must put

$$F = \frac{G}{2\,T},$$

and with eight arms,

$$F = \frac{G}{4\,T},$$

as can be easily shown by a similar investigation.

In general, if the number of the arms of a wheel is $m$, and the total weight of the wheel is $G$, the section of an arm is determined by the simple formula,

$$F = \frac{2\,G}{m\,T}.$$

For wooden arms, according to Table I., § 212, Vol. I., $T$ would be 2600, while for cast-iron $T$ would be 9480, and, by Table II., for wrought-iron, 18,700. As, however, long arms are apt to bend under compression, and as the strain is constantly changing during a revolution, of the first value only a tenth and of the last two only a fifth part is to be used in practice, and hence, for wooden arms, we put

$$F = \frac{2}{260}\,\frac{G}{m} = 0{\cdot}0077\,\frac{G}{m},$$

while for cast-iron

$$F = \frac{2}{1896}\cdot\frac{G}{m} = 0{\cdot}00105\,\frac{G}{m},$$

and for wrought-iron

$$F = \frac{2}{3740}\cdot\frac{G}{m} = 0{\cdot}00054\,\frac{G}{m},$$

all being in square inches.

If the crowns of a water-wheel are fastened to the shaft by wrought-iron ties, the wheel is carried by those arms or rods alone, which point downwards. Accordingly, in this case,

$$G_1 = \tfrac{2}{3} G \qquad \text{and} \qquad G_2 = \tfrac{1}{6} G,$$

and $N$ and $F$ are also twice as great as in the case of a rigid-arm system.

REMARK.—With the aid of the foregoing theory, the requisite size of the crown of a wheel can also be ascertained. Each half of the wheel is subjected to a couple $(H, -H)$, which produces at the points $B, B$ the forces $R, -R$, which the crown must be strong enough to resist. Putting the moment $R . 2\,a$ of the couple $R, -R$, equal to the moment $H a$, of the couple $H, -H$, we have

$$R = \tfrac{1}{2} H = \tfrac{1}{2} \sqrt{3} \,.\, G_2 = \tfrac{1}{24} \sqrt{3}\, G = 0{\cdot}072\, G,$$

and hence $f d$, the necessary section of the crown, is given by the formula, $f d = \frac{R}{T}$, and its thickness by

$$f = \frac{0{\cdot}072\; G}{d\,T}.$$

To obtain as rigid a crown as possible, we must assume the following values for $T$:

For wooden crowns, $T = 52$,
For cast-iron crowns, $T = 364$, and
For wrought-iron crowns, $T = 624$.

§ **191.—Thickness of the Shaft.**—The size of the shaft of a water-wheel is determined from the moment of the power, $P r$, by considering its torsional strength, and also from the weight of the wheel, by taking into consideration its relative strength. If we consider merely the moment $P r$, we have by the theory of torsional strength (see Vol. I., § 264), for the size of a cast-iron cylindrical shaft, the formula,

$$P r = 374\, d^3 \text{ inch lbs.},$$

and therefore the diameter is

$$d = \sqrt[3]{\frac{P\,r}{374}} = 0{\cdot}1388 \sqrt[3]{P\,r} \text{ inches};$$

or when, as is generally the case, $Pr$ is expressed in ft. lbs.,

$$d = 0 \cdot 3178 \sqrt[3]{Pr} \text{ inches.}$$

If $L$, the power of the wheel in horse-powers, is given, and the number of revolutions per minute $u$, we have

$$Pr = \frac{33{,}000}{2 \pi u} \cdot L = 5252 \frac{L}{u},$$

and hence the required diameter of the shaft is

$$d = 0 \cdot 3178 \sqrt[3]{5252} \sqrt[3]{\frac{L}{u}} = 5 \cdot 52 \sqrt[3]{\frac{L}{u}},$$

instead of which we take in practice

$$d = 0 \cdot 357 \sqrt[3]{Pr} = 6 \cdot 2 \sqrt[3]{\frac{L}{u}} \text{ inches.}$$

For wrought-iron shafts, on the other hand, we take as a mean

$$d = 0 \cdot 297 \sqrt[3]{Pr} = 5 \cdot 16 \sqrt[3]{\frac{L}{u}} \text{ inches.}$$

Wooden shafts are made in practice from three to four times as thick as cast-iron ones, although, on theoretical grounds, double the thickness is amply sufficient.

For shafts having a square section $s$, the side of the square must be determined from the equation $s = 0 \cdot 94\, d$; and for hollow cylindrical shafts with an exterior diameter $d_1$ and interior diameter $d_2$, we have instead

$$d^3 = \frac{d_1^4 - d_2^4}{d_1} = d_1^3 \left[1 - \left(\frac{d_2}{d_1}\right)^4\right] = \left[1 - \psi^4\right] d_1^3,$$

and therefore

$$d_1 = \frac{d}{\sqrt[3]{1 - \psi^4}},$$

in which $\psi$ denotes the ratio $\frac{d_2}{d_1}$. This ratio is usually taken equal to 0·6, and, in that case, we have

$$d_1 = \frac{d}{\sqrt[3]{1 - \psi^4}} = \frac{d}{\sqrt[3]{1 - 0 \cdot 13}} = 1 \cdot 05\ d,$$

and

$$d_2 = 0 \cdot 63\ d.$$

The direct application of these formulæ presupposes that the driving-wheel is set upon the shaft of the water-wheel; if, however, it is fastened to one of the sets of arms, or to one of the crowns, then the other sets of arms transmit only a part of the whole moment $Pr$ to the shaft, and the thickness required is smaller than in the former case. If, in this case, the wheel has two arm systems, we can assume that the second set of arms transmits to the shaft one half of the whole moment $Pr$, which is then communicated by means of the other set to the cog-wheel, which is in the same plane with this set; on the other hand, if the wheel has three sets of arms, it may be assumed that, by means of the shaft, the middle set transmits two quarters and the third set one quarter of the whole moment $Pr$ to the first set; accordingly, in determining the necessary thickness of the shaft, we must, in this case, introduce into the formulæ, instead of $P$, either $\frac{1}{2} P$ or $\frac{3}{4} P$, and, in like manner, instead of $L$, either $\frac{1}{2} L$ or $\frac{3}{4} L$.

When the driving-wheel is put upon a crown midway between the outer crowns of the wheel, and also when the transmission of power is effected by two cog-wheels placed upon the outermost crowns of the wheel, the shaft has to sustain hardly any torsion, and its size is to be computed from the weight of the wheel, according to the theory of flexure.

EXAMPLE.—If an overshot-wheel 24 ft. high, making five revolutions per minute, gives a performance of 20 horse-powers, and if its power is transmitted by means of a cog-wheel fastened to its shaft, which is of cast-iron, the requisite diameter of this shaft is

$$d = 6\cdot2 \sqrt{\frac{L}{u}} = 6\cdot \sqrt[3]{4} = 9\cdot84 \text{ inches.}$$

If we wanted to use instead of this a round wooden shaft, we should have to give this shaft, at least, this diameter,

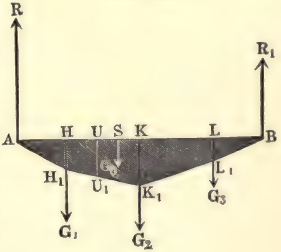
$$d = 3 \times 9\cdot84 = 29\cdot52 \text{ inches};$$

and if the shaft were to be square in section, the side of the square would have to be

$$s = 0\cdot94\, d = 27\cdot75 \text{ inches.}$$

§ **192.**—In order to be able to determine the thickness of shaft required by the weight of a wheel, it is necessary first to ascertain the pressures on the two journals of this shaft. In Fig. 386 let $A\,K\,B$ be the axis of the shaft, having the length $A\,B = l$, and loaded at the points $H$, $K$, $L$ with the weights $G_1$, $G_2$, $G_3$; let $l_1$, $l_2$, $l_3$ be the distances of the points of application $H$, $K$, $L$ of these weights from the journal $A$, and, finally, let $R$ denote the pressure on the journal at $A$, and $R_1$ the pressure at $B$. Regarding the whole shaft as a lever with the fulcrum at $A$, we can put (according to Vol. I., § 92)

FIG. 386.



$$R_1 = \frac{G_1\, l_1 + G_2\, l_2 + G_3\, l_3}{l},$$

and, if we put in place of $l_1$, $l_2$, $l_3$ : $l - l_1$, $l - l_2$, $l - l_3$, and in place of $R_1$, $R$, we have

$$R = \frac{G_1\,(l - l_1) + G_2\,(l - l_2) + G_3\,(l - l_3)}{l} = G_1 + G_2 + G_3 - R_1.$$

We can easily infer what additions these formulæ must receive when the number of weights is greater. When we have determined the journal pressures $R$ and $R_1$, we can also determine the moments of flexure of the shaft at different points $H$, $K$, $L$. For the point $H$ this moment is

$$M_1 = R\, l_1,$$

and for the point $K$ it is

$$M_2 = R\, l_2 - G_1 (l_2 - l_1) = R\, l_1 + (R - G_1)(l_2 - l_1),$$

and therefore greater or smaller than $M_1$, as $R$ is greater or smaller than $G_1$.

For the point $L$ this moment is

$$\begin{aligned} M_3 &= R\, l_3 - G_1 (l_3 - l_1) - G_2 (l_3 - l_2) \\ &= R\, l_1 + (R - G_1)(l_2 - l_1) + [R - (G_1 + G_2)]\,(l_3 - l_2), \end{aligned}$$

and this is greater or less than the moment at $K$, according as $R$ proves to be greater or less than $G_1 + G_2$.

If we erect these moments at the corresponding points $H$, $K$, $L$, as ordinates $H\,H$, $K\,K$, $L\,L$, and join the extremities of these ordinates, $A$, $H$, $K$, $L$, $B$, by straight lines, then the ordinates $U\,U_1 \ldots$ will also measure the moments of the intermediate points $U \ldots$

The maximum moment $M$ occurs, therefore, only at one of the points of application $H$, $K$, $L$. Besides the weights $G_1$, $G_2$, $G_3 \ldots$, which the sets of arms transmit to the shaft in definite points $H$, $K$, $L$, we have also the weight $G_0$ of the shaft itself, uniformly distributed along the whole axis $A\,B$. If $l_0$ is the distance $A\,S$ of the centre of gravity $S$ of the shaft from the end $A$, we have for the increase of the pressure $R_1$ at $B$, caused by the weight $G_0$:

$$Z_1 = \frac{l_0}{l}\, G_0,$$

and for the increase at $A$,

$$Z = \frac{l - l_0}{l} \cdot G_0.$$

The weight $G_0$ of the shaft is generally but a small part of the weight $G$ of the wheel, and accordingly the pressures on the journals are but little increased by this weight. It is therefore sufficient to regard the shaft as a prismatic body, and put

$$l_0 = \tfrac{1}{2}\, l,$$

and

$$Z = Z_1 = \tfrac{1}{2}\, G_0.$$

In this case, the weights of the portions *A H*, *A K*, *A L* of the shaft can be taken equal to

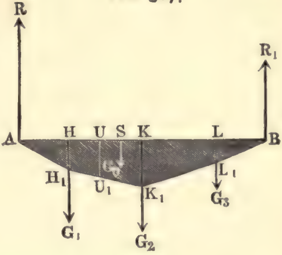


FIG. 387.

$$\frac{l_1}{l}\, G_0, \quad \frac{l_2}{l} \cdot G_0, \quad \frac{l_3}{l}\, G_0,$$

and their moments with respect to the points *H*, *K*, *L* are successively

$$\tfrac{1}{2}\, G_0 \frac{l_1^2}{l}, \quad \tfrac{1}{2}\, G_0 \frac{l_2^2}{l}, \quad \tfrac{1}{2}\, G_0 \frac{l_3^2}{l},$$

so that finally the increase of the bending moments with respect to the points *H*, *K*, *L*, produced by the weight of the shaft, is respectively as follows:

$$Z\, l_1 - \tfrac{1}{2}\, G_0 \frac{l_1^2}{l} = \tfrac{1}{2}\, G_0 \frac{l_1 (l - l_1)}{l},$$

$$Z\, l_2 - \tfrac{1}{2}\, G_0 \frac{l_2^2}{l} = \tfrac{1}{2}\, G_0 \frac{l_2 (l - l_2)}{l},$$

$$Z\, l_3 - \tfrac{1}{2}\, G_0 \frac{l_3^2}{l} = \tfrac{1}{2}\, G_0 \frac{l_3 (l - l_3)}{l}, \text{ and so on.}$$

If we now add these moments to those given above, which correspond to the weights $G_1$, $G_2$, $G_3$, we shall obtain the complete bending moments by which the requisite thickness of the shaft at *H*, *K*, *L* is to be determined. If the shaft is to have at every point the same diameter, this must, of course, be computed from the greatest of these moments. It is better, however, to have the thickness of the shaft gradually diminish towards the ends, in conformity with the decrease of the bending moment. With this object in view, the shaft is very often provided with wings or ribs, whose height gradually diminishes towards the ends of the shaft.

EXAMPLE.—The shaft *A K B*, Fig. 387, is 10 ft. long, weighs $G_0 = 3000$ lbs., and carries at *H* and *K* the sets of arms of a water-wheel weighing 20,000 lbs., and at *L* a driving-wheel of 2000; if the distance $A H = B L = 2 \cdot 0$ ft., and the length *H K*, or width of the water-wheel, is 3·0 ft., what are the pressures on the journals and the bending moments of this wheel?

We have here

$$G_1 = G_2 = 10{,}000, \quad G_3 = 2000,$$

and

$$l_1 = 2, \quad l_2 = 5, \quad l_3 = 8, \quad l = 10.$$

Accordingly, the pressure on the journal at *B* is

$$R_1 = \frac{G_1 l_1 + G_2 l_2 + G_3 l_3}{l} = \frac{20{,}000 + 50{,}000 + 16{,}000}{10} = 8600 \text{ lbs.},$$

while that at *A* is

$$R = G_1 + G_2 + G_3 - R_1 = 13{,}400 \text{ lbs.}$$

The bending moment at *H* is

$$M_1 = R\, l_1 = 13{,}400 \times 2 = 26{,}800 \text{ ft. lbs.};$$

that at *K* is

$$M_2 = R\, l_1 + (R - G_1)(l_2 - l_1) = 26{,}800 + 3400 \times 3 = 37{,}000 \text{ ft. lbs.};$$

and, on the other hand, that at *L* is

$$M_3 = R_1 (l - l_3) = 8600 \times 2 = 17{,}200 \text{ ft. lbs.}$$

By the weight of the shaft, each of the two journal pressures is approximately increased by the amount $\frac{1}{2} G_0 = 1500$, and therefore these pressures amount in all to

$$R_1 = 8600 + 1500 = 10{,}100 \text{ lbs.};$$

$$R = 13{,}400 + 1500 = 14{,}900 \text{ lbs.}$$

Moreover, this last weight increases the bending moment—

at *H* by

$$\frac{1}{2} G_0 \frac{l_1 (l - l_1)}{l} = \frac{2 \cdot 8}{10} \cdot 1500 = 2400 \text{ ft. lbs.};$$

at $K$ by

$$\tfrac{1}{2} G_0 \frac{l_2(l-l_2)}{l} = \frac{5 \times 5}{10} \cdot 1500 = 3750 \text{ ft. lbs.};$$

at $L$ by

$$\tfrac{1}{2} G_0 \frac{l_3(l-l_3)}{l} = \frac{8 \times 2}{10} \cdot 1500 = 2400 \text{ ft. lbs.};$$

so that the total amount is

at $H$ $26{,}800 + 2400 = 29{,}200$ ft. lbs.;
at $K$ $37{,}000 + 3750 = 40{,}750$ " "
at $L$ $17{,}200 + 2400 = 19{,}600$ " "

§ **193.**—It now remains in what follows to state how, from the journal pressures just found, and from the maximum bending moment ($M$), the thickness of the shaft is to be computed.

If the section of the shaft is a circle, and its diameter is $d$, we have from Vol. I., § 236, the allowable moment of flexure with this form of section:

$$M = \frac{\pi}{4} \cdot \left(\frac{d}{2}\right)^3 T = \frac{\pi}{32} \cdot d^3 T,$$

and therefore, inversely, it follows that the corresponding diameter of the shaft is

$$d = \sqrt[3]{\frac{32}{\pi} \frac{M}{T}} = 2 \cdot 17 \sqrt[3]{\frac{M}{T}},$$

or, if $M$ is given, not in inch lbs., but in ft. lbs.,

$$d = 4 \cdot 963 \sqrt[3]{\frac{M}{T}}.$$

Substituting for $T$ its value for cast-iron, 7000 lbs., according to Vol. I., § 240, we obtain

$$d = \frac{4 \cdot 963}{\sqrt[3]{7000}} \sqrt[3]{M} = 0 \cdot 260 \sqrt[3]{M} \text{ inches.}$$

For safety, however, we take $T$ at 4500 only, and therefore put the required thickness of cast-iron shafts at

$$d = 0{\cdot}131 \sqrt[3]{M}, \text{ when } M \text{ is in inch lbs.;}$$
$$= 0{\cdot}300 \sqrt[3]{M}, \text{ when } M \text{ is in ft. lbs.}$$

For wrought-iron shafts, it is sufficient to take

$$d = 0{\cdot}250 \sqrt[3]{M} \text{ inches.}$$

Wooden shafts are to be made two and a half to three times as thick as cast-iron ones.

For shafts square in section, the length of side of the square is

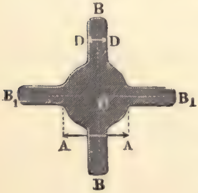
$$s = 0{\cdot}94\, d$$
$$= 0{\cdot}282 \sqrt[3]{M},$$

when they are made of cast-iron.

For a hollow shaft of the diameters $d_1$ and $d_2$, we must again take, instead of $d$,

$$d_1 = \frac{d}{\sqrt[3]{1 - \psi^4}},$$

FIG. 388.



where $\psi$ denotes the ratio $\frac{d_2}{d_1}$.

If the shaft has ribs, and the diameter $A\,A$, Fig. 388, of the cylindrical core is $d_1$, and the total depth $B\,B$ of the rib is $h_1$, and its thickness $D\,D$ is $s_1$, we have for the moment of resistance of this shaft (compare Vol. I., § 228, etc.),

$$M = \left(\frac{\pi}{64} d_1^4 + \frac{(h_1^3 - d_1^3)\, s_1 + (h_1 - d_1)\, s_1^3}{12}\right) \frac{T}{\frac{1}{2} h_1},$$

or, when we put $h_1 - \mu\, d_1$ and $s_1 = \nu\, d_1$,

$$M = \left[\frac{\pi}{16} + \frac{(\mu^3 - 1)\, \nu + (\mu - 1)\, \nu^3}{3}\right] \frac{T d_1^3}{2 \mu}.$$

For a cylindrical shaft of the diameter $d$, this moment is

$$M = \frac{\pi d^3}{32} \cdot T;$$

so that, by equating these moments, we obtain the following for the diameter of the cylindrical core of the ribbed shaft:

$$d_1 = \frac{d \sqrt[3]{\mu}}{\sqrt[3]{1 + \frac{16}{3\pi}\left[(\mu^3 - 1)\nu + (\mu - 1)\nu^3\right]}}.$$

In most cases, $\nu$ is so small in comparison with $\mu$ that the term $(\mu - 1)\nu^3$ may be neglected, and we can put more simply,

$$d_1 = \frac{d \sqrt[3]{\mu}}{\sqrt[3]{1 + \frac{16(\mu^3 - 1)\nu}{3\pi}}}.$$

The ratio

$$\mu = \frac{h_1}{d_1}$$

is usually taken equal to 3, and

$$\nu = \frac{s_1}{d_1}$$

equal to $\frac{1}{9}$, so that we have simply,

$$d_1 = \frac{d \sqrt[3]{3}}{\sqrt[3]{1 + \frac{16 \times 26}{9\pi}}} = 0 \cdot 576\, d,$$

and

$$h_1 = 1 \cdot 727\, d \quad \text{and} \quad s_1 = 0 \cdot 192\, d.$$

If one of the moments $Pr$ and $M$ is much greater than the other, we can leave the smaller moment entirely out of consideration, and consequently compute the size of the shaft from the *larger moment* only—*i. e.*, by the formula

$$d = 0 \cdot 355 \sqrt[3]{Pr},$$

when the moment of torsion $Pr$ is the greater, and, on the other hand, from the formula

$$d = 0 \cdot 300 \sqrt[3]{M},$$

when the maximum bending moment $M$ is the greater. But when the two moments do not differ appreciably, we must compute the size of the shaft by the theory of combined strength (see Vol. I., § 277), which gives the fundamental formulæ,

$$d^3 = \frac{16}{\pi} \frac{Pr}{T} \left( 1 - \frac{32 M}{\pi d^3 T} \right)^{-\frac{1}{2}}$$

and

$$d^3 = \frac{32}{\pi} \frac{M}{T} \left[ 1 - \left( \frac{16 P r}{\pi d^3 T} \right)^2 \right]^{-1}.$$

But if we leave out of consideration

(1) the flexure, and put the necessary size of shaft in that case equal to

$$d_1 = \sqrt[3]{\frac{16}{\pi} \frac{P r}{T}},$$

and

(2) the torsion, and put the corresponding size of shaft equal to

$$d_2 = \sqrt[3]{\frac{32}{\pi} \cdot \frac{M}{T}};$$

we can then put

$$d^3 = d_1^3 \left( 1 - \left( \frac{d_2}{d} \right)^3 \right)^{-\frac{1}{2}},$$

and

$$d^3 = d_2^3 \left( 1 - \left( \frac{d_1}{d} \right)^6 \right)^{-1},$$

or, approximately, according as $d_1$ is greater or less than $d_2$, we have either

(1) $$d = d_1 \left[ 1 + \tfrac{1}{6} \left( \frac{d_2}{d_1} \right)^3 \right],$$

or

$$(2) \qquad d = d_2 \left[ 1 + \tfrac{1}{3} \left( \frac{d_1}{d_2} \right)^6 \right].$$

With the aid of the formula given above,

$$d = 0 \cdot 300 \sqrt[3]{M},$$

we can finally determine the diameter required by the journals, when we suppose these journals to be in their most unfavorable positions, viz., with nothing but their ends resting upon the bearings. If, in this case, $l_0$ be the length, $d_0$ the diameter of the journal, and $R$ the journal pressure, to be determined according to § 192, we have to put

$$d_0 = 0 \cdot 300 \sqrt[3]{R\, l_0}.$$

If the length $l_0$ stands in a certain ratio $\lambda$ to the diameter of the journal, $d_0$, we can put

$$d_0^3 = (0 \cdot 300)^3 \frac{R\, \lambda\, d_0}{12},$$

so that we have, finally,

$$d_0 = \sqrt{\frac{(0 \cdot 300)^3}{12}} \cdot \sqrt{R\, \lambda} = 0 \cdot 0474 \sqrt{R\, \lambda} \text{ inches.}$$

As a rule, $\lambda$ is equal to from 1 to 1·25, and hence the diameter of the journal is

$$d_0 = 0 \cdot 0474 \sqrt{R} \qquad \text{to} \qquad 0 \cdot 0530 \sqrt{R} \text{ inches.}$$

Example.—If the maximum bending moment of a cast-iron water-wheel is $M = 40{,}750$ ft. lbs. (see the example in the previous article), the necessary diameter for the shaft is

$$d = 0 \cdot 300 \sqrt[3]{M} = 0 \cdot 300 \sqrt[3]{40{,}750} = 10 \cdot 3 \text{ inches};$$

and if the pressures upon the two journals of this shaft are respectively $R = 14{,}900$ and $R_1 = 10{,}100$, the requisite diameters of the journals are

$$d_0 = 0{\cdot}0530 \sqrt{14{,}700} = 6{\cdot}43 \text{ inches,}$$

and

$$d_0 = 0{\cdot}0530 \sqrt{10{,}100} = 5{\cdot}33 \text{ inches.}$$

If the shaft had to sustain only the moment of torsion,

$$P\,r = \frac{33{,}000\,L}{2\,\pi\,u} = \frac{5252}{5}\,L = \frac{5252 \times 20}{5} = 21{,}008 \text{ ft. lbs.}$$

(see example in Art. 191), the requisite diameter of the shaft would be

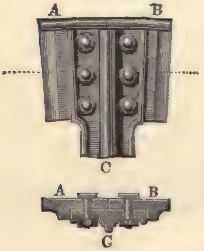
$$d = 0{\cdot}350 \sqrt[3]{21{,}008} = 9{\cdot}66 \text{ inches.}$$

Accordingly, in formula (2) above we put $d_2 = 10{\cdot}3$ inches, and $d_1 = 9{\cdot}66$ inches, and we have the diameter of shaft which is to sustain at the same time the moment of torsion and the bending moment given above,

$$d = d_2\left[1 + \tfrac{1}{8}\left(\frac{d_1}{d_2}\right)^6\right] = 10{\cdot}3\left[1 + \tfrac{1}{8}\left(\frac{9{\cdot}66}{10{\cdot}3}\right)^6\right] = 12{\cdot}63 \text{ inches.}$$

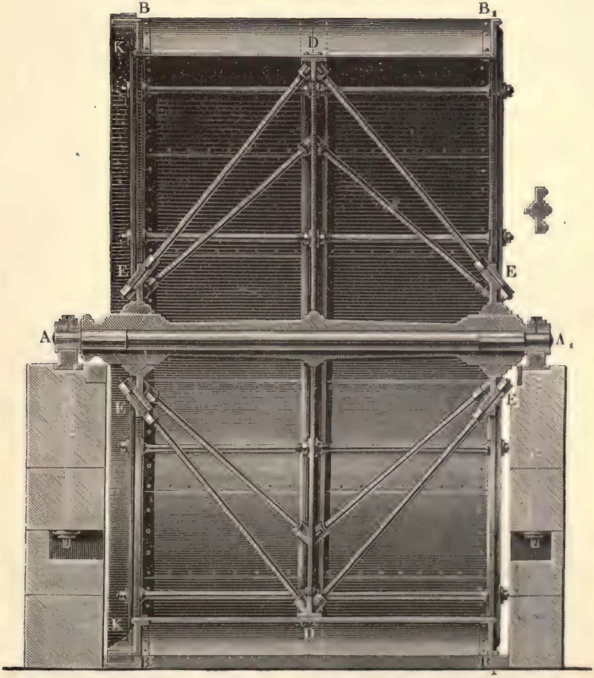
§ **194.—Construction of Water-Wheels.**—In what follows, a somewhat more special treatment of the construction and setting up of overshot water-wheels may be given. The construction of wooden crowns, consisting of a double layer of circular pieces (felloes), has already been considered (§ 172). Wrought-iron crowns are constructed in a similar manner, but cast-iron crowns consist of only one layer of circular pieces. The mode of fastening is, for wooden crowns, by wooden pins or iron nails, for those of wrought-iron by rivets, and for cast-iron crowns by bolts. The main arms, usually wholly, or almost wholly, radial, are generally bolted to the outer faces of the crowns. If the crown is of cast-iron, the bolts by which the felloes *A B*, Fig. 389, are united can be used at the same time for fastening the arm *C*. In the same way the arms are bolted to the rosette. In

FIG. 389.



order that these bolts may have to sustain no force except in the direction of their axes, the ends of the arms are not allowed to rest freely upon the crown, but are to be let into grooves or between lugs on the crowns. To prevent any lateral motion, the wheels are also provided with diagonal arms which extend from the rosette of one crown to the other crown. Such diagonal arms are also used when the wheel is very wide, in which case it carries an intermediate crown, as is shown in Fig. 390. The arms are fastened, at one end by a pin and at the other by screws, in naves or boxes.

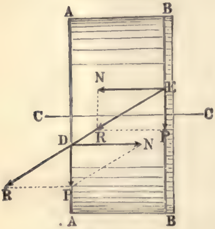
FIG. 390.



When the power is transmitted by a cog-wheel fastened to the crown, instead of thick, rigid arms of wood or cast-iron, slender ties of wrought-iron are often used.

These ties, when put in, are likewise, by means of screws or wedges, stretched so tight that they support the wheel by their tensile strength alone. In order to give a wheel having such arms the necessary rigidity, it must be furnished not only with diagonal ties, but also with special rods, running from one circumference to the other. The latter must not be confounded with the rods (spikes) by which the crowns or arms are joined together; they are rods running round on the inside of the sole, and inclined obliquely towards the crowns, and their object is to transmit the load of one crown *A A*, Fig. 391, to the other *B B*, which carries the driving wheel.

FIG. 391.



Let *P* be a part of the load on the wheel *A A*, and *D E* the tie which has to transmit this load to the crown *B B*. The force *P* is decomposed into a component *N*, parallel to the axis of the wheel *C C*, and into a component *R* in the direction of the tie *D E*. The latter is transmitted through *D E* to the end *E* in the second crown *B B*, and is here decomposed again into the components,

$$\overline{EN} = -N \quad \text{and} \quad \overline{EP} = P.$$

The forces $N$, $-N$ are resisted by the whole system of floats, and the force $\overline{EP} = P$ is added to the load on the crown *B B*, which transmits them both to the driving-wheel.

For wooden shafts, it is best to use oak, but deal and pitch pine are also often used. For star wheels and rosette wheels, the shafts are made polygonal in shape, but for saddle wheels they are made square. The journals of wooden shafts are either wrought-iron and pointed, as in Fig. 392, or wrought-iron hooks, like that in Fig. 393, or cast-iron tenons. The latter consist either of a single plate, as *C D*, Fig. 394, or of several plates. In order that the neck of the shaft may be secured against springing open, it is made somewhat conical, and over it are driven iron rings *A A*, *B B* . . . (Fig. 393), from a quarter to half of an inch thick, and one and a half to three inches wide. Instead of three rings, one single ring *A A* is also used, which embraces the whole neck of the shaft, and forms one piece, with the four flanges of the journal, as is shown in Fig. 395.

FIG. 392.

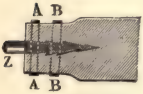


FIG. 393.

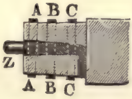
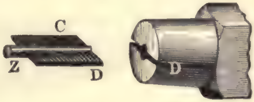


FIG. 394.



In Fig. 396 an octagonal wooden shaft is shown. On the left is seen the end of the journal *A* and the neck *B B* with the three iron rings, and on the right the rear half of the neck *C C* and the journal *E F*, with four flanges *K*, *L* . . . and the tail piece *F G*. We can also see in *a a* and *b b* the wedges which are driven into the neck of the shaft between the rings and flanges from the face outwards.

FIG. 395.

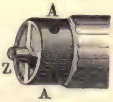
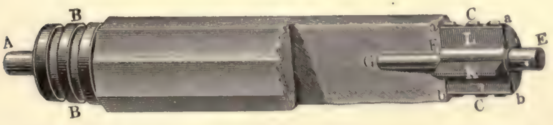
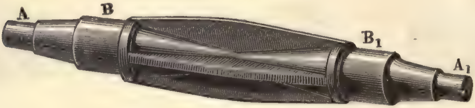


FIG. 396.



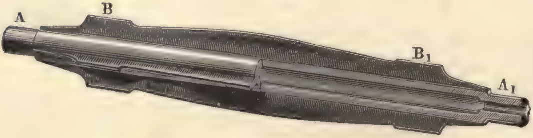
Cast-iron shafts are either solid or hollow. The journals of solid shafts form a part of the shaft itself, and must be accurately turned off in a lathe; those of hollow shafts, on the other hand, are fastened to the body of the shaft, or are inserted in it. The bosses or parts of the shaft, where the naves of the rosettes and cog-wheels are to come, are either simply cylindrical or ribbed, and must have their surfaces accurately turned off. With shafts having cylindrical bosses, the fastening is made with one or two wedges, which enter half into the boss and half into the nave; with shafts having ribbed bosses, each rib is wedged independently into the nave.

FIG. 397.



A solid ribbed shaft with cylindrical bosses is represented in Fig. 397, and a hollow shaft with ribbed bosses in Fig. 398. In both figures, $A$ and $A_1$ are the journals, and $B$ and $B_1$ the bosses.

FIG. 398.



A simple hollow shaft of cast-iron, with inserted journals $A$, $A_1$, is represented in Fig. 390.

The journals rest in pillow blocks, which must be fastened to strong foundations or frames, in order that the wheel may be kept secure in its revolution. Each pillow block consists of a socket and of a bed piece.

The socket is generally made of cast-iron, less often of stone, wood, glass, or composition (eight parts copper to one part tin); it is provided with a cap, as the case may be, and is also used with or without metal lining or brasses.

A pillow block with a wooden bed piece is shown in Fig.

354, and one with an iron foot plate and cap in Fig. 355. A simple, open pillow block is shown in Fig. 399; one with a removable brass *F* in Fig. 400, and a covered pillow block with brasses in Fig. 401. These pillow blocks, together with their foot plates *B B*, are bolted with screw bolts *A A*, either directly to the foundation or to a bed plate *D D*, which is itself firmly fastened to the foundation. In the cap of the pillow block in Fig. 401 an oil-hole *L* is also made, in which an oil-cup can be set. Channels are cut crosswise in the inner surfaces of the brasses, in order that the oil, which flows down through the oil-hole, may be distributed better.

FIG. 399.

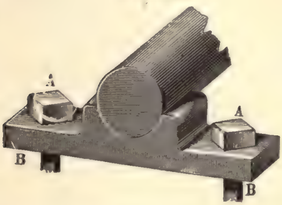


FIG. 400.

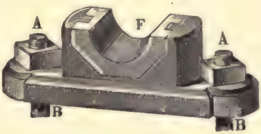
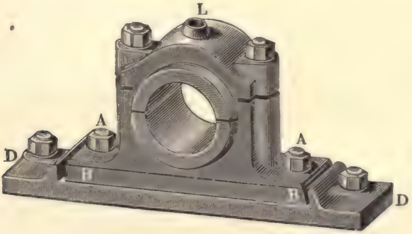


FIG. 401.



§ 195.—**Friction of Water-Wheels.**—A part of the mechanical effect of an overshot-wheel, which is not altogether inconsiderable, is lost in the work consumed by friction.

This friction depends chiefly upon the weight of the wheel $G$, and is $F = \phi G$, where $\phi$ denotes the coefficient of friction. If $r$ is the radius of the journal, and $u$ the number of revolutions per minute, the surface velocity of the journal is

$$v = \frac{\pi u r}{30},$$

and accordingly the work done by friction is

$$L_1 = F v = \phi G v = \frac{\pi u r}{30} \cdot \phi G = 0 \cdot 1047 \, \phi u r G.$$

For journals accurately turned off, according to Vol. I., § 181, $\phi$ is to be taken equal to 0·075, if the journals are lubricated with oil, tallow, or grease; but when the lubrication is carefully attended to, this coefficient falls to $\phi = 0 \cdot 054$, while, on the other hand, with poorer unguents—*e. g.*, with graphite and lard—it may rise to $\phi = 0 \cdot 110$.

The size of a wheel, and hence also its weight, always depend upon its power, and we can assume, if we do not care for other than approximate values, that the weight increases proportionally with the power of the wheel. This weight, however, depends also upon the ratio in which the buckets are filled, and upon the number of revolutions of the wheel; for, if the buckets are filled twice as full, the weight of the wheel is thereby only slightly increased, but its power is nearly doubled; and, on the other hand, if twice as much water is thrown upon a wheel, it will make with the same load or resistance about twice as many revolutions, and therefore give nearly twice the power also. Assuming, then, that the weight of the wheel increases equally with the power $L$, and inversely with the coefficient $\varepsilon$ and the number of revolutions $u$, and introducing another empirical coefficient $\iota$, we can put

$$G = \iota \frac{L}{\varepsilon u}.$$

According to Redtenbacher, for a small iron wheel, with $\varepsilon = \frac{1}{8}$, and $u = 9 \cdot 3$, and $G = 3175$ kilogrammes (6999·6 pounds), the power $L$ is equal to 6·3 (6·2408 English) horse-powers, and hence it follows that

$$\iota = \frac{\varepsilon\, u\, G}{L} = \frac{\frac{1}{8} \times 9 \cdot 3 \times 3175}{6 \cdot 3} = 1560;$$

$$[\text{or} = \tfrac{1}{8} \times \frac{9 \cdot 3 \times 6999 \cdot 6}{6 \cdot 24} = 3477, \text{ English measures}].$$

On the other hand, for a Freiberg wheel of wood, with iron floats, with $\varepsilon = \frac{1}{4}$, $u = 5$, $G = 20{,}000$ kilogrammes (44,092 pounds), $L$ is equal to 20 (19·81 English) horse-powers, and hence we have

$$\iota = \tfrac{1}{4}\,\frac{5 \times 20{,}000}{20} = 1250;$$

$$[\text{or} = \tfrac{1}{4}\,\frac{5 \times 44{,}092}{19 \cdot 81} = 2782, \text{ English measures}].$$

Taking now for $\iota$ the mean of these two values, we obtain the following formulæ for the weight of a wheel:

$$G = 1400\,\frac{L}{\varepsilon\, u} \text{ kilogrammes,}$$

or

$$= 3130\,\frac{L}{\varepsilon\, u} \text{ pounds.}$$

The diameter of the journals depends upon the weight $G$, and as the work of friction depends upon the diameter, this weight has a double influence upon the friction. We have stated (§ 193) the mean diameter of the journals to be

$$2\,r = 0 \cdot 0474 \sqrt{\frac{G}{2}} \text{ inches} = 0 \cdot 00279 \sqrt{G} \text{ ft.};$$

we can therefore put $G\,r = 0 \cdot 00140 \sqrt{G^3}$, and hence the work done by friction is

$$L_1 = 0 \cdot 1047\, \phi\, u \times 0 \cdot 00140 \sqrt{G^3} = 0 \cdot 00015\, u\, \phi \sqrt{G^3},$$

or, when we introduce $L$, the power of the wheel in horse-powers, we have

$$L_1 = 0{\cdot}00015\,\phi\,u\sqrt{\left(\frac{3130\,L}{\varepsilon\,u}\right)^3} = 26{\cdot}27\,\phi\sqrt{\frac{L^3}{\varepsilon^3\,u}}\text{ ft. lbs.}$$

$$= 0{\cdot}0478\,\phi\sqrt{\frac{L^3}{\varepsilon^3\,u}}\text{ horse-powers;}$$

and its ratio to the power left, or useful power, of the wheel is

$$\frac{L_1}{L} = 0{\cdot}0478\,\phi\sqrt{\frac{L}{\varepsilon^3\,u}}.$$

EXAMPLE 1.—What is the mechanical effect consumed by the friction of a water-wheel weighing 25,000 pounds, with journals six inches in diameter when it makes six revolutions per minute? Taking the coefficient of friction $\phi = 0{\cdot}08$, we have for the friction with journals, $\phi\,G = 0{\cdot}08 \times 25{,}000 = 2000$ lbs., and for the statical moment of the same we have $\phi\,G\,r = \frac{1}{4} \times 2000 = 500$ ft. lbs., and, finally, for the work consumed by the friction we have

$$L_1 = 0{\cdot}1047 \times 6 \times \phi\,G\,r = 314\text{ ft. lbs.}$$

2. How much work is lost by the friction of the journals of a water-wheel of 30 horse-powers, the buckets being one third filled, and the number of revolutions being four? It would be

$$L_1 = 0{\cdot}0478\,\phi\sqrt{\frac{L^3}{\varepsilon^3\,u}}\cdot = 0{\cdot}0478 \times 0{\cdot}08\sqrt{\frac{30 \times 27}{4}}\cdot L = 0{\cdot}054\,L,$$

*i. e.*, about $5\frac{1}{2}$ per cent of the useful work, or $1\frac{2}{3}$ horse-powers.

NOTE.—The friction of the journals may also be increased or diminished by the manner of making connections with the rest of the machinery. If, as is shown in Fig. 402, the power $P$ and the load $Q$ act on the same side, the

FIG. 402.

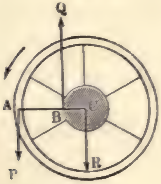
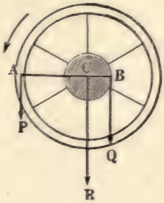


FIG. 403.



pressure on the journals is diminished; in this case then the friction comes out smaller; if, however, the power and load act on opposite sides of the wheel, as is shown in Fig. 403, then the pressure $R$ is increased by the load $Q$, and therefore the friction is, in this case, as much greater than what would be produced by either $P$ or $Q$ alone as it was in the former case smaller. If, in the first case, we make the lever arm $CB$ of the load equal to the lever arm $CA$ of the power, by transmitting the power, for example, by means of a cog-wheel fastened directly to one of the crowns, as is shown, for example, in Fig. 355, Art. 172, the effect of the power upon the journals is then almost entirely counteracted by the load. The other advantages of this mode of construction have already been mentioned above (Art. 190).

**§ 196.—Total Delivery.**—We may now write for the total delivery of an overshot-wheel,

$$L = \left(\frac{(c_1 \cos. \alpha_1 - v_1)v_1}{g} + h_3 + \xi h_4\right) Q\gamma - \phi \frac{\rho}{r} G v,$$

where $\rho$ is the radius of the axle, or if the water is introduced in a direction nearly tangential and with the velocity $c_1 = 2 v_1$, and if we take approximately $v_1 = v$ so that

$$\frac{(c_1 \cos. \alpha_1 - v_1) v_1}{g} = \frac{v^2}{g},$$

we have

$$L = \left(\frac{v^2}{g} + h_3 + \xi h_4\right) Q\gamma - \phi \frac{\rho}{r} G v.$$

If, now, according to the preceding, we put the wheel weight

$$G = 3130 \frac{L}{\varepsilon u} \text{ pounds},$$

and hence the work consumed by axle friction

$$L_1 = 26 \cdot 27\, \phi \sqrt{\frac{L^3}{\varepsilon^3 u}} \text{ ft. lbs.},$$

we have for the total delivery of the wheel

$$L = \left(\frac{v^2}{g} + h_3 + \xi h_4\right) Q\gamma - 26 \cdot 27\, \phi \sqrt{\frac{L^3}{\varepsilon^3 u}}.$$

Since for the generation of the velocity $c = 2\,v$, the fall

$$4 \times 1 \cdot 1 \times \frac{v^2}{2\,g} = \frac{4 \cdot 4}{2\,g}\left(\frac{\pi\,u\,r}{30}\right)^2 = 0 \cdot 000748\,u^2\,r^2$$

is necessary, we have remaining the fall $h - \frac{4 \cdot 4}{2\,g}\left(\frac{\pi\,u\,r}{30}\right)^2$.

If we put, for the sake of simplicity,

$$h_3 + \xi\,h_4 = \chi\left[h - \frac{4 \cdot 4}{2\,g}\left(\frac{\pi\,u\,r}{30}\right)^2\right]$$

where $\chi$ is a fraction (about $\frac{2}{3}$ or $\frac{3}{4}$, etc.). we have for the delivery of the wheel

$$L = \left(\frac{1}{g}\left(\frac{\pi\,u\,r}{30}\right)^2 + \chi\left[h - \frac{4 \cdot 4}{2\,g}\left(\frac{\pi\,u\,r}{30}\right)^2\right]\right) Q\,\gamma - 26 \cdot 27\,\phi\sqrt{\frac{L^3}{\varepsilon^3\,u}},$$

or approximately, if we put $4 \cdot 4\,\chi\frac{v^2}{2\,g} - \frac{v^2}{g} = \frac{v^2}{g} = \frac{1}{g}\left(\frac{\pi\,u\,r}{30}\right)^2$;

$$L = \chi\left[h - \frac{1}{g}\left(\frac{\pi\,u\,r}{30}\right)^2\right] Q\,\gamma - 26 \cdot 27\,\phi\sqrt{\frac{L^3}{\varepsilon^3\,u}}\,.$$

But we can also put for the work of friction, approximately,

$$L = \chi\,h\,Q\,\gamma \text{ ft. lbs.} = \frac{\chi\,h\,Q\,\gamma}{550} \text{ horse-power,}$$

hence

$$L = \left[h - \frac{1}{g}\left(\frac{\pi\,u\,r}{30}\right)^2 - 26 \cdot 27\,\phi\sqrt{\frac{\chi\,h^3\,Q\,\gamma}{(550\,\varepsilon)^3\,u}}\right]\chi\,Q\,\gamma,$$

or, if $g = 32 \cdot 2$ ft. and $\gamma = 62 \cdot 5$ lbs.,

$$L = \left[h - 0 \cdot 000340\,(u\,r)^2 - 0 \cdot 01602\,\phi\sqrt{\left(\frac{h}{\varepsilon}\right)^3\frac{\chi\,Q}{u}}\right]\chi\,Q\,\gamma \text{ ft. lbs.}$$

We see, from the way in which $u$ occurs in this expression, that the delivery $L$ is a maximum for some value of $u$ between $u = 0$ and $u = \infty$. This value is found by the calculus

$$u = \sqrt[5]{\frac{\chi\, Q\, \gamma}{550^3}(5 \cdot 55\, \phi\, g)^2 \left(\frac{h}{\varepsilon}\right)^3 \left(\frac{30}{\pi\, r}\right)^4},$$

or, in English measures,

$$u = 2 \cdot 511 \sqrt[5]{\frac{\chi\, \phi^2\, Q}{r^4}\left(\frac{h}{\varepsilon}\right)^3},$$

or if we put approximately $r = \frac{1}{2}\, h$,

$$u = 4 \cdot 372 \sqrt{\frac{\chi\, \phi^2\, Q}{\varepsilon^3\, h}}.$$

In practice, $u$ is generally taken somewhat greater, in order to obtain a uniform rate of revolution.

If we insert this value of $u$ in the expression for $L$, we have for the *maximum delivery* of the wheel,

$$L = \left[h - 0 \cdot 002144 \sqrt[5]{(\chi Q\, r)^2 \phi^4 \left(\frac{h}{\varepsilon}\right)^6}\right.$$

$$\left. - 0 \cdot 01011 \sqrt{(\chi\, Q\, r)^2 \phi^4 \left(\frac{h}{\varepsilon}\right)^6}\right] \chi\, Q\, \gamma$$

$$= \left[h - 0 \cdot 012254 \sqrt[5]{(\chi\, Q\, r)^2 \phi^4 \left(\frac{h}{\varepsilon}\right)^6}\right] \chi\, Q\, \gamma.$$

The *efficiency* of the overshot water-wheel is then, since the total work at disposal is $Q\, h\, \gamma$, equal in general to

$$\eta = \frac{\left(h_1 + \xi\, h_2 + \frac{(c_1\, cos.\, \alpha_1 - v_1)\, v_1}{g}\right) Q\, \gamma - \phi \frac{\rho}{r} G\, v}{Q\, h\, \gamma}.$$

The maximum value is from the above

$$\eta = \frac{L}{Q\, h\, \gamma} = \chi \left( 1 - \frac{0 \cdot 012254 \sqrt[5]{(\chi\, Q\, r)^2 \phi^4 \left(\frac{h}{\varepsilon}\right)^6}}{h} \right)$$

EXAMPLE. 1.—For an overshot water-wheel, which utilizes the fall $h = 35$ ft. and uses $Q = 5$ cubic ft., the coefficient of fill $\varepsilon$ being $\frac{1}{4}$, the coefficient of friction $\phi = 0 \cdot 1$ and the coefficient of fall $\chi = \frac{5}{6}$; we have for the best number of revolutions,

$$u = 4 \cdot 372 \sqrt[5]{\tfrac{5}{6} \cdot \frac{0 \cdot 01 \times 5 \times 64}{35}} = 4 \cdot 372 \sqrt[5]{0 \cdot 0762} = 2 \cdot 61.$$

2. For $h = 10$, $Q = 15$, $\varepsilon = \frac{1}{3}$ and $\chi = \frac{4}{5}$, we have the best number of revolutions per second,

$$u = 4 \cdot 372 \sqrt[5]{\tfrac{4}{5} \cdot \frac{0 \cdot 01 \times 15 \times 27}{10}} = 450 \sqrt[5]{0 \cdot 324} = 3 \cdot 50.$$

§ **197.—Effective Delivery.** — Many experiments have been made upon the overshot water-wheel by Smeaton, Nordwall, Morin, and others, but much still remains to be investigated, especially in the case of very high wheels, and those of the best construction. The delivery of the first is not yet satisfactorily determined by experiment, and is generally assumed too small. Smeaton made experiments with a model of 75 inches circumference and 36 buckets. He found the greatest efficiency $0 \cdot 74$ for $u = 20$. D'Aubisson gives, in his Hydraulics, an efficiency of $0 \cdot 76$ for a wheel $11\frac{1}{3}$ metres high, running at a velocity $u = 2\frac{1}{2}$ metres. The author found, for a wheel 7 metres high, $\frac{6}{7}$ metre wide and with 48 buckets, the greatest efficiency $0 \cdot 78$ for $u = 12$. For wheels from 10 to 11 metres high and $u = 5$, he found the efficiency $0 \cdot 80$, and often still higher. It may, indeed, be easily shown that the efficiency of a very high overshot-wheel, making from 3 to 4 revolutions per minute, may rise as high as $0 \cdot 83$, thus leaving for the entrance velocity 3 per cent, for the premature emptying 9 per cent, and for the axle friction 5 per cent of lost work. Small wheels have always a smaller efficiency, not only because they make a greater number of revolutions, but also because the water-retaining arc is smaller. The most numerous and complete experiments upon water-wheels are by Morin (" Expériences sur les roues hydrauliques à aubes planes et sur les roues hydrauliques à augets." Metz, 1836). Of these, however, only three can be noticed in this connection, and these upon small wheels. The first of these wheels was of wood, 3·425 metres diameter and 30 buckets, and for $1\frac{1}{2}$ metres velocity gave an efficiency of 0·65, while the coefficient of fall was $\chi = 0 \cdot 775$. The second wheel, also of wood, was only 2·28 metres in diameter, and had 24 curved sheet-iron buckets. The efficiency for 1·5 metres velocity was $\eta = 0 \cdot 69$ and the coefficient of fall $\chi = 0 \cdot 762$. The third wheel was of iron, 4 metres high, with 20 buckets, and at least 1 metre fall above the crown. It gave for $1\frac{1}{2}$ metres velocity the efficiency 0·55 to 0·60, for a velocity of $3\frac{1}{2}$ metres the efficiency was $\eta = 0 \cdot 40$, and for 4 metres velocity $\eta$ was only 0·25, because the centrifugal force impeded the entrance of the water. Morin draws from his experiments the conclusion that for wheels less than 2 metres diameter, with a velocity, at most, of 2 metres, as also for wheels above 2 metres diameter, with at most a velocity of $2\frac{1}{2}$ metres, the coefficient $\chi$ of the pressure fall is in the mean 0·78, and, therefore, the delivery of the overshot-wheel, without regard to axle friction, is

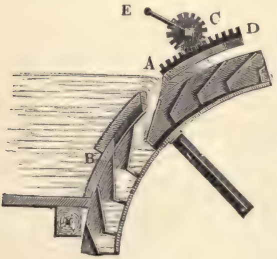
$$P v = \left(\frac{(c \cos. \alpha - v) v}{g} + 0 \cdot 78\, h\right) Q \gamma$$

where $h$ is the height of the point of entrance above the lowest point of wheel, and hence $0 \cdot 78\, h$ is the mean height of the water-retaining arc. This coefficient $\chi = 0 \cdot 78$ is, however, to be used only when the coefficient of fill $\varepsilon$ is below $\frac{1}{2}$. It becomes, according to Morin, 0·65 when $\varepsilon$ is near $\frac{2}{3}$. It is certain that for high wheels $\chi$ is greater than this, even in some cases as high as 0·9. Morin also concludes that wheels which have a very great velocity (over 2 metres) or whose coefficient of fill is above $\frac{2}{3}$, the coefficient $\chi$ for the water-retaining arc cannot be definitely given, because in such cases small variations in $\chi$ and $\varepsilon$ have great influence upon the delivery. We may, however, remark, in passing, that it is not the velocity but the number of rotations $u$ (Art. 187) which determine a limit, for high wheels give, even for two metres velocity, a high and tolerably definite efficiency.

§ **198. — Back-pitch Water-Wheels.** — The *back-pitch* water-wheel (Fr. roue par derrière; Ger. rückenschlägiges Rad) differs from the overshot only in the way in which the water is introduced. In the overshot, the water enters at or near the summit; in the back-pitch, between the summit and middle, but nearer the summit, upon the same side of the summit as the entering water. In the first case, the race passes over the wheel; in the second it only comes close up to it. In the first case, the height of wheel is less than the total fall, but here it may be, and generally is, greater. Finally, in the first case, the wheel revolves in the direction of the entering water as it leaves the head race; in the second, in the opposite direction. We therefore apply the back-pitch wheel, more especially in situations where the feed and exit water levels are subject to considerable variations, because the wheel revolves in the direction in which the water departs, and drowning of the wheel is prevented, and because an arrangement of sluice is possible, which renders the aperture for feed adjustable, so that it may always be kept at a given distance below the surface of water in head race, and thus, even for variable levels, the same entrance velocity be preserved. Sluices for the back-pitch wheel are shown in Figs. 404 and 405.

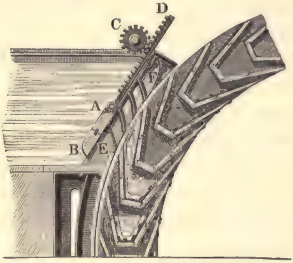
In Fig. 404 the water flows over the head *A* of the sluice board *A B*, which is concentric with the wheel, in order that the water may be properly directed in all positions of the sluice. The sluice board is moved by a rack *A D* and a pinion *C*. In Fig. 405 the water flows over the head *A* of the sluice, as in the previous case. But in order that the water may enter in the proper direction, it is further directed by guide boards over which it glides. These guide boards must have a

FIG. 404.



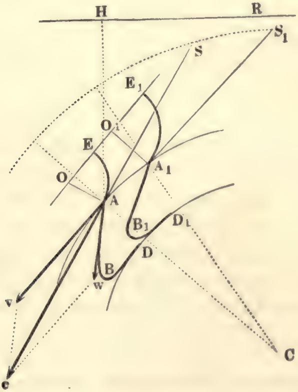
certain direction in order that the water in entering may not strike against the outer float ends.

FIG. 405.



If *A w*, Fig. 406, is the direction of the outer float end, and *A v* the velocity at the circumference, then we can find precisely, as in Art. 178, the necessary direction *A c* of the entrance water, by drawing *v c* parallel to *A w*, and making *A c* equal to the entrance velocity *c*, given by the head above *A*. If *h* is this head *A H*, or distance of *A* below the water surface *H* in the head race, then we have at least $c = 0{\cdot}82\sqrt{2\,g\,h}$, as for discharge through short tubes (see Weisbach, Vol. I., Art. 421). If, however, the canal formed by the guide boards is rounded off, the coefficient of discharge is greater, so that we may have

FIG. 406.



$c = 0 \cdot 9 \sqrt{2gh}$. If we use plane guide buckets, we must put them in the direction $c A S$. If, however, we curve the guide boards so as to change the direction of the water gradually, they must be tangent to $A S$ at $A$; thus we may draw $A O$ perpendicular to $A S$ and describe a circular arc with radius $A O$.

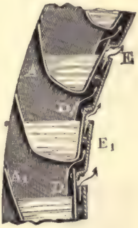
Since we may have several such guide buckets instead of one, there will be several values of $h$ and $c$, and the construction must be made for each. Ordinarily, we make the entrance velocity $c = 9$ to 10 ft. and the wheel velocity $v = \frac{1}{2} c$ to at most $= \frac{2}{3} c$. We then make the construction for the mean height of water in the head race, in order that the deviation for the highest and lowest levels may not be too great.

The air in these sluices can escape less readily than for those without guide boards, wherefore the sluice should be smaller than the wheel opening, or the wheel itself should be *ventilated*—*i. e.*, furnished with air-holes in the sole plate. It is also not advisable to make the angle $\beta$ for the floats too sharp, but rather to retain the water in the floats by a breast or *apron* or *curb* rather than by the curvature of the floats themselves. When the back-pitch wheel is thus furnished with a *curb* or apron, we may distinguish it as a *high-breast* wheel.

Finally, as to the *efficiency* of the back-pitch, it is at least equal to the overshot; on account of the method of introducing the water, it is often even greater than for the overshot-wheel, other things being the same. Morin found for a wheel of $9 \cdot 1$ metres high, with 96 buckets, water entering at 50° from summit, $1\frac{1}{2}$ metres velocity at circumference ($v$) and $2\frac{1}{2}$ metres entrance velocity ($c$) $\eta = 0 \cdot 69$, the height of the water-retaining arc being, however, $0 \cdot 78\, h$.

§ **199.—Ventilated Back-pitch Water-Wheels.**—If the back-pitch water-wheel is ventilated, so that the air can escape through canals $D E D_1 E_1$, Fig. 407, we can put the floats nearer, and thus have a greater number of buckets than in the unventilated wheel, whereby, other things being equal, we have more water capacity than for the overshot-wheel, so that we may take the coefficient of fill $\varepsilon$ from $\frac{1}{3}$ to $\frac{1}{2}$. For the ordinary float construction, we have approximately for the capacity or water cross-section of the bucket $A\,B\,D\,F$, Fig. 408, $A\,B\,D\,H = A\,E\,D\,F$ minus $A\,B\,E$ minus $A\,F\,H = \psi\, r_1\, d - \frac{1}{4}\, r_1\, \psi\, d - \frac{1}{2}\, d^2\, tan.\, \lambda$ where $\psi$ is the angle $A\,C\,B$ and $\lambda = C\,A\,H = A\,C\,M$, and $B\,E = \frac{1}{2}\,D\,E = \frac{d}{2}$. On the other hand, the entire cross-section of a bucket is $E\,D\,D_1\,E_1 = \phi\, r_1\, d$ where $\phi$ is the angle $E\,C\,E_1 = A\,C\,A_1$. Hence the coefficient of fill is

FIG. 407.

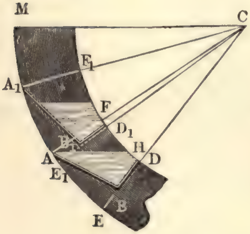


$$\varepsilon = \frac{area\; A\,B\,D\,H}{area\; E\,D D_1 E_1} = \frac{\frac{3}{4}\psi\, r_1 - \frac{1}{2}\, d\, tan.\,\lambda}{\phi\, r_1},$$

and hence

$$tan.\,\lambda = (\tfrac{3}{4}\,\psi - \varepsilon\,\phi)\,\frac{2\,r_1}{d}\,.$$

FIG. 408.



The greatest utilization of space will then occur when the water surface $A\;H$ just touches the following float at $B_1$. When this is the case, then, since $B\,D = B\,E$, $B_1\,E_1 = B_1\,D_1$, $B_1\,H = B_1\,A$, $D_1\,H = D_1\,F$,

$$\tfrac{1}{2}\,d\; tan.\,\lambda = (\psi - \phi)\,r_1,$$

or also

$$tan.\,\lambda = (\psi - \phi)\,\frac{2\,r_1}{d}.$$

From these two values for $\lambda$ we have the simple formula

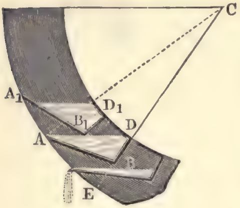
$$\tfrac{3}{4}\psi - \varepsilon\,\phi = \psi - \phi \quad \text{or} \quad \phi = \frac{\psi}{4\,(1 - \varepsilon)}.$$

If we take $\varepsilon = \frac{1}{2}$, we have

$$\phi = \frac{\psi}{2},$$

and the cross-section of the water forms a triangle $A\,B\,D$, Fig. 409, whose sides are $A\,B$ and $B\,D$.

FIG. 409.



The angle $A\,C\,B = \psi$ is determined by the entrance angle

$$B\,A\,E = \beta$$

by means of the formula

$$sin.\ A\,B\,C = \frac{C\,A\ sin.\ C\,A\,B}{C\,B}$$

or

$$cos.\ (\beta - \psi) = \frac{r\ cos.\ \beta}{r - \frac{1}{2}\,d} \quad . \quad . \quad (1),$$

From this we have for the angle $A\,C\,B = \psi$,

$$\psi = \beta - (\beta - \psi) \quad . \quad . \quad . \quad . \quad . \quad (2),$$

or, from the above formula,

$$\phi = \frac{\psi}{4\,(1 - \varepsilon)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and finally, for the number of floats,

$$n = \frac{2\,\pi}{\phi} = \frac{360^\circ}{\phi^\circ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

EXAMPLE.—For a back-pitch wheel of 15 ft. radius, 1 ft. depth of crown and $\beta = 20^\circ$, we have

$$cos. (\beta - \psi) = \frac{15\ cos.\ 20^\circ}{14 \cdot 5},$$

or

$$\beta - \psi = 13^\circ\ 34'$$

and

$$\psi = 20^\circ - 13^\circ\, 34' = 6^\circ\ 26'\ ;$$

finally, for

$$\varepsilon = \tfrac{1}{2},\ \phi = \frac{6^\circ\ 26'}{2} = 3^\circ\ 13'$$

and

$$n = \frac{360 \times 60}{3 \times 60 + 13} = \frac{21600}{193} = 112.$$

For the point of overflow,

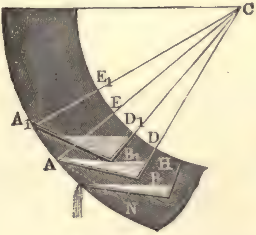
$$tan.\ \lambda = (\psi - \phi)\,\frac{2\,r_1}{d} = 30 \text{ arc } 3^\circ\ 13' = 1 \cdot 684,$$

and hence

$$\lambda = 59^\circ\ 18'.$$

§ **200.**—If the coefficient of fill $\varepsilon$ is less than $\frac{1}{2}$, then, at the moment of exit the water does not fill the space *A B D*, Fig. 410, and the formula for the water-retaining arc may be obtained as follows:

FIG. 410.



The cross-section of the water in a bucket is

$$\triangle\, A\,B\,H = \triangle\, A\,N\,H - \triangle\, A\,N\,B = \tfrac{1}{2}\, A\,N\,(N\,H - N\,B).$$

But

$$AN = CA\ sin.\ ACB = r\ sin.\ \psi,$$

$$NB = AN\ cot.\ ABN = r\ sin.\ \psi\ tan\ (\beta - \psi),$$

and

$$NH = AN\ cot.\ AHN = r\ sin.\ \psi\ cot.\ (\lambda + \psi).$$

Hence

$$\triangle\ ABH = \tfrac{1}{2} r^2\ sin.^2\ \psi\ [cot.\ (\lambda + \psi) - tan.\ (\beta - \psi)],$$

and the coefficient of fill

$$\varepsilon = \frac{\triangle\ ABH}{AEE_1A_1} = \frac{\frac{1}{2} r^2\ sin.^2\ \psi\ [cot.\ (\lambda + \psi) - tan.\ (\beta - \psi)]}{d\ r\ \phi},$$

or inversely

$$cot.\ (\lambda + \psi) = tan.\ (\beta - \psi) + \frac{2\ \varepsilon\ \phi\ d}{r\ sin.^2\ \psi}.$$

If here also the surface of the departing water just touches the following float, we have approximately

$$tan.\ \lambda = (\psi - \phi)\frac{2\ r}{d},$$

and from these two equations $\phi$ and $\lambda$ may be determined.

We may put

$$cot.\ (\lambda + \psi) = \frac{cot.\ \lambda\ cot.\ \psi - 1}{cot.\ \lambda + cot.\ \psi} = \frac{1 - tan.\ \lambda\ tan.\ \psi}{tan.\ \psi + tan.\ \lambda},$$

and therefore, substituting the above value for *tan.* $\lambda$,

$$cot.\ (\lambda + \psi) = \frac{1 - (\psi - \phi)\frac{2\ r}{d}\ tan.\ \psi}{tan.\ \psi + (\psi - \phi)\frac{2\ r}{d}} = \frac{d - 2\ r\ (\psi - \phi)\ \psi}{d\ \psi + 2\ r\ (\psi - \phi)},$$

if we put approximately $tan.\, \psi = \psi$. Hence

$$\frac{d - 2\,r\,(\psi - \phi)\,\psi}{d\,\psi + 2\,r\,(\psi - \phi)} = tan.\,(\beta - \psi) + \frac{2\,\varepsilon\,\phi\,d}{r\,\psi^2},$$

and

$$\phi = \frac{r\,\psi^2}{2\,\varepsilon\,d}\left(\frac{d - 2\,r\,(\psi - \phi)\,\psi}{d\,\psi + 2\,r\,(\psi - \phi)} - tan.\,(\beta - \psi)\right),$$

whence we have for the number of floats

$$n = \frac{2\,\pi}{\phi} = \frac{6{\cdot}28}{\phi}.$$

EXAMPLE.—Thus, if in the example of the preceding article we have $\varepsilon = \frac{1}{4}$ instead of $\frac{1}{2}$, we have

$$\phi = \frac{15 \times 0{\cdot}1123^2}{2 \times \frac{1}{4}}\left(\frac{1 - 30 \times 0{\cdot}1123\,(0{\cdot}1123 - \phi)}{0{\cdot}1123 + 30\,(0{\cdot}1123 - \phi)} - 0{\cdot}2413\right)$$

$$= 30 \times 0{\cdot}012611\left(\frac{1 - 3{\cdot}369\,(0{\cdot}1123 - \phi)}{0{\cdot}1123 + 30\,(0{\cdot}1123 - \phi)} - 0{\cdot}2413\right)$$

$$= 0{\cdot}37833\left(\frac{0{\cdot}62166 + 3{\cdot}369\,\phi}{3{\cdot}4813 - 30\,\phi} - 0.2413\right).$$

If we assume as a first approximate value $\phi = \frac{1}{20} = 0{\cdot}05$, and insert, we have $\phi = 0{\cdot}3783 \times 0{\cdot}1575 = 0{\cdot}0596$. If we assume again $\phi = 0{\cdot}04$, we have $\phi = 0{\cdot}3783 \times 0{\cdot}0903 = 0{\cdot}0342$. We may accordingly take

$$\phi = 0{\cdot}044 \quad \text{or} \quad \phi^0 = 2^\circ\ 31'.$$

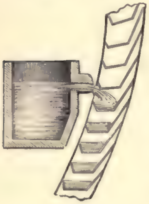
Then we have for the number of floats

$$n = \frac{6{\cdot}28}{\phi} = \frac{6{\cdot}28}{0{\cdot}044} = 143,$$

or for more convenient division, $n = 136$.

§ 201.—**Middleshot or Breast Wheel.**—Middleshot wheels are either ordinary middleshot or breast wheels. The first are bucket wheels, like the overshot and back-pitch. The latter are, however, float wheels enveloped by an apron or curb (Art. 170). Since the greatest loss of fall or work is due to the untimely exit of the water from the buckets, it is apparent that, other things being the same, middleshot wheels have a less efficiency than the overshot or back-pitch. For this reason, we have to keep the water as long as possible in the wheel, and for this purpose may introduce the water upon the interior of the wheel, as shown in Fig. 411, or may surround it with an apron, *breast*, or curb, and make the buckets simply of radial floats. Such wheels we may therefore designate as *breast* wheels. Thus the back-pitch water-wheel, if furnished with an apron or curb, we may term a high *breast* wheel, and its efficiency, etc., is to be determined by the following. If without such breast or apron, it is an overshot wheel simply. In like manner, a wheel is middleshot according as it receives its water at or near the centre, but if furnished with a curb, its buckets become radial floats, and it becomes a *breast* wheel. The apron or curb should allow as little clearance as possible between it and the floats, say $\frac{1}{2}$ to 1 inch, in order that as little water as may be may escape. As to the floats themselves, they may be quite radial, since they have no longer to retain the water; that office being performed by the apron. In order, however, that they may offer no resistance in rising out of the water in the tail race, it is advisable to set them at such an inclination that upon exit from the water they may be vertical. As to the number of floats, it is here also advantageous to make it large, not only because the loss of water between the curb and float ends is thereby diminished, but also because for a larger number the impact fall is less and the pressure fall greater. Ordinarily, we make the outer distance between

FIG. 411.



every two floats equal to the crown depth, or take it at from 10 to 15 inches; we may also apply the rules already given for the determination of the number of floats (Art. 175). It is, however, essential that the breast wheel should be ventilated, because the entering water here fills nearly the whole cross-section between two floats, so that the air has no other outlet. We must, therefore, have openings in the sole plate for the escape of the air. This is the more necessary as in these wheels the buckets are filled from $\frac{1}{2}$ to $\frac{5}{8}$ of their capacity. Breast wheels occur in falls of from 5 to 15 ft., and for a feed of from 5 to 80 cubic ft. per second.

REMARK.—Theoretical investigations and experiments upon middleshot and undershot wheels, the water being introduced upon the interior, made in Sweden, will be found very completely detailed in the work, "Hydrauliska Försök," etc., of Lagerhjelm, of Forselles och Kallstenius, Andra Delen, Stockholm, 1822. Egen describes such a wheel in his "Untersuchungen über den Effect einiger Wasserwerke," etc., Berlin, 1831. This wheel was erected by the Count de Thiville at the Salt Works near Werl, with the expectation that a high efficiency would be attained. Egen found, however, an efficiency of only 59 per cent, although the wheel had a fall of 13·42 ft. After this wheel, another of only 2 metres height was built in France (see "Bulletin de la Société d'Encouragement," No. 282), and examined by Mallet. The efficiency of this wheel was found not more than 60 per cent. Egen concludes therefore, with justice, that wheels with interior feed are but rarely to be recommended.

**§ 202.—Overflow Sluice.**—The methods of letting on the water in breast wheels are various; either by a dam or overflow sluice, or guide-bucket sluice, or underflow sluice; more rarely the water flows on freely. In the overflow sluice *A S*, Figs. 412, 413, the water flows over the top *A* of the sluice board. In order that it may have the proper direction, it is necessary to round off the top or make use of a guide board. This guide board *A B*, Fig. 414, should be curved according to the parabola which the lowest elements of the water describe; for if more curved, the water would not follow it, and if less, either its breadth or the friction of the water would be greater, or the water would not enter in the desired direction.

The theory of discharge through an overflow weir (see Art. 411) gives us for the discharge, when $e_1$ is the width, $h_0$ the pressure height $HA$, Fig. 414, over the sill, and $\mu$ the coefficient of discharge,

FIG. 412.

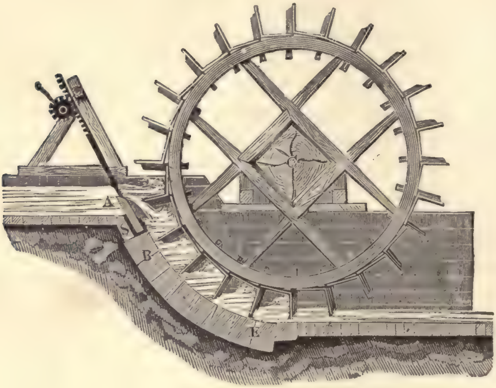
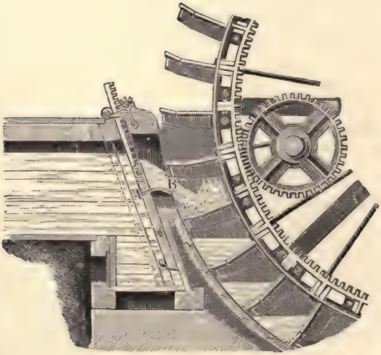


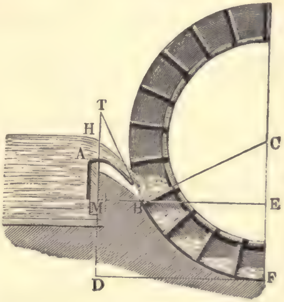
FIG. 413.



$$Q = \tfrac{2}{3} \mu e_1 h_0 \sqrt{2 g h_0}.$$

If, now, $Q$ is given as also $e_1$, since this last is somewhat less (3 to 4 inches) than the wheel width $e$, we have for the height

FIG. 414.



$$h_0 = \left(\frac{\frac{3}{2} Q}{\mu e_1 \sqrt{2 g}}\right)^{\frac{2}{3}}$$

$$= 0 \cdot 3269 \left(\frac{Q}{\mu \varepsilon_1}\right)^{\frac{2}{3}}.$$

Now, the velocity $c$ of the water entering at $B$ is determined by the ratio $\kappa = \frac{c}{v}$, hence the fall necessary for this velocity is

$$H M = h_1 = \frac{c^2}{2 g} = \frac{(\kappa v)^2}{2 g},$$

or, by reason of the loss due to friction,

$$h_1 = 1 \cdot 1 \frac{(\kappa v)^2}{2 g}.$$

Ordinarily we have $\kappa = 2$, and hence $h_1 = 4 \cdot 4 \frac{v^2}{2 g}$. Hence for the height $A M = H M - H A$, we have $x = h_1 - h_0$; and if the total fall $H D = h$, there remains for the pressure fall $M D = E F = h_2 = h - h_1$. We have also from the theory of projectiles the angle $T B M = \nu$ of the guide board with the horizon given by

$$x = \frac{c^2 \sin.^2 \nu}{2 g},$$

therefore

$$\sin. \nu = \sqrt{\frac{x}{h_1}} = \sqrt{\frac{h_1 - h_0}{h_1}},$$

and the distance

$$MB = y = \frac{c^2 \, sin. \, 2\,v}{2\,g} = h_1 \, sin. \, 2\,v.$$

Finally, if the water enters tangentially,

$$CB = CF = r$$

is given by the equation,

$$r\,(1 - cos.\,v) = h - h_1, \qquad \text{or} \qquad r = \frac{h - h_1}{1 - cos.\,v}.$$

Inversely, we have for the angle $BCF = \theta$ of the water-holding arc,

$$cos.\,\theta = 1 - \frac{h - h_1}{r}.$$

If $v$ is not $= \theta$, we have for the deviation of the entering stream from the direction of motion of the bucket,

$$\alpha = \theta - v.$$

EXAMPLE.—For a breast wheel with overflow sluice, and feed $Q = 6$ cubic ft., with total fall $h = 8$ ft., and velocity $v = 5$ ft., $\varepsilon$ being $\frac{2}{5}$, we have for 1 ft. depth of crown,

$$e = \frac{5}{2}\,\frac{Q}{d\,v} = \frac{5 \times 6}{2 \times 1 \times 5} = 3 \text{ ft.}$$

If therefore we make the width of the entering stream $2\frac{3}{4}$ ft., and put $\mu = 0{\cdot}6$, we have

$$h_0 = 0{\cdot}3269 \left(\frac{6}{0{\cdot}6 \times \frac{11}{4}}\right)^{\frac{2}{3}} = 0{\cdot}3269 \left(\frac{4}{1{\cdot}1}\right)^{\frac{2}{3}} = 0{\cdot}781 \text{ ft.}$$

Taking $\kappa = \frac{8}{5}$, we have for the fall necessary for the velocity

$$c = \tfrac{8}{5} \times 5 = 8 \text{ ft.}, \; h_1 = 1{\cdot}1 \times 0{\cdot}016 \times 8^2 = 1{\cdot}126 \text{ ft.},$$

and hence for the height

$$x = 1{\cdot}126 - 0{\cdot}781 = 0{\cdot}345 \text{ ft.} = 4\tfrac{1}{7} \text{ inches.}$$

Further, for the angle of inclination of the end of guide board,

$$sin.\,v = \sqrt{\frac{0{\cdot}345}{1{\cdot}126}} = 0{\cdot}5539, \qquad \text{or} \qquad v = 33^\circ \, 38',$$

and for the length

$$y = 1 \cdot 126 \, sin. \, 67^\circ \, 16' = 1 \cdot 039 \text{ ft.} = 12\tfrac{1}{2} \text{ inches.}$$

In order to bring the water in tangentially, we must have

$$r = \frac{h - h_1}{1 - cos. \, v} = \frac{8 - 1 \cdot 126}{1 - cos. \, 33^\circ \, 38'} = \frac{6 \cdot 874}{0 \cdot 1674} = 41 \cdot 06 \text{ ft.}$$

If, however, we make it only 25 ft. high, or $r = 12 \cdot 5$ ft., we have for the angle $\theta$ of the water arc,

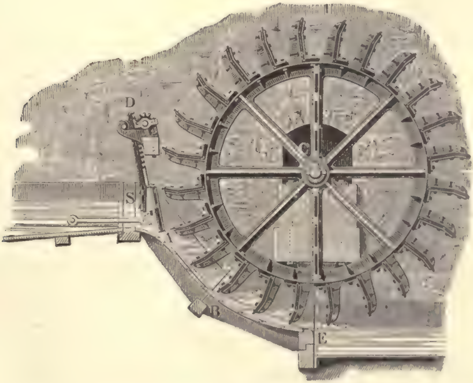
$$cos. \, \theta = 1 - \frac{6 \cdot 875}{12 \cdot 5} = 0 \cdot 450, \qquad \text{or} \qquad \theta = 63^\circ \, 16',$$

and for the deviation of the direction of the water from that of the motion of the wheel, at the point of entrance,

$$\alpha = \theta - v = 63^\circ \, 16' - 33^\circ \, 38' = 29^\circ \, 38'.$$

**§ 203.—Underflow Sluice, and Sluice with Guide Buckets.**—The bringing on of the water by an underflow sluice is represented in Fig. 415. The sluice board *A H* is placed as near the wheel as possible, and well rounded off below, so that the water may enter in the desired direc-

FIG. 415.



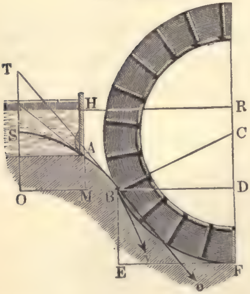
tion without contraction. The height $B E = h_2$ (Fig. 416) of the apron or breast is determined from the total fall $R F = h$, and the height due to the velocity

$$M H = h_1 = 1 \cdot 1 \, \frac{c^2}{2g} = 1 \cdot 1 \, \frac{\kappa^2 v^2}{2g}$$

by means of the formula $h_2 = h - h_1$. The corresponding central angle $B C F = \theta$ is given by

$$\cos. \theta = \frac{C D}{C B} = \frac{r - h_2}{r} = 1 - \frac{h - h_1}{r}.$$

FIG. 416.



If we bring the water in tangentially, the angle $T B O = \nu$ of the stream with the horizon must equal $\theta$, and accordingly the co-ordinates $S O = x$ and $O B = y$ of the vertex of the parabola $S$ are given by

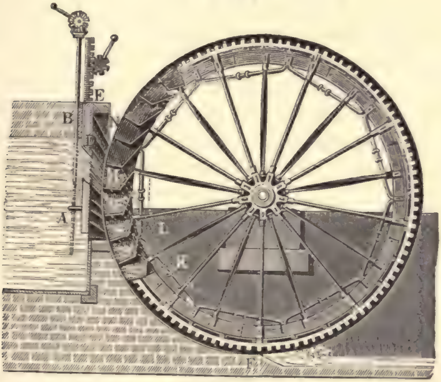
$$x = \frac{c^2 \sin.^2 \theta}{2g} \text{ and } y = \frac{c^2 \sin. 2\theta}{2g}.$$

It is not, however, necessary to have the sluice opening exactly at the vertex $S$ of the parabola. We may have it at any other point, as $A$, and must only have care that the axis of the opening is tangential to the parabola. (See Art. 33.)

Still a third method of bringing on the water consists in sluices with guide buckets (Fig. 417). These are used with the best advantage when the level of the water is variable. The water passes through the guide buckets according to the same law as for short tubes. Hence, as a rule, the coefficient of efflux is $\mu = 0 \cdot 82$, or only for very exact rounding $\mu = 0 \cdot 9$. Thus the coefficient of resistance is greater than for over and underflow sluices. If we take for $\mu$ the mean value 0·85, we have for the generation of the velocity $c$ the height

$$h_1 = \left(\frac{1}{0 \cdot 85}\right)^2 \frac{c^2}{2g} = 1 \cdot 384 \frac{c^2}{2g},$$

FIG. 417.



and, accordingly, the remaining height, or the height of the water-retaining arc, is

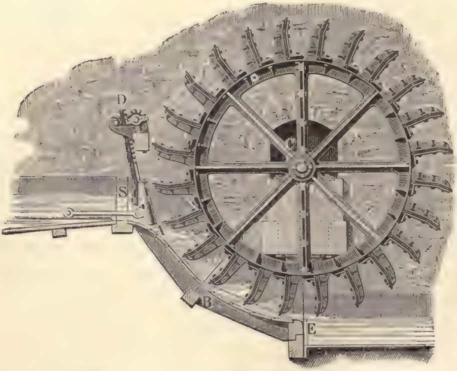
$$h_2 = h - h_1 = h - 1 \cdot 384 \frac{\kappa^2 v^2}{2g}.$$

In case of variable level, the calculation and arrangement is made for the mean level, by placing the end $M$ of the centre guide curve at the height $h_2$ above the foot of the wheel $F$. In order to have all the guides, whose distance apart is generally three inches, make the same angle with the circumference, they are all made tangential to a circle $KL$, which is determined by the direction $DK$ of the first guide.

§ **204.—Construction of Wheel and Curb.**—The curb by which breast wheels are surrounded, in order to retain the water as long as possible, is formed of masonry (Fig.

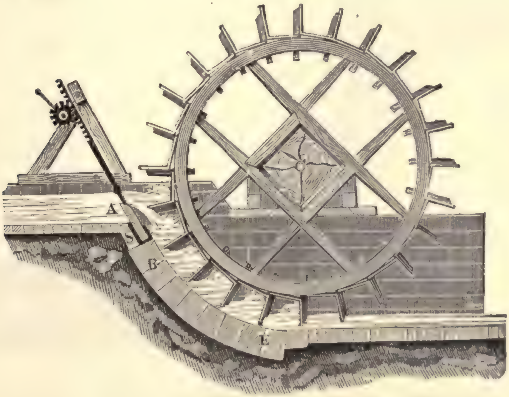
412), of wood (Fig. 415), or, more rarely, of iron. The purpose of the curb is better fulfilled the less the play between the outer edge of the buckets and the cylindrical surface of the curb, because the water escapes by the free space or clearance thus left. This play or clearance is about half an inch in the best constructions, but not infrequently amounts to an inch, or even two inches. When the wheel is of wood as well as the curb, a play of half an inch is scarcely sufficient, because the curb is apt to lose its symmetry, and then friction between it and the wheel may ensue. For iron wheels and stone curbs, there is no likelihood of such derangement, and therefore the clearance can be made very small. Injury may be caused, however, to close-fitting wheels by stray pieces of wood or ice, and it is hence necessary to place a screen in front of the sluice, otherwise a much larger clearance must be allowed. Stone curbs are constructed of carefully dressed stone laid in hydraulic mortar or cement. Wooden curbs, *A E*, Fig. 418, are composed of curb sills *A*, *B*, *E*, and of curved beams *A B* and *B E*, planked with deals. In general, the curb is enclosed by side walls, so that the water cannot

FIG. 418.



escape laterally. If the water can flow off by the tail race with the same velocity with which it issues from the wheel, the curb may be finished flush with the bottom, as shown in Fig. 419. If, however, the water flows away more slowly, the race must be cut out deeper, as shown at *E*, Fig. 418. Finally, as concerns the wheel construction, it differs essentially from the overshot or undershot, in that, in the last, we have buckets or cells, while in the present case we have merely straight paddles or floats. This also gives rise to a different mode of connecting the floats with the wheel rim. Thus we may have *shroudings* enclosing the floats, as in Fig. 417, or the floats may project radially from the circumference, and be supported by

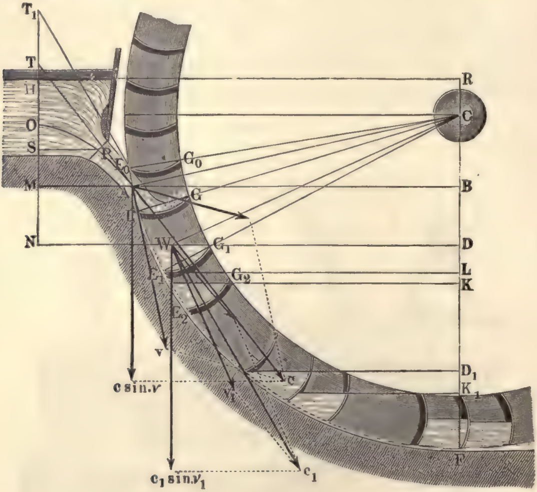
FIG. 419.



short arms or cantilevers, as in Figs. 418 and 419. Fig. 419 is a wooden wheel, and Fig. 418 one of iron. Very narrow float wheels have only a single narrow ring by which the floats are attached to cantilevers; broader wheels have two. In wooden wheels, the supports for the floats are passed between the two sides of a compound ring which forms the shrouding. In iron wheels, they are either cast in one piece with the wheel segments or are bolted on to these segments. The buckets or floats are usually of wood, fastened to the above supports. The sole is placed upon the outside of the crown or shrouding, and has in it slits to allow the escape of the air, as shown in Figs. 418 and 419.

§ **205.**—**Entrance of the Water.**—The rules for the introduction of water in a breast wheel, Fig. 420, are in general the same as for bucket wheels. From the velocity $c = \kappa v$ of the water entering at $A$, we have for the fall necessary for this velocity

FIG. 420.



$$h_1 = 1 \cdot 1 \frac{c^2}{2g},$$

and hence for the remaining fall $FB$ of the breast

$$FB = h_2 = h - h_1 = h - 1 \cdot 1 \frac{c^2}{2g}.$$

If, now, the radius $CA = CF = r$ is given, we have for the angle $ACF = \theta$,

$$cos.\ ACF = \frac{CB}{CA} = \frac{CF - FB}{CA},$$

or

$$cos.\ \theta = \frac{r - h_2}{r} = 1 - \frac{h_2}{r}.$$

Since the angle of approach $vAc = \alpha$ (10 to 20 degrees) can be considered as given, we can determine the angle of inclination of the stream at $A$,

$$cAB = v = \theta - \alpha,$$

from whence the co-ordinates of the vertex $O$ of the parabola are

$$OM = x = \frac{c^2\ sin.^2\ v}{2g} \quad \text{and} \quad MA = y = \frac{c^2\ sin.\ 2v}{2g}.$$

If the centre $P$ of the sluice opening is at a distance $MS = z$ above $A$, we have for the co-ordinates of $P$ with reference to $O$,

$$OS = x_0 = x - z$$

and

$$SP = y_0 = y\sqrt{\frac{x - z}{x}} = y\sqrt{1 - \frac{z}{x}};$$

also for the inclination of the axis of the stream at $P$,

$$tan.\ v = \frac{2x_0}{y_0} = \frac{2\sqrt{x(x - z)}}{y}.$$

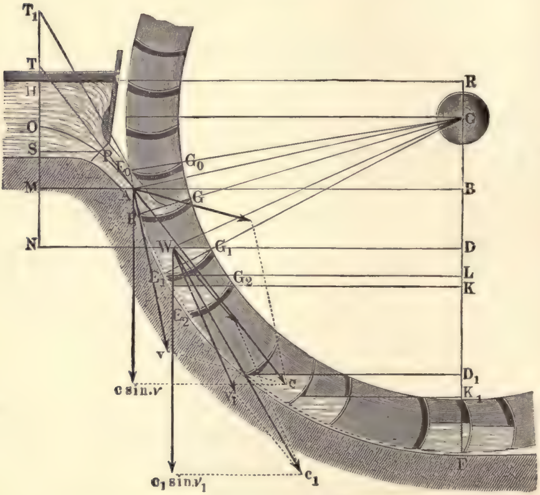
If we know the distance $MN = z_1$ which the water in the wheel sinks during continuance of impact, then we have for the co-ordinates of the point $W$, where impact ends,

$$ON = x_1 = x + z_1$$

and

$$N W = y_1 = y \sqrt{\frac{x_1}{x}} = y \sqrt{1 + \frac{z_1}{x}};$$

FIG. 421.



also for the angle of inclination $D W c_1$ of the stream with the horizon at $W$,

$$tan.\, v_1 = \frac{2\, x_1}{y_1} = \frac{2 \sqrt{x\,(x + z_1)}}{y}.$$

Further, for the angle $W C F = \theta_1$, if $r_1$ is the mean radius $C W$,

$$cos.\, \theta_1 = \frac{C D}{C W} = \frac{r\, cos.\, \theta + z_1}{r_1};$$

and for the angle $c_1 W v_1 = \alpha_1$, which the final velocity $c_1$ of the water at $W$ makes with the wheel velocity $v_1$,

$$\alpha_1 = \theta_1 - v_1.$$

Finally, the velocity at $W$ is

$$c_1 = \sqrt{c^2 + 2\,g\,z_1}.$$

These latter formulæ assume that $z_1 = MN$ is known. This is, therefore, to be first determined by the following approximate method:

In the time

$$t = \frac{E\,E_1}{v} = \frac{s}{v}$$

the float $E\,G$ passes through a space $E\,E_1 = s$, while the water passes through $A\,W$, the vertical projection of which is $MN = z_1$. Since the vertical projections of the velocities at $A$ and $W$ are

$$c \sin. v \quad \text{and} \quad c_1 \sin. v_1,$$

the mean velocity for $z_1$ is

$$\frac{c \sin. v + c_1 \sin. v_1}{2},$$

and hence

$$t = \frac{2\,z_1}{c \sin. v + c_1 \sin. v_1}.$$

Accordingly,

$$\frac{s}{v} = \frac{2\,z_1}{c \sin. v + c_1 \sin. v_1},$$

and hence the distance $s$ passed through by the float during entrance is

$$s = \frac{2\,z_1\,v}{c \sin. v + c_1 \sin. v_1}.$$

If we take first an approximate value for $z_1$, and calculate from this formula $s$, we can draw the corresponding position of the bucket $E_1\,G_1$. If upon this we draw the cross-section

$$F = \frac{V}{e} = \frac{60\,Q}{n\,u\,e},$$

we can test if the surface of this cross-section $W$ lies at the distance $MN = z_1$ already assumed. If not, we can take another value of $z_1$, determine $s$ again, and test again, and so on until $z_1$ is found with sufficient exactness.

§ **206.—Mechanical Effect or Delivery of Curb Wheels.** —The delivery of the breast wheel may, as in the overshot wheel, be divided into an impact and pressure delivery. The formula for the delivery of each is the same, therefore; only the determination of the loss of water requires different calculations, because, while in the one case this loss is due to the gradual emptying of the buckets, in the other it is due to the escape of the water in the space between the wheel and breast. We have therefore to estimate the method of escape and the quantity escaping in this space, which we may call the *prejudicial space*, as well as the work thereby lost to the wheel. If, now, we put, as in the overshot wheel, the entrance velocity at $W = c_1$, the velocity of the wheel at this point $v_1$ and the angle $c_1 W v_1 = \alpha_1$, Fig. 422, we have again for the work of impact

$$\frac{(c_1 \cos. \alpha_1 - v_1) v_1}{g} Q \gamma.$$

If, further, we denote the level $DK_1$ between $W$ and the surface of the tail water by $h_3$, and assume that of the whole feed $Q$, only the portion $Q_1 = \xi Q$ takes effect in the breast; then we have for the work of pressure $h_3 Q \gamma$, and, just as in the overshot wheel, then

$$L = Pv = \left[\frac{(c_1 \cos. \alpha_1 - v_1) v_1}{g} + \xi h_3\right] Q \gamma.$$

In order, then, from the above formula to determine the delivery of a breast wheel, it is necessary to determine the ratio

$$\xi = \frac{Q_1}{Q}.$$

The loss of work due to the escape of water through the space between the wheel and breast, or the so-called "back lash," is during the impact of the water insignificant. During the action of the pressure, however, we have a continual loss, while any float $E\,G_1$, Fig. 418, takes the lower positions, $E_2\,G_2$, $E_3\,G_3$, etc., until the lowest position is reached. Here, then we have openings $E_1$, $E_2 \ldots$ through which the water flows with varying height.

If we denote the width of wheel by $e$, and put the width of clearance between the floats and breast $= \sigma$, then the cross-section of the opening through which the water flows from one bucket into the next lower is $\sigma\,e$, and if now during the gradual sinking of the float the depth $D\,L$ of the opening below the surface of water in the bucket is $l_1$, $l_2$, etc., then the corresponding velocities of exit are

$$v_1 = \sqrt{2\,g\,l_1}, \quad v_2 = \sqrt{2\,g\,l_2}, \quad \text{etc.},$$

and the discharges are

$$V_1 = \sigma\,e\,\tau\sqrt{2\,g\,l_1}, \quad V_2 = \sigma\,e\,\tau\sqrt{2\,g\,l_2}, \quad \text{etc.},$$

or, introducing the coefficient of discharge $\mu$,

$$V_1 = \mu\,\sigma\,e\,\tau\sqrt{2\,g\,l_1}, \quad V_2 = \mu\,\sigma\,e\,\tau\sqrt{2\,g\,l_2}, \quad \text{etc.}$$

These quantities of water sink without producing work through the distances $D\,K = k_1$, $k_2$, etc., between the water surfaces in two successive buckets. The losses of work are then

$$V_1\,k_1\,\gamma = \mu\,\sigma\,e\,\tau\sqrt{2\,g\,l_1}\,k_1\,\gamma, \quad V_2\,k_2\,\gamma = \mu\,\sigma\,e\,\tau\sqrt{2\,g\,l_2}\,k_2\,\gamma, \text{ etc.}$$

The sum of these gives the total loss of work

$$A_1 = \mu\,\sigma\,e\,\tau\,\sqrt{2g \times \gamma}\,(k_1\sqrt{l_1} + k_2\sqrt{l_2} + \cdots).$$

But, now, the length of the breast is $r\,\theta$, and the time in which a float with velocity $c$ passes over it is

$$t = \frac{r\,\theta}{v}.$$

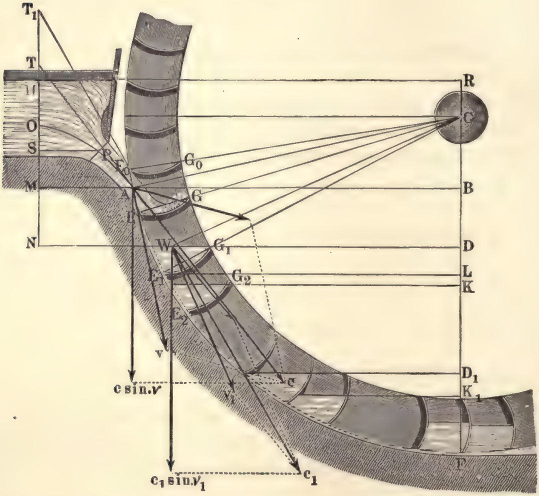
If, then, we put

$$\tau = \frac{t}{n_1},$$

we have

$$A_1 = \mu \sigma e \frac{r \theta}{v} \sqrt{2 g} \gamma \left(\frac{k_1 \sqrt{l_1} + k_2 \sqrt{l_2} + \cdots}{n_1}\right)$$

FIG. 422.



If we subtract this from the whole work,

$$A = v h_3 \gamma = F e h_3 \gamma,$$

which the water of a single bucket performs in sinking through the height of the breast, we have the actual work of the water in a bucket,

$$A - A_1 = F e h_3 \gamma \left(1 - \mu \sigma \frac{r \theta}{F v} \sqrt{2 g} \frac{k_1 \sqrt{l_1} + k_2 \sqrt{l_2} + \cdots}{n_1 h_3}\right),$$

and hence the corresponding work of the water by pressure is

$$L_2 = \frac{n\,u}{60}(A - A_1) = \frac{n\,u}{60} F\,e\,h_3\,\gamma$$

$$\left(1 - \frac{\mu\,\sigma\,r\,\theta\sqrt{2\,g}\,(k_1\sqrt{l_1} + k_2\sqrt{l_2} + \ldots)}{F\,v\,n_1\,h_3}\right)$$

$$= \left(1 - \frac{\mu\,\sigma\,r\,\theta\sqrt{2\,g}}{F\,v}\;\frac{k_1\sqrt{l_1} + k_2\sqrt{l_2} + \cdots}{n_1\,h_3}\right) Q\,h_3\,\gamma,$$

or applying Simpson's rule,

$$L_2 = \left(1 - \frac{\mu\,\sigma\,r\,\theta\sqrt{2\,g}}{F\,v}\right.$$

$$\left.\times\frac{k_0\sqrt{l_0} + 4\,k_1\sqrt{l_1} + 2\,k_2\sqrt{l_2} + 4\,k_3\sqrt{l_3} + k_4\sqrt{l_4}}{12\,h_3}\right) Q\,h_3\,\gamma.$$

Therefore the work of the water by pressure is greater, the greater the wheel velocity $v$ and the cross-section $F$ of the water in a bucket—*i. e.*, the greater the fill. In order to make the calculation, we must divide the arc $E_1\,F$ into $n_1$, or, for example, four equal parts, draw floats through the points of division, then construct the surface of cross-section, and take off the heights $k_1, k_2 \ldots$ and $l_1, l_2 \ldots$ with the dividers. Here we must not forget that when the water passes from one bucket under the water of the next, the values of $l_1$, $l_2$, etc., become $k_1, k_2 \ldots$. (Weisbach, Vol. I., Art. 399).

There is also another loss, due to the escape of water between the crowns and the breast. The area of the exit openings is here $b\,\sigma$ when $b$ is the arc of the breast in contact with the water in a bucket. The heights of pressure are the distances $m_1\,m_2 \ldots.$ of the surface of water in the descending bucket above the under end of the float, and the falls are $p_1\,p_2 \ldots.$, or the distances from water surfaces to $k_1$. From these falls, we have the loss of work

$$A_2 = \tfrac{2}{3}\,\mu\,\sigma\,b\,\frac{r\,\theta}{v}\sqrt{2\,g} \times \gamma\left(\frac{p_1\sqrt{m_1} + p_2\sqrt{m_2} + \ldots}{n_1}\right),$$

and hence, taking into account both causes of loss, and considering only three bucket divisions, we have

$$L_2 = \left[1 - \frac{\mu \sigma r \theta \sqrt{2g}}{6 F v h_3} \left(k_0 \sqrt{l_0} + 4 k_1 \sqrt{l_1} + k_2 \sqrt{l_2}\right.\right.$$

$$\left.\left. + \tfrac{2}{3} \frac{b}{c} (p_0 \sqrt{m_0} + 4 p_1 \sqrt{m_1} + p_2 \sqrt{m_2})\right)\right] Q h_3 \gamma.$$

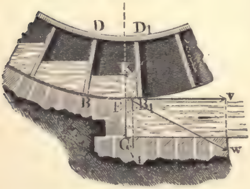
If we put this work $L_2 = \xi Q h_3 \gamma$, we have, finally,

$$\xi = \left[1 - \frac{\mu \sigma r \theta \sqrt{2g}}{6 F v h_3} \left(k_0 \sqrt{l_0} + 4 k_1 \sqrt{l_1} + k_2 \sqrt{l_2}\right.\right.$$

$$\left.\left. + \tfrac{2}{3} \frac{b}{e} (p_0 \sqrt{m_0} + 4 p_1 \sqrt{m_1} + p_2 \sqrt{m_2})\right)\right].$$

§ **207.—Other Losses of Work.**—A further loss of work takes place when the surface of the water in the tail race is not at the same level as that in the lowest bucket, as shown in Fig. 423, because then the water flows at once out of the bucket $B D D_1 B_1$ as soon as the float $B_1 D_1$ passes the sill $F G$, and it has, therefore, besides the velocity of the wheel $v$, a velocity due to the head $F K$. This head is, however, variable; it has its greatest value when the water just commences to run out, and becomes less and less, and finally zero, when the discharge is ended. We may put the mean value of this head $= \frac{1}{2} h_4$, if $h_4$ is the original head $F K$, and hence for the velocity of the issuing water we have no longer the head $\frac{v^2}{2g}$, but $\frac{w^2}{2g} = \frac{v^2}{2g} + \frac{1}{2} h_4$.

FIG. 423.



Since we have already considered the loss corresponding to $\frac{v^2}{2g}$ in impact, we have only to subtract from the delivery

$$L_4 = \tfrac{1}{2} Q h_4 \gamma.$$

We see, therefore, that it is not advantageous to have a fall under the wheel, which is only justified by very changeable level, so that in high water there is danger of drowning the wheel.

There are still other losses of work in the breast wheel. In the first place, the water moving over the breast experiences friction which must be overcome. The coefficient of friction $\zeta$ we can take according to Vol. I., Art. 476, Weisbach's Mechanics, for velocities of 4 to 6 ft. equal to 0·00769. The corresponding loss of fall is (Weisbach's Mechanics, Vol. I., Art. 475),

$$h_5 = \zeta \frac{l p}{F} \frac{v^2}{2g},$$

where $l$ is the length of the breast, $p$ the circumference, and $F$ the area of the water profile. Therefore,

$$\frac{p}{F} = \frac{e + d}{\frac{1}{2} d e},$$

and approximately $= \frac{2}{d}$, and hence

$$h_5 = \zeta \frac{2 l}{d} \frac{v^2}{2g} = 0{\cdot}0002384 \frac{l}{d} v^2.$$

The corresponding loss of work is therefore

$$L_5 = 0{\cdot}0002384 \frac{l v^2}{d} Q \gamma.$$

Finally, we must also take into account the resistance of the air to the motion of the floats, and also even to that of the arms. The coefficient of resistance of air is here (Weisbach, Vol. I., Art. 512) $\zeta = 1 \cdot 25$, and hence the resistance is

$$\zeta F \gamma \frac{v^2}{2g}$$

where $F$ is the area and $\gamma$ the density of the air. If, according to Weisbach's Mechanics, Vol. I., Art. 393, we put for $\gamma$, 0·08076 lbs., we have for this resistance

$$0 \cdot 0016 \, F v^2,$$

or, when for $F$ we put the area $n\,d\,e$ of the floats,

$$0 \cdot 0016 \, n\, d\, e\, v^2,$$

and therefore the corresponding loss of work

$$L_6 = 0 \cdot 0016 \, n\, d\, e\, v^3.$$

In general, all these losses together form but a small percentage of the entire wheel delivery, as we shall see in an example further on.

§ **208.**—**Formula for Delivery.**—We can now give an expression for the complete delivery of a breast wheel, if we take into account, besides the above losses of work, also the work consumed by axle friction. According to the above, the work by pressure of the water is $\zeta \, Q \, h_3 \, \gamma$, and if, as before, for the overshot water-wheel we put the work of the axle friction

$$\phi \frac{\rho}{r} G v,$$

we have for the useful work

$$L = P v = \left[ \frac{(c_1 \cos. \alpha_1 - v_1) v_1}{g} + \xi h_3 \right] Q \gamma - \phi \frac{\rho}{r} G v.$$

If we denote the total fall from the surface of water in head to that in tail race by $h$, we can put

$$h_2 = h - 1 \cdot 1 \frac{c_1^2}{2g},$$

and have then

$$L = \left[\frac{(c_1 \cos. \alpha_1 - v_1) v_1}{g} + \xi \left(h - 1 \cdot 1 \frac{c_1^2}{2g}\right)\right] Q\gamma - \phi \frac{\rho}{r} G v.$$

In order, now, to find that value of the entrance velocity $c_1$, for which the delivery is greatest, we have to find that value which gives a maximum to the expression

$$\left(\frac{c_1 v_1 \cos. \alpha_1}{g} - 1 \cdot 1 \xi \frac{c_1^2}{2g}\right) Q\gamma = \frac{1 \cdot 1 \xi c_1}{2g} \left(\frac{2 v_1 \cos. \alpha_1}{1 \cdot 1 \xi} - c_1\right) Q\gamma,$$

or

$$c_1 \left(\frac{2 v_1 \cos. \alpha_1}{1 \cdot 1 \xi} - c_1\right).$$

The value of $c_1$ for a maximum is then

$$c_1 = \frac{v_1 \cos. \alpha_1}{1 \cdot 1 \xi},$$

and hence the corresponding maximum delivery is

$$L = \left[\xi h - \left(2 - \frac{\cos.^2 \alpha_1}{1 \cdot 1 \xi}\right) \frac{v_1^2}{2g}\right] Q\gamma - \phi \frac{\rho}{r} G v.$$

The formula

$$c_1 = \frac{v_1 \cos. \alpha_1}{1 \cdot 1 \xi}$$

gives, since $\alpha_1$ is small, and therefore $\cos. \alpha_1 = 1$ nearly, also $1 \cdot 1 \xi$ nearly $= 1$, and $c_1$ nearly $= v_1$. For the more ready introduction of the water, we make $c_1 \cos. \alpha_1 = 2 v_1$, *i. e.*, allow the water to enter as fast again as the wheel revolves, and hence have

$$L = \left[\xi h - \left(\frac{4 \cdot 4 \xi}{\cos.^2 \alpha_1} - 2\right) \frac{v_1^2}{2g}\right] Q\gamma - \phi \frac{\rho}{r} G v.$$

Since this expression for the delivery of a breast wheel is not essentially different from that for an overshot wheel, we see at once that the most advantageous number of revolutions is nearly the same (Art. 196).

§ **209.—Efficiency of Breast Wheels.**—A number of experiments have been made by Morin upon breast wheels of good construction. He compares his experimental results with those given by the theoretical formula,

$$Pv = \left(\frac{(c\,cos.\,\alpha - v)\,v}{g} + h_2\right) Q\gamma,$$

and finds a tolerable agreement when this expression is multiplied by an experimental coefficient $\chi$. We have then

$$Pv = \chi\left(\frac{(c\,cos.\,\alpha - v)\,v}{g} + h_2\right) Q\gamma.$$

One of the wheels experimented upon by Morin was of cast-iron, with wooden floats placed obliquely to the curb, and the curb itself fitting quite close. The diameter was 6½ metres (21·3 ft.) and 1½ metres (4·9 ft.) wide. The fall was 1⅝ metres (5·5 ft.). There were 50 floats, and the velocity of rotation was from 3⅛ to nearly 8 ft. per second; the velocity of the water from a well-constructed sluice being from 9 ft. to 10½ ft. The coefficient $\chi$ was found to be about 0·75, and the efficiency, including axle friction, nearly 0·60. A second wheel, also of iron, was hung in a close-fitting sandstone curb. It was 13 ft. in diameter and also 13 ft. wide. There were 32 floats, and the fall was 6½ ft. So long as the velocity of the wheel did not differ more than from 47 to 100 per cent of the entering water—*i. e.*, for speeds of from 1⅝ to 6 ft.—the coefficient $\chi$ was nearly constant and about 0·788, the efficiency being 0·70. A third wheel was almost entirely of wood, and was also hung in a close-fitting curb. Two sets of experiments were made upon it: one with an ordinary sluice, the other with an overfall sluice. The diameter was 20 ft., and the number of floats was 40. In the first case, with common sluice, the coefficient $\chi$ was 0·792, and in the second case, with overfall sluice, the value of $\chi$ rose to 0·809. The efficiency in the first case, however, was only 0·54, and in the second it was 0·67. If we take a mean value from these experiments, we obtain for middle-shot curb wheels with "penstock" sluice,

$$L = 0 \cdot 77 \left(\frac{(c\, cos.\, \alpha - v)\, v}{g} + h_2\right) Q \gamma,$$

and for overflow sluice,

$$L = 0 \cdot 80 \left(\frac{(c\, cos.\, \alpha - v)\, v}{g} + h_2\right) Q \gamma\, ;$$

from which, however, the mechanical effect due to axle friction is to be deducted. The greater efficiency in the second case is due to the fact that the water enters more slowly, and hence there is less loss by impact.

It follows, also, from the experiments of Morin, that the efficiency diminishes if the water fills more than from ½ to ⅝ of the space between the floats, and that it does not vary much so long as the circumferential velocity is within the limits of from 1·5 to 6½ ft.

Egen made experiments upon a breast wheel of 23 ft. diameter and 4⅛ ft. in width. This wheel had two peculiarities: the 69 well-ventilated buckets were constructed precisely as in the overshot wheel, and the sluice was in two divisions, so that the upper or under one could be drawn according to the supply of water. The curb fitted very closely, but still the efficiency was found by Egen to be at most only 0·52, and, upon an average, with 6 cubic ft. of water per second and 4 revolutions per minute, the efficiency was only 0·48.

Experiments upon a breast wheel are described in the "Bulletin de la Société indust. de Mulhouse," T. XVIII. (see "Polytech. Centralblatt," Bd. IV., 1844). This wheel was of wood, 16·4 ft. in diameter and 13 ft. wide, made in three divisions, with 2 centre shroudings or crowns. The curb started from a parabolic saddle beam 8 inches in height, and the water was laid on by an overfall sluice 8 inches high. The velocity of entrance $c$ was therefore about 8·8 ft., and the velocity of the wheel was from 5 ft. to 6½ ft. per second. The buckets were filled from ½ to ⅔, and the efficiency increased as the buckets were more filled. Thus, when the buckets were well filled, the efficiency was 0·80; when half filled, it was 0·73, and with still less water it was only 0·52. The experiments upon the efficiency of the wheel for different degrees of fill were in this case easily and accurately made, because the water could be led on to each division of the wheel separately.

By friction-brake experiments upon an iron breast wheel of 20 ft. diameter, 3 ft. width, and having 48 floats, the water being let on at about the centre by a guide-bucket sluice, made by the author in connection with Profs. Brückmann and Zeuner (see "Civilingenieur," Bd. II.), the following was determined:

For the coefficient of fill $\varepsilon = \frac{1}{2}$ and the velocity ratio $\kappa = \frac{1}{2}$, the wheel made 8 to 9 revolutions per minute, and performed 12½ to 12 horse-power, while the whole work at disposition $Q h \gamma$ was 19 horse-power. The efficiency of this wheel was, therefore,

$$\eta = \frac{12 \cdot 5}{19} = 0 \cdot 65.$$

EXAMPLE.—Required, for a feed of $Q = 20$ cubic ft. per second, and for a fall of $h = 9$ ft., to determine the proportions for a breast wheel, Fig. 424, 16 ft. high and revolving with a circumferential velocity of 8 ft. per second. If we take the depth of crown $d = 1\frac{1}{4}$ ft. and take the buckets half full, or $\varepsilon = \frac{1}{2}$, we have for the breadth of wheel,

$$e = \frac{2\,Q}{d\,v} = \frac{2 \times 20}{\frac{5}{4} \times 8} = 4 \text{ ft.}$$

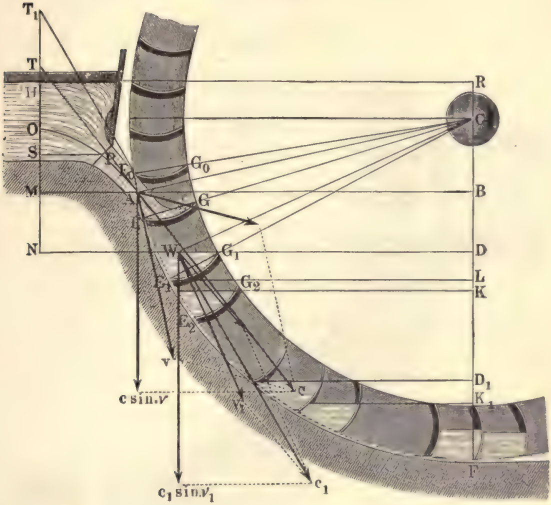
If we let the water enter with the velocity

$$c = \kappa\, v = \tfrac{3}{2}\, v = \tfrac{3}{2} \cdot 8 = 12 \text{ ft.},$$

we have for the fall necessary for this velocity,

$$h_1 = 1 \cdot 4 \frac{c^2}{2g} = 1 \cdot 4 \times 0 \cdot 015 \times 12^2 = 2 \cdot 54 \text{ ft.}$$

FIG. 424.



If we subtract this fall from the total fall $h$, we have for the fall in the breast,

$$h_2 = h - h_1 = 9 - 2 \cdot 54 = 6 \cdot 46 \text{ ft.},$$

and hence for the angle $\theta$ of the point of entrance from the summit,

$$\cos. \theta = 1 - \frac{h_2}{r} = 1 - \frac{6 \cdot 46}{8} = 1 - 0 \cdot 8075 = 0 \cdot 1925,$$

and therefore

$$\theta = 78^\circ \, 54'.$$

If, now, we allow the water to enter at an angle $\alpha = 25\frac{1}{2}^\circ$ with the circumference, we have for the inclination of the entering stream with the horizon,

$$\nu = \theta - \alpha = 78^\circ \, 54' - 25^\circ \, 30' = 53^\circ \, 24'.$$

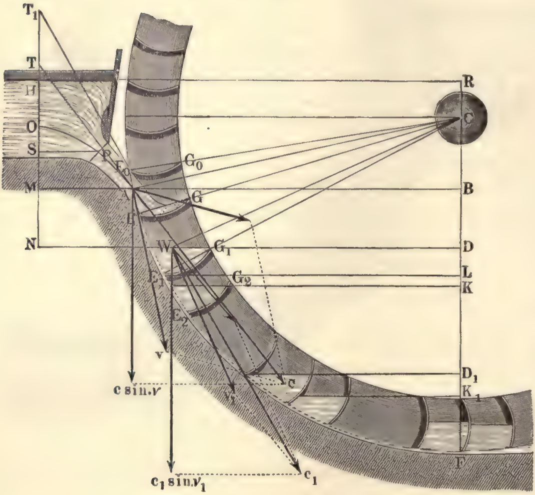
Hence for the vertex of the parabola in which the water moves,

$$x = \frac{c^2 \sin.^2 v}{2g} = 0 \cdot 015 \times 144 (\sin.\ 53^\circ\ 24')^2 = 1 \cdot 48 \text{ ft.},$$

and

$$y = \frac{c^2 \sin.\ 2v}{2g} = 0 \cdot 015 \times 144 \sin.\ 73^\circ\ 12' = 2 \cdot 21 \text{ ft.}$$

FIG. 425.



The centre of the sluice opening must be upon this parabola, and as near as possible to the wheel. It must also be so disposed that its axis is tangent at this point to the parabolic arc. If we put the sluice opening at 0·54 ft. above the point of entrance, then the height of water above it is

$$h_0 = h_1 - 0 \cdot 54 = 2 \cdot 54 - 0 \cdot 54 = 2 \text{ ft.}$$

The velocity of exit then is

$$c_0 = 0 \cdot 95 \sqrt{2 g h_0} = 0 \cdot 95 \sqrt{128 \cdot 8} = 11 \cdot 35$$

and if we take the width of the sluice opening,

$$e_0 = e - 0 \cdot 25 = 3 \cdot 75 \text{ ft.},$$

then the height of this opening must be

$$d_0 = \frac{Q}{c_0 e_0} = \frac{20}{11 \cdot 35 \times 3 \cdot 75} = 0 \cdot 47 \text{ ft.}$$

If we give the wheel 48 floats, we have for the outer distance between any two,

$$b = \frac{2 \pi r}{n} = \frac{2 \pi \times 8}{48} = \frac{3 \cdot 1416}{3} = 1 \cdot 047 \text{ ft.}$$

If we assume that a float passes through the distance $s = 0 \cdot 9$ ft. while receiving water, draw the corresponding new position of the float and determine the cross-section of the water area, we can thus find the depth $z_1$ of the surface of water in the bucket below the point of entrance. Suppose in this manner we have found $z_1 = 1 \cdot 25$ ft. Then for the velocity $c_1$ of the water in the bucket at the moment entrance ceases, we have

$$c_1 = \sqrt{c^2 + 2 g z_1} = \sqrt{144 + 64 \cdot 4 \times 1 \cdot 25} = \sqrt{222} = 14 \cdot 9 \text{ ft.,}$$

and for the abscissa of the point $W$ (see figure to Art. 51),

$$x_1 = O N = x + z_1 = 1 \cdot 48 + 1 \cdot 25 = 2 \cdot 73 \text{ ft.,}$$

and for the ordinate,

$$y_1 = N W = y \sqrt{\frac{x_1}{x}} = 2 \cdot 21 \sqrt{\frac{2 \cdot 73}{1 \cdot 48}} = 3 \cdot 00 \text{ ft.,}$$

and for the angle of inclination $c_1 W D = v_1$,

$$tan.\, v_1 = \frac{2 x_1}{y_1} = \frac{5 \cdot 46}{3 \cdot 00} = 1 \cdot 82,$$

or

$$v_1 = 61^\circ\ 13'.$$

Since, therefore,

$$c\, sin.\, v = 12\, sin.\, 53^\circ\ 24' = 9 \cdot 634,$$

and

$$c_1\, sin.\, v_1 = 14 \cdot 9\, sin.\, 61^\circ\ 13' = 13 \cdot 059,$$

we have

$$\frac{s}{v} = \frac{2 z_1}{c\, sin.\, v + c_1\, sin.\, v_1} = \frac{2 \times 1 \cdot 25}{22 \cdot 693} = 0 \cdot 1102,$$

while for our assumed value of $s$ and given value of $v$, we have

$$\frac{s}{v} = \frac{0 \cdot 9}{8} = 0 \cdot 1125.$$

The difference is so small that we may regard our assumption of $s = 0 \cdot 9$ and therefore $z_1 = 1\cdot25$ as sufficiently correct. Were it not so, another approximation could be made.

Further, for the angle $\theta_1$ of $W$ from $F$, we have

$$\cos.\,\theta_1 = \frac{CB + z_1}{r_1} = \frac{1\cdot54 + 1\cdot25}{76} = 0\cdot3671,$$

or

$$\theta_1 = 68^\circ\ 28',$$

and therefore for the deviation of the stream at $W$ from the direction of motion of the wheel,

$$\alpha_1 = \theta_1 - \nu_1 = 68^\circ\ 28' - 61^\circ\ 13' = 7^\circ\ 15'.$$

Since, then, the effective fall is

$$FD = h_3 = h_2 - z_1 = 6\cdot46 - 1\cdot25 = 5\cdot21 \text{ ft.},$$

and the velocity at $W$ is

$$v_1 = \frac{r_1}{r}v = \frac{7\cdot6}{8} \times 8 = 7\cdot6 \text{ ft.},$$

we have for the delivery of the wheel, without reference to water loss, etc.,

$$L = \left[\frac{(c_1 \cos.\,\alpha_1 - v_1)\,v_1}{g} + h_3\right] Q\gamma$$

$$= [0\cdot032\,(14\cdot9 \cos.\,7^\circ 15' - 7\cdot6)\,7\cdot6 + 5\cdot21]\ 20 \times 62\cdot5$$

$$= (0\cdot032 \times 7\cdot10 \times 7\cdot6 + 5\cdot21)\,1250 = (1\cdot75 + 5\cdot21)\,1250$$

$$= 6\cdot96 \times 1250 = 8700 \text{ ft. lbs.}$$

If, now, the clearance between wheel and breast is $\sigma = \frac{1}{2}$ inch, and if we take $\mu = 0\cdot7$, we have

$$\mu\sigma\theta r\sqrt{2g} = 0\cdot7 \times \tfrac{1}{24} \times 8 \times 8\cdot025 \text{ arc } 78^\circ\ 54' = \frac{0\cdot7 \times 8\cdot025}{3} \times 1\cdot377 = 2\cdot54.$$

Since, further,

$$Fv = \frac{60\,Q}{n\,u\,e} \times \frac{2\,\pi\,u\,r}{60} = \frac{2\,\pi\,r\,Q}{n\,e} = \frac{6\cdot28 \times 8 \times 20}{48 \times 4} = 5\cdot23,$$

and $h_3 = 5\cdot21$, we have

$$\frac{\mu\sigma\theta r\sqrt{2g}}{F\,v\,h_3} = \frac{2\cdot54}{5\cdot23 \times 5\cdot21} = 0\cdot0932.$$

If, now, the mean value of $k\sqrt{l} = 0\cdot5$, and if $b = 1$, and the mean value of $\frac{2}{3}p\sqrt{m} = 1\cdot0$, we have

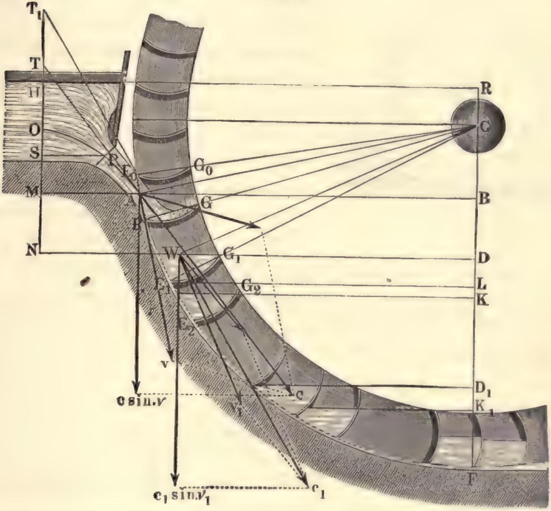
$$\xi = 1 - 0\cdot0932\,(0\cdot5 + \tfrac{1}{4} \times 1) = 1 - 0\cdot0932 \times 0\cdot75 = 1 - 0\cdot07 = 0\cdot93,$$

and hence the effective delivery is

$$L = \left[\frac{(c_1 \, cos. \, \alpha_1 - v_1)\, v_1}{g} + \xi\, h_3\right] Q \gamma = (1 \cdot 75 + 0 \cdot 930 \times 5 \cdot 21)\, 1250$$

$$= 6 \cdot 595 \times 1250 = 8243 \text{ ft. lbs.}$$

FIG. 426.



If the remaining resistances, resistance of air and axle friction, etc., consume still 743 ft. lbs., we have for the useful effect of the wheel $L = 7500$, and for the efficiency,

$$\eta = \frac{L}{Q\, h\, \gamma} = \frac{7500}{20 \times 9 \times 62 \cdot 5} = 0 \cdot 66.$$

§ **210.** — **The Undershot Water-Wheel.** — Undershot water-wheels hang, in general, in a race, the bottom and sides of which enclose the wheel, so that as little loss of water as possible can occur. For this reason, a curb concentric, with the wheel at bottom, for a small distance, is more advantageous than a perfectly straight race which is merely tangent to the wheel. Moreover, such a curb, when upon one side only of the wheel, gives some effect due to pressure, which is entirely lost in the straight race. The calculation of such a wheel, Fig. 427, when the curb includes at least three or four floats, is exactly the same as for a low breast-wheel. Also middle and undershot curb-wheels are constructed according to the same rules, since they do not essentially differ. Generally simple radial floats are used; sometimes these are inclined somewhat in order to escape from the water with less resistance. Sometimes we have elbow floats composed of two straight portions which meet under an angle of 100° to 120°. We may also have considerable openings in the sole plate, without fear that the water will escape through them, and therefore we can fill the buckets half or even two thirds full, and accordingly take $\varepsilon = \frac{1}{2}$ to $\frac{2}{3}$. In order to prevent the escape of the water, or to obtain a greater capacity, we may often make use of a greater depth of crown, from $1\frac{1}{4}$ to $1\frac{1}{2}$ ft. It is easier also to bring the water on tangentially than for middleshot-wheels. In order to bring the sluice opening as near the wheel as possible, we use an inclined sluice board *S*, Fig. 428, whose under edge should be rounded off to avoid partial contraction of the stream.

FIG. 427.

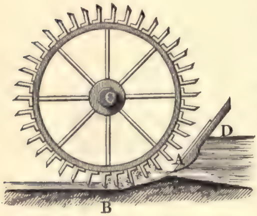
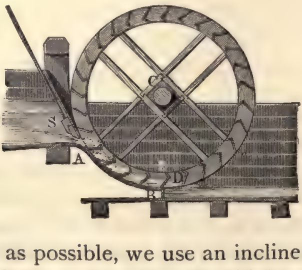


FIG. 428.



§ **211.—Undershot Curb-Wheels.**—In any case, the delivery of the undershot curb wheel is less than the middleshot, where the pressure fall is always greater. The reason is easy to see, since, by the action of water by impact, at least half of the disposable work is lost, while, by the loss of water in the prejudicial space or clearance between the wheel and curb, at most one quarter of the disposable work is lost. Experiments also prove this satisfactorily. One wheel experimented upon by Morin was 6 metres (19½ ft.) high and 1·6 metres (5½ ft.) wide, and had 36 radial floats. The sluice board was inclined 34½° to the horizon, and the opening was 0·78 metre (2½ ft.) back from the beginning of the curb. The total fall was 1·9 metres (6¼ ft.), the height in front of sluice opening 1·4 metres (4½ ft.), and accordingly the pressure fall in the wheel 0·5 metre (1⅝ ft.). The velocity of the wheel was 2 to 4 metres (6½ to 13 ft.), and the velocity of the entering water 5 to 5½ metres (16 to 18 ft.). As long as $\frac{v}{c}$ did not exceed 0·63, the efficiency was $\eta = 0 \cdot 41$. When, however, $\frac{v}{c}$ was between 0·5 and 0·8, $\eta$ was 0·33. We have accordingly, retaining our former notation, for the delivery of this wheel in the first case, without reference to axle friction,

$$Pv = 0 \cdot 74 \left[\frac{(c - v)\,v}{g} + h_2\right] Q\,\gamma,$$

and in the second

$$Pv = 0 \cdot 60 \left[\frac{(c - v)\,v}{g} + h_2\right] Q\,\gamma.$$

Another wheel upon which Morin experimented was nearly 4 metres (13 ft.) high, about 0·8 metre (2⅝ ft.) wide, 0·3 metre deep (11·8 inches), and had 24 floats. The water issued under a vertical sluice board, and passed through a guide race 0·8 metre (2⅝ ft.) long to the wheel. This race, as well as the curb, was of cut stone, and the clearance was only 0·005 metre (0·2 inches). The fall was from 0·78 to 1 metre (3 ft.), the height back of the sluice was, however, 0·15 to 0·45 metre (6 to 18 inches). The experiments were made for very different velocities of circumference. For very small velocities, the efficiency was also very small; for the velocity of 1·5 metres (5 ft.) it was the greatest, and when the velocity of the entering water was not far from this the greatest efficiency was 0·49. For $\frac{v}{c}$ within the limits $\frac{1}{4}$ and $\frac{2}{4}$, the mean was exactly as for the preceding wheel $\eta = 0 \cdot 74$, and hence we have here also

$$Pv = 0 \cdot 74 \left[ \frac{(c - v)\, v}{g} + h_2 \right] Q\gamma.$$

Morin gives the following as the results of his experiments upon curb-wheels:

$\eta = 0 \cdot 40$ to $0 \cdot 45$, when $h_1 = \frac{1}{4}\, h$,
$\eta = 0 \cdot 42$ to $0 \cdot 49$, when $h_1 = \frac{2}{5}\, h$,
$\eta = 0 \cdot 47$, when $h_1 = \frac{2}{3}\, h$, and
$\eta = 0 \cdot 55$, when $h_1 = \frac{3}{4}\, h$,

$h$ being the total fall and $h_1$ the fall necessary for the velocity of entrance.

EXAMPLE.—What is the delivery of an undershot curb-wheel 15 ft. in diameter, which makes $u = 8$ revolutions per minute, and has a fall $h = 4$ ft. and $Q = 20$ cubic ft. per second?

The velocity of circumference is

$$v = \frac{\pi\, u\, r}{30} = \frac{\pi \times 8 \times 15}{60} = 6 \cdot 283 \text{ ft.}$$

If, now, the entering velocity of the water is twice this, we have for the height of water behind the sluice opening

$$h_1 = 1 \cdot 1 \frac{c^2}{2\, g} = 1 \cdot 1 \times 0 \cdot 016 \times 12 \cdot 566^2 = 2 \cdot 779 \text{ ft.}$$

The fall of water in the wheel is then

$$h_2 = 4 - 2 \cdot 779 = 1 \cdot 221 \text{ ft.}$$

The theoretical delivery is then

$$L = \left( \frac{c^2}{g} + h_2 \right) Q\gamma = (0 \cdot 032 \times 6 \cdot 283^2 + 1 \cdot 221)\, 20 \times 62 \cdot 50$$

$$= (1 \cdot 263 + 1 \cdot 221) \times 1250 = 3067 \text{ ft. lbs.}$$

Now, we have here

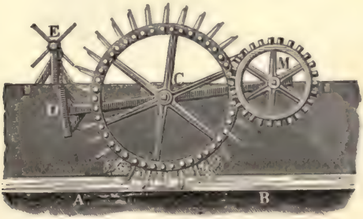
$$h_1 = \frac{1 \cdot 221}{4} h = 0 \cdot 3 \, h,$$

hence $\eta$ is only 0·42, and therefore, without reference to axle friction, the delivery is

$$L = 0 \cdot 42 \times 3067 = 1288 \text{ ft. lbs.}$$

§ **212.—Wheels in Straight Race.**—The least delivery or mechanical effect is furnished by the undershot-wheel in straight race, because here the water acts by impact alone, and because a considerable quantity of water escapes without producing full effect. Such wheels are applied to small falls of less than 4 ft., because, in such case, a curb would give but little increase of effect. By reason of their small delivery, they are best replaced by the Poncelet wheel or by turbines, which we shall notice hereafter. We may make these wheels from 12 to 24 ft. high, with from 24 to 48 floats, either radial or somewhat inclined. The floats must be three times as high as the approaching stream of water is deep, because the water, after impact, takes the velocity of the wheel, which, for the best action, is 35 to 40 per cent of the velocity of the water before impact. Hence, since the width of the race is invariable, the issuing stream is 2½ to 3 times as deep as the approaching stream. In general, the approaching stream is 4 to 6 inches deep, hence the issuing water is 10 to 18 inches, and the necessary length of float, in order that the water may not escape inwardly, is 12 to 20 inches. The straight race for an ordinary undershot-wheel may be either horizontal, as *A B*, Fig. 429, or

FIG. 429.



inclined, as *A B*, Fig. 430. In order to obtain all the effect of the water, the clearance should be as small as possible, 1 to 2 inches, or even less. It is still better, as shown in Fig. 431, to give the race a slight curvature, and have the floats close, so that four or five may always be in the water. The sluice is best inclined so as to bring the orifice close up, and to avoid as far as possible contraction. Under the wheel, the race often falls off suddenly to avoid drowning the wheel in backwater. Sometimes an apparatus is used for raising and lowering the wheel or the race. Such an arrangement is shown in Fig. 429. Such contrivances for accommodating the wheel to variable level of water are best replaced by turbines, the more so as the delivery of these latter wheels is greater.

FIG. 430.

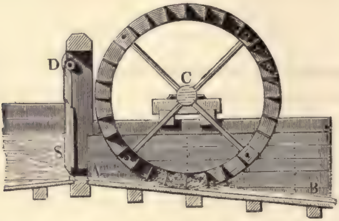
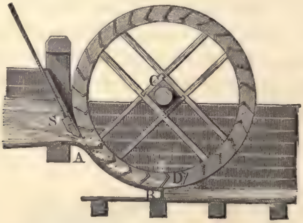


FIG. 431.



§ 213.—**Loss of Water in Straight Race.**—If $c$ is the velocity of the water, and $v$ that of the wheel, then, for the theoretical delivery of an undershot-wheel in straight race, we have

$$P\,v = \frac{(c - v)\,v}{g}\,Q_1\,\gamma,$$

and hence the force of rotation,

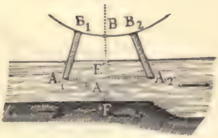
$$P = \frac{c - v}{g}\,Q_1\,\gamma\,. = 1{\cdot}937\,(c - v)\,Q_1$$

(see Vol. I., Art. 501).

Here $Q_1$ denotes the amount of water which actually strikes the floats, and we have, therefore, to find the ratio which this bears to the whole amount of water. The loss of water for a wheel in a straight race is a double one. First, the water escapes through the clearance between wheel and race, and, second, certain deeper elements of the water do not strike at all upon some of the foremost floats.

Let us consider first the loss through the clearance between the race and lowest float end. The height of this clearance is variable. Thus, if the float $A\,B$, Fig. 432, is at its lowest position, the clearance $A\,F = \sigma$ is the least possible. If, however, the two floats $A_1\,B_1$ and $A_2\,B_2$ are equally distant from $F$, the height $E\,F$ of the clearance is greatest. If the radius $C\,A = r$ and the number of floats is $n$, we have

FIG. 432.



$$E\,A_1 = E\,A_2 = \frac{2\,\pi\,r}{2\,n} = \frac{\pi\,r}{n},$$

and hence for $E\,A$ approximately,

$$E\,A = \frac{\overline{E\,A_1}^2}{2\,r} = \left(\frac{\pi}{n}\right)^2 \frac{r}{2}.$$

Therefore, the greatest height of clearance

$$E\,F = \sigma + \left(\frac{\pi}{n}\right)^2 \frac{r}{2}.$$

The mean value between these extremes is, then,

$$= \sigma + \left(\frac{\pi}{n}\right)^2 \frac{r}{4}.$$

This multiplied by width of race $e_1$ gives for the cross-section through which water escapes,

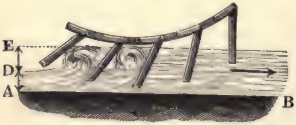
$$e_1\left[\sigma + \left(\frac{\pi}{n}\right)^2 \frac{r}{4}\right],$$

and we have only to determine the velocity $w$ with which the water passes through this cross-section. If the level of the issuing water is the same as that of the approaching water, the velocity through $E\,F$ is the same as that of approach or $= c$, and hence the quantity of water escaping is

$$Q_2 = \left[\sigma + \left(\frac{\pi}{2\,n}\right)^2 r\right] e_1\,c.$$

If, however, the level of the issuing water is greater than that of the approaching, which is always the case when the race $A\,B$, Fig. 433, does not fall off abruptly beyond the wheel, the velocity required is less than $c$, because there is a back pressure. If

FIG. 433.



$$A\,D = d_1 \text{ and } A\,E = d_2,$$

we must have $d_1\,c = d_2\,v$, and hence

$$d_2 = \frac{d_1\,c}{v}.$$

Also

$$d_2 - d_1 = \left(\frac{c - v}{v}\right) d_1.$$

Accordingly, the velocity with which the water passes through the clearance is

$$w = \sqrt{c^2 - 2\,g\left(\frac{c - v}{v}\right)d_1}.$$

Hence the loss of water is, for this case,

$$Q_2 = e_1 \left[\sigma + \left(\frac{\pi}{2\,n}\right)^2 r\right] \sqrt{c^2 - 2\,g\left(\frac{c-v}{v}\right) d_1}.$$

Both these expressions for $Q_2$ must, however, be multiplied by an efflux coefficient $\mu$, which, as in curb-wheels, may be taken at 0·7.

There is a still further loss of water between the sides of race and wheel. The cross-section of water lost in this way is $\sigma_1\, d_1$, and hence for the first case,

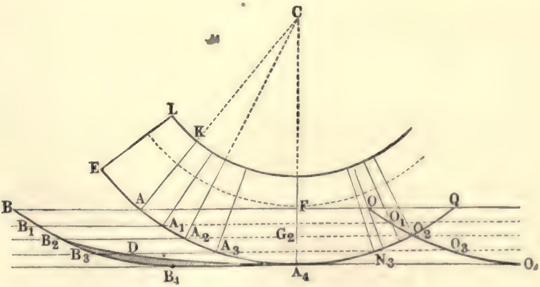
$$Q_3 = 2\,\mu\, d_1\, \sigma_1\, c,$$

and for the second case,

$$Q_3 = 2\,\mu\, d_1\, \sigma_1 \sqrt{c^2 - 2\,g\left(\frac{c-v}{v}\right) d_1}.$$

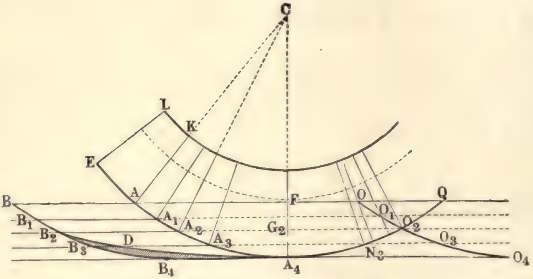
§ **214.**—**Gerstner's Formula.**—The quantity of water lost in the second way above noticed, viz., by passing under the foremost floats without impact, can be approximately determined, according to Gerstner, as follows: From the distance apart of two floats $A\,E = b$, Fig. 434, and the velocities $c$ and $v$ of the water and wheel, we can find the length $A\,B = A_1\,B_1 = A_2\,B_2$, etc., of the water elements between

FIG. 434.



any two floats, viz., $l = \frac{c}{v}\, b$. If, now, the first particle of the element $A\,B$ strikes the float $A\,K$ at $A$, the last particle $B$ of

FIG. 435.



this same element will strike the float $A\,K$ at a point $O$, whose distance $A\ O$ from $A$ is determined by the equation

$$\frac{A\,O}{v} = \frac{B\,O}{c} \quad \text{or} \quad \frac{A\,O}{v} = \frac{A\,O}{c} + \frac{B\,A}{c}.$$

Accordingly

$$\overline{A\,O} = \left(\frac{v}{c-v}\right) \overline{B\,A} = \frac{v\,l}{c-v}.$$

In like manner, for deeper elements, we have

$$\overline{A_1 O_1} = \overline{A_2 O_2} = \overline{A\,O} = \frac{v\,l}{c-v}.$$

The last particle $B_2$ of the element $A_2\,B_2$ will strike the float at $O_2$ just on the circumference. Any deeper element, as $A_3\ B_3$, would strike at $O_3$, but as the float, by reason of its circular motion, has moved higher, the last particle $B_3$ will not strike at all. This holds not only for the last particle $B_3$, but also for a number of particles $B_3\ D$, because the particle at $D$ strikes just on the circumference. The length

$A_3 D$ of that portion of the element $A_3 B_3$ which *does* strike the float is as above

$$\overline{A_3 D} = \frac{c - v}{v} A_3 N.$$

This applies to all elements between $A_2 B_2$ and $A_4 B_4$. The area included then between $A_2 B_2 D A_4 A_3 A_2$, or all the water striking the float, is equal to $\frac{c - v}{v}$ times the circular segment $A_2 O_2 A_4$. The area of this segment is (see "Ingenieur," p. 189),

$$\tfrac{2}{3} A O \times A_4 G_2,$$

hence the area

$$A_2 B_2 D A_4 = \frac{c - v}{v} \times \tfrac{2}{3} \frac{v l}{c - v} A_4 G_2 = \tfrac{2}{3} l A_4 G_2.$$

The ratio, therefore, of the actually impinging water to the whole amount of water is

$$\frac{Q_1}{Q} = \frac{\textit{area } A B B_2 A_2 + \textit{area } A_2 B_2 D A_4}{\textit{area } A B B_4 A_4} = \frac{l F G_2 + \tfrac{2}{3} l A_4 G_2}{l A_4 F}$$

$$= \frac{A_4 F - \tfrac{1}{3} A_4 G_2}{A_4 F} \quad \text{or} \quad \frac{Q_1}{Q} = 1 - \frac{A_4 G_2}{3 A_4 F}.$$

Now $A_4 F$ is approximately

$$\frac{A F_2}{2 r}$$

and

$$A_4 G_2 = \frac{\overline{A_2 G_2}^2}{2 r},$$

consequently

$$\frac{A_4 G_2}{A_4 F} = \frac{\overline{A_2 G_2}^2}{\overline{A F}^2}.$$

But

$$A_2 G_2 = \tfrac{1}{2} A O = \tfrac{1}{2} \frac{v l}{c - v}$$

and

$$A F = \tfrac{1}{2} A Q = \tfrac{1}{2} n_1 b = \tfrac{1}{2} n_1 \frac{v}{c} l,$$

when $n_1$ is the number of immersed floats. Hence

$$\frac{A_4 G_2}{A_4 F} = \frac{1}{n_1^2} \left(\frac{c}{c - v}\right)^2,$$

and finally

$$Q_1 = \left[1 - \frac{1}{3 n_1^2} \left(\frac{c}{c - v}\right)^2\right] Q.$$

We see from this that the loss of water is less the greater the number $n_1$ of immersed floats, and therefore the greater the whole number of floats $n$. Since this number increases with the radius, the greater also the radius.

Example.—If an undershot-wheel in straight race dips 3 floats in the water, and if its velocity is half that of the approaching water, we have for the ratio of the striking water to the whole amount,

$$\frac{Q_1}{Q} = 1 - \tfrac{1}{27} \left(\frac{1}{\tfrac{1}{2}}\right)^2 = 1 - \tfrac{4}{27} = \tfrac{23}{27} = 0{\cdot}85 \text{ per cent,}$$

or 15 per cent of water produces no effect.

Remark.—The above investigation assumes that each water element, after impact, gives place to the next. Since each element, during impact, runs up on the float, this assumption is admissible.

When the race falls off abruptly at $A_4$, we have, instead of $A_2 O_2 A_4$, only its half or $\tfrac{1}{2} l A_4 G_2$, and hence for this case,

$$Q_1 = \left[1 - \frac{2}{3 n_1^2} \left(\frac{c}{c - v}\right)^2\right] Q.$$

§ **215.—Delivery of Undershot-Wheel.**—If, now, we take account of the preceding causes of loss as well as of friction of axle, we can determine the delivery of an undershot water-wheel with considerable exactness. Thus we have

$$L = P v = \frac{(c - v) v}{g} (Q_1 - Q_2) \gamma - \phi \frac{\rho}{r} G v,$$

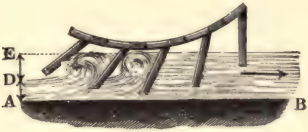
or approximately putting

$$Q_2 = \sigma e c = \frac{\sigma}{d_1} Q, \quad \text{and} \quad Q_1 = \left[1 - \frac{1}{3 n_1^2} \left(\frac{c}{c - v}\right)^2\right] Q,$$

$$P v = \frac{(c - v) v}{g} \left[1 - \frac{\sigma}{d_1} - \frac{1}{3 n_1^2} \left(\frac{c}{c - v}\right)^2\right] Q \gamma - \phi \frac{\rho}{r} G v.$$

In the case where the bottom of race is level, as in Fig. 436,

FIG. 436.



and hence the depth $d_2$ of the issuing stream is greater than the depth of the entering stream, or

$$d_2 = \frac{c}{v} d_1,$$

we have also a reaction of the water upon the floats, the work of which is

$$L_1 = (d_2 - d_1) Q \gamma = \left(\frac{c - v}{v}\right) d_1 Q \gamma.$$

This work is greater the greater the difference $c - v$, and the greater $d_1$. In order then to lose as little work as may be from this cause, the wheel should revolve rapidly and the depth of the approaching stream should be small, or the water should flow in in a thin, broad stream. We can regard this loss due to reaction as a relative loss merely, since, by reason of the elevation of the water level, the total fall is diminished by the amount $d_2 - d_1$, and hence the disposable work is less by $(d_2 - d_1) Q \gamma$. In any case, we shall commit no noticeable error in neglecting this loss in our calculations.

The question now arises, for what ratio of $\frac{v}{c}$ is the delivery a maximum? Relatively, the loss due to axle friction is here very small; we can therefore neglect it in the determination of our maximum, and have therefore only to find the maximum of

$$(c - v)\, v \left(1 - \frac{\sigma}{d_1} - \frac{c^2}{3\, n_1^2\, (c - v)^2}\right),$$

or of

$$\left(1 - \frac{\sigma}{d_1}\right)(c\, v - v^2) - \frac{c^2\, v}{3\, n_1^2\, (c - v)}.$$

Differentiating and putting the first differential coefficient equal to zero, we have

$$\left(1 - \frac{\sigma}{d_1}\right)(c - 2\, v) = \frac{c^3}{3\, n_1^2\, (c - v)^2};$$

whence we find

$$v = \frac{c}{2}\left(1 - \frac{c^2}{3\, n_1^2 \left(1 - \frac{\sigma}{d_1}\right)(c - v)^2}\right),$$

when the delivery is a maximum. We see therefore that the maximum delivery occurs *when the velocity of the circumference is somewhat less than half the velocity of the entering water.*

EXAMPLE.—What is the delivery of an undershot-wheel in a confined straight race, which has 3 ft. fall and uses $Q = 20$ cub. ft. per second? The theoretical velocity of the water is

$$c = \sqrt{2\, g\, h} = 8{\cdot}025\, \sqrt{3} = 13{\cdot}899 \text{ ft.}$$

The effective velocity is, however,

$$0{\cdot}95 \times 13{\cdot}899 = 13{\cdot}204 \text{ ft.}$$

If we make the breadth of the entering stream such that the depth

$$d_1 = 4 \text{ in.} = \tfrac{1}{3} \text{ ft.},$$

then this breadth is

$$e_1 = \frac{Q}{d_1\,c} = \frac{20}{\frac{1}{3} \times 13 \cdot 899} = \frac{60}{13 \cdot 9} = 4 \cdot 615.$$

We can take the wheel breadth $e$ then $= 4 \cdot 75$ ft. If, now, the clearance is $\sigma = \frac{3}{4}$ in., we have

$$\frac{\sigma}{d_1} = \frac{3/4}{4} = \frac{3}{16}.$$

If the radius of the wheel is 10 ft., we can give it 48 floats, each 1 ft. long. If the immersed portion then is

$$\frac{2\sqrt{d_1 \times 2\,r}}{2\,\pi\,r} = \frac{1}{\pi}\sqrt{\frac{2\,d_1}{r}} = 0 \cdot 318\sqrt{\frac{2}{30}} = 0 \cdot 0822,$$

we have 48 × 0·0822 = 3·95, or 4 immersed floats. The most advantageous velocity is then

$$v = \frac{13}{2}\left(1 - \frac{c^2}{3 \times 16 \times \frac{13}{16}\,(c - v)^2}\right) = \frac{13}{2}\left[1 - \frac{1}{39}\left(\frac{c}{c - v}\right)^2\right].$$

We find easily thus $v = 0 \cdot 45\,c$. If, however, we take on account of axle friction $v = 0 \cdot 43\,c$, we have for the effective delivery of the wheel

$$L = \frac{(c - v)\,v}{g}\left[1 - \frac{\sigma}{d_1} - \frac{1}{3\,n_1^2}\left(\frac{c}{c - v}\right)^2\right] Q\,\gamma$$

$$= \frac{0 \cdot 57 \times 0 \cdot 43\,c^2}{g}\left[\frac{13}{16} - \frac{1}{48}\left(\frac{1}{0 \cdot 57}\right)^2\right] 20 \times 62 \cdot 5$$

$$= 0 \cdot 032 \times 0 \cdot 2451 \times 169\,(0 \cdot 8125 - 0 \cdot 0641)\,1235 = 1225 \text{ ft. lbs.}$$

If, now, the weight of the wheel is 7200 lbs., and the radius of the axle 1·75 in., and if the coefficient of friction $\phi = 0 \cdot 1$, we have for the work of the axle friction

$$\phi\,\frac{\rho}{r}\,G\,v = 0 \cdot 1\,\frac{1 \cdot 75}{10 \times 12}\,7200 \times 0 \cdot 43 \times 13 \cdot 9 = 61 \text{ ft. lbs.}$$

The effective delivery is then

$$L = 1225 - 61 = 1164 \text{ ft. lbs., or } 2 \cdot 12 \text{ horse-power,}$$

and finally the efficiency is

$$\eta = \frac{1164}{3 \times 20 \times 62 \cdot 5} = \frac{1164}{3750} = 0 \cdot 313.$$

**§ 216.—Effective Delivery of the Undershot-Wheel.**—Experiments upon the delivery of undershot-wheels in confined straight races, have been made only upon models, by Parcieux, Bossut, Smeaton, Nordwall, and Lagerhjelm. Those by Smeaton and Bossut are of special value. The results of all these investigations agree not only among themselves, but also coincide with theory. The action of the wheels in each case was measured by means of a string, which, passing over the axis, was made to lift weights. Smeaton made his experiments ("Recherches expériment. sur l'eau et le vent," etc.) upon a small wheel of 75 inches circumference, 24 4-inch long and 3-inch broad floats. The chief result at which he arrived was, that the greatest efficiency of an undershot water-wheel in confined straight race occurs for the velocity ratio $\frac{v}{c} = 0 \cdot 34$ to $0 \cdot 52$, and amounts to $0 \cdot 165$ to $0 \cdot 25$. Bossut used in his investigations a wheel of 3 ft. height, with 48, 24, and 12 floats, 5 inches long and 4 to 5 inches wide. He found, entirely in accordance with theory, the effect for 48 floats greater than for 24, and for 24 greater than for 12. He also concluded that it was advantageous to immerse about 25° of the circumference, or

$$\frac{25}{360} \times 48 = \frac{10}{3},$$

or more than 3 floats, in the water. From the investigations of Bossut upon the wheel of 48 floats, a somewhat greater efficiency is found than is given by Smeaton's. Gerstner, who also finds that the results of Bossut correspond better with his theory than those of Smeaton, accounts for the deviation by the fact that the wheel of Smeaton had a less number of floats, and that a considerable backing up of the water occurred. From the results of these two experimenters, we have as a mean, for the effective delivery of such a wheel, without reference to friction,

$$L = 0{\cdot}61\,\frac{(c - v)\,v}{g}\,Q\,\gamma = 1{\cdot}182\,(c - v)\,v\,Q \text{ ft. lbs.}$$

This formula is, however, only sufficiently exact when the clearance does not exceed 1½ inches. Beyond this, we have instead of $Q$, $F\,c$, where $F$ is the area of the immersed float surface, and 0·76 instead of 0·61. According to Christian, therefore (see his "Mécanique industr."),

$$L = 0{\cdot}76\,F\,\gamma\,\frac{(c - v)}{g}\,c\,v = 1{\cdot}472\,(c - v)\,F\,c\,v \text{ ft. lbs.}$$

Moreover, it follows from all these investigations, that the greatest delivery, as indicated by theory, is for the velocity ratio $\frac{v}{c} = 0{\cdot}4$, and that for great velocities this ratio is somewhat less, and for great quantities of water somewhat greater.

In Sweden, experiments upon models, one of 3 and one of 6 ft. diameter, of 72 and 144 floats respectively, are given in the works of Lagerhjelm, Forselles, and Kallstenius. It follows from these that the efficiency of a wheel in confined straight race is still greater, viz., without reference to friction of axle, 0·3 to 0·35, for velocity ratio $\frac{v}{c}$ nearly ½. Since here the number of immersed floats was very great, it was to be expected that but little loss of water should occur, and this high efficiency is hence in agreement with theory.

EXAMPLE.—The empirical formula

$$L = 0{\cdot}61\,\frac{(c - v)\,v}{g}\,Q\,\gamma = 1{\cdot}182\,(c - v)\,v\,Q$$

gives for the example to Art. 215, where

$$c = 13,\ v = 0{\cdot}43,\ c = 5{\cdot}59 \text{ and } Q = 20,$$

for the delivery of the wheel,

$$1{\cdot}182 \times 0{\cdot}57 \times 0{\cdot}43 \times 20 \times 13^2 = 998 \text{ ft. lbs.},$$

while by the theoretical formula we have there found 1225 ft. lbs.

§ **217.—Partition of Water-Power.**—Very often an existing water-power is divided among several wheels, not only because one wheel alone would be too large, but also and especially in order to set in motion the connected machines independently of each other, and thus avoid the connection and disconnection of several machines with the same prime mover. In such division, we may have two cases. We may either divide the water or the fall. In general, we may assert that for pressure-wheels a division of the water, and for impact-wheels, a division of the fall, is best; inasmuch as we have already seen that the efficiency of a high overshot-wheel is greater than that of a smaller overshot or middleshot wheel. Inversely, we can conclude that the loss of water by impact and through clearance space is less for two wheels, one behind the other, than for two side by side, because the height for the lost work $\frac{(c-v)^2}{2g}$ (Vol. I., Art. 436), and the ratio $\frac{\sigma}{d_1}$ of the clearance to the water depth is less than in the last case. For middleshot curb-wheels, where the water acts by pressure and impact, and where the loss of water depends especially upon $\frac{\sigma}{d_1}$, the advantage of the one method of division above the other is indeterminate, and a special investigation is requisite in order to determine the advantage in each special case. In the following, we shall consider only the division of the water-power for *undershot-wheels in confined straight* race.

Let us take *two* wheels behind each other in a horizontal race, and assume the water to arrive at the second with the velocity $v_1$ with which the first revolves. If, now, $c$ is the velocity of the water entering the first wheel, and $v_2$ the velocity of the second wheel, also $Q$ the feed for both wheels, and $\chi$ an empirical constant, then the delivery of these wheels is

$$L_1 = \chi (c - v_1) v_1 Q \quad \text{and} \quad L_2 = \chi (v_1 - v_2) v_2 Q.$$

If both wheels are to have the same delivery, we have

$$(c - v_1)\, v_1 = (v_1 - v_2)\, v_2,$$

and if now for the maximum delivery we put $v_2 = \frac{1}{2} v_1$ we have

$$(c - v_1)\, v_1 = \tfrac{1}{4} v_1^2 \qquad \text{or} \qquad c - v_1 = \tfrac{1}{4} v_1,$$

and hence

$$v_1 = \tfrac{4}{5} c \qquad \text{and} \qquad v_2 = \tfrac{2}{5} c.$$

Therefore for the delivery of these two wheels together,

$$L = L_1 + L_2 = 2\,\chi\, (c - \tfrac{4}{5} c)\, \tfrac{4}{5} c\, Q = \tfrac{8}{25}\, \chi\, c^2\, Q = 0{\cdot}32\, \chi\, c^2\, Q.$$

If we had only one wheel, we should have

$$L = \tfrac{1}{4}\, \chi\, c^2\, Q, \qquad \text{or} \qquad L = 0{\cdot}25\, \chi\, c^2\, Q.$$

Accordingly, by the use of two wheels, we have a gain of 32 – 25 = 7 per cent. For *three* wheels, the gain is still greater. For the third wheel, we have

$$L_3 = \chi\, (v_2 - v_3)\, v_3\, Q,$$

where $v_3$ is the velocity of this third wheel. If again $v_3 = \frac{1}{2} v_2$, and if again this wheel is to have the same delivery as the others, we have

$$v_2 = \tfrac{4}{5} v_1 \qquad \text{and} \qquad c - v_1 = \tfrac{4}{25} v_1,$$

hence

$$v_1 = \tfrac{25}{29} c \qquad \text{and} \qquad v_2 = \tfrac{20}{29} c \qquad \text{and} \qquad v_3 = \tfrac{10}{29} c.$$

The delivery of all three wheels together, then, is

$$L = L_1 + L_2 + L_3 = 3\,\chi\, (c - v_1)\, v_1\, Q = 3\,\chi \times \tfrac{4}{29} \times \tfrac{25}{29}\, c^2\, Q$$
$$= \tfrac{300}{841}\, \chi\, c^2\, Q = 0{\cdot}356\, \chi\, c^2\, Q.$$

With reference to a single wheel, we have a gain of work of 35·6 – 25 = 10·6 per cent.

REMARK.—If we discard the condition that the wheels shall all have the same delivery, the gain is still greater. We assume, in the treatment of this case, the loss of water in a *concentric and close-fitting* race, including three or four floats, small enough to be disregarded. Then we have for the delivery of the first wheel

$$L_1 = \frac{(c - v_1)\, v_1}{g}\, Q\, \gamma,$$

and for the second

$$L_2 = \frac{(v_1 - v_2)\, v_2}{g}\, Q\, \gamma;$$

hence the delivery of both together is

$$L = [(c - v_1)\, v_1 + (v_1 - v_2)\, v_2]\, \frac{Q\, \gamma}{g}.$$

In order that this may be a maximum, we have first $v_2 = \frac{1}{2}\, v_1$, and thus

$$L = (c - \tfrac{3}{4}\, v_1)\, v_1\, \frac{Q\, \gamma}{g}.$$

Here, again, we put $\frac{3}{4}\, v_1 = \frac{1}{2}\, c$, and therefore $v_1 = \frac{2}{3}\, c$ and $v_2 = \frac{1}{3}\, c$; hence

$$L = (\tfrac{2}{3} \times \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{1}{3})\, \frac{Q\, c^2\, \gamma}{g} = \tfrac{1}{3}\, \frac{c^2\, Q\, \gamma}{g} = 0{\cdot}333\, \frac{c^2\, Q\, \gamma}{g}.$$

while one wheel alone gives only $0{\cdot}25\, \frac{c^2\, Q\, \gamma}{g}$, and two wheels for equal delivery only $0{\cdot}32\, \frac{c^2\, Q\, \gamma}{g}$. For three wheels, the advantage is still greater. Here we have

$$v_1 = \tfrac{3}{4}\, c,\ v_2 = \tfrac{2}{4}\, c,\ v_3 = \tfrac{1}{4}\, c,$$

and hence for all three wheels together

$$L = (\tfrac{3}{4} \times \tfrac{1}{4} + \tfrac{2}{4} \times \tfrac{1}{4} + \tfrac{1}{4} \times \tfrac{1}{4})\, \frac{c^2\, Q\, \gamma}{g} = \tfrac{3}{8}\, \frac{c^2\, Q\, \gamma}{g} = 0{\cdot}375\, \frac{c^2\, Q\, \gamma}{g},$$

while one wheel alone $= 0{\cdot}25\, \frac{c^2\, Q\, \gamma}{g}$, and three wheels for equal delivery $= 0{\cdot}356\, \frac{c^2\, Q\, \gamma}{g}$. For four wheels,

$$v_1 = \tfrac{4}{5}\, c,\ v_2 = \tfrac{3}{5}\, c,\ v_3 = \tfrac{2}{5}\, c,\ v_4 = \tfrac{1}{5}\, c,$$

and

$$L = \frac{(4 + 3 + 2 + 1)}{25}\, \frac{Q\, c^2\, \gamma}{g} = \tfrac{4}{5}\, \frac{Q\, c^2\, \gamma}{2g} = \tfrac{4}{5}\, Q\, h\, \gamma.$$

For five wheels,

$$L = \tfrac{5}{6}\, \frac{Q\, c^2\, \gamma}{2g},$$

and for $n$ wheels

$$L = \frac{n}{n+1} \frac{Q c^2 \gamma}{2g}.$$

For an infinite number of wheels, therefore,

$$L = \frac{Q c^2 \gamma}{2g} = Q h \gamma,$$

while one wheel gives only $\frac{1}{2} Q h \gamma$. Considered, therefore, merely theoretically, we see that many wheels, one behind the other, give the entire work of the fall or of the water, while one wheel can only give half as much.

Several wheels, side by side, give together of course the same work as a single wheel for the same feed as all together.

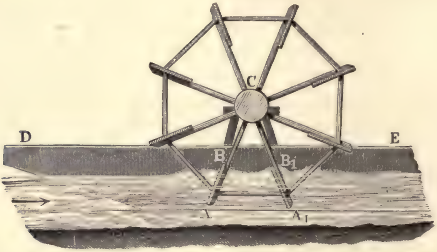
## § 218.—Paddle-Wheels.

—We have still to consider wheels hanging free, not enclosed by a race, but moving in a wide canal or river, and therefore occupying only a portion of the breadth of the flowing stream. Such wheels are sometimes placed on barges and anchored off in the stream, where they serve as motors for mills.

The construction of paddle-wheels varies from that of other wheels, inasmuch as these wheels often have no crown, but have the floats or paddles fastened directly to the arms. These wheels are only 12 to 15 ft. high, and have often only six paddles. It is, however, better to give them 12 or more. The paddles should be made long and broad, that they may intercept a considerable stream, since otherwise, owing to the very moderate velocity of the water, the *vis viva* would be not very great. The length of the paddles then is from 6 to 18 ft., and their breadth 1 to 2 ft. It is also advisable to give the paddles an inclination of 10° to 20° to the stream, and not to have them dip more than half in the water. Fig. 437 represents a floating mill-wheel; $A\ C$ is the wheel, which has 8 paddles, $A\ B\ A_1\ B_1$, etc., and $D\ E$ is the barge which carries the axle $C$. Often we may have two such wheels, the common axis being carried by the barge.

The delivery of paddle-wheels is smaller than for wheels in confined race, for two reasons, for here not only a part of the water escapes at the sides and under the paddles, but also a larger amount goes through the wheel

FIG. 437.



without impact, because the number of the immersed paddles is very small, sometimes only from 1½ to 2.

§ **219.—Delivery of Paddle-Wheels.**—We can put the theoretical mechanical effect or delivery of a free hanging paddle-wheel, just as for a wheel in confined race, equal to

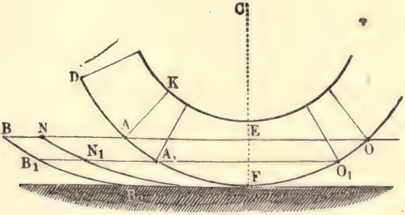
$$L = P v = \frac{(c - v)\, v\, c}{g} F \gamma,$$

where, again, $c$ and $v$ are the velocities of water and wheel, and $F$ is the area of the immersed portion of a float (without reference to the piling up of water on the float). On account of the loss of water, we must, however, multiply this expression by a coefficient, whose value we can, according to Gerstner, at least partially determine. If the number $n_1$ of immersed floats is not very small, we have here, as for the undershot-wheel, the amount of water actually producing impact,

$$Q_1 = \left(1 - \frac{c^2}{3\, n_1^2\, (c - v)^2}\right) Q.$$

If, however, the number $n_1$ of immersed floats is small, it may happen that even the uppermost elements at $A\,B$, Fig. 438, may not all strike against the float $A\,K$, but rather a portion $A\,N$ only may really produce impact, the rest going through without touching. In this case, there is a loss of water for all the elements, and the ratio of the impinging water to the approaching is

FIG. 438.



$$\frac{Q_1}{Q} = \frac{area\ A\,N\,N_1\,F\,A_1}{area\ A\,B\,B_2\,F\,A},$$

or since from Art. 214, *area* $A\,N\,N_1\,F\,A_1 = \frac{c-v}{v}$ times the segment $A\,O\,F$;

$$\frac{Q_1}{Q} = \frac{\frac{2}{3}\left(\frac{c-v}{v}\right)A\,O}{A\,B} = \frac{2}{3}\left(\frac{c-v}{v}\right)\frac{n_1\,A\,D}{\frac{c}{v}\,A\,D} = \frac{2\,n_1}{3}\left(\frac{c-v}{c}\right).$$

We have, therefore, for this case, for the delivery of the wheel,

$$\text{I. } L = \frac{(c-v)\,v}{g} \cdot \frac{2\,n_1}{3}\left(\frac{c-v}{c}\right)Q\,\gamma = \frac{2}{3}\,n_1\,\frac{(c-v)^2\,v}{g\,c}\,Q\,\gamma$$

$$= \frac{2}{3}\,n_1\,\frac{(c-v)^2\,v}{g}\,F\,\gamma.$$

The greatest delivery is accordingly not for $v = \frac{1}{2} c$, but for $v = \frac{1}{3} c$, and this delivery is

$$L = \tfrac{2}{3} n_1 \times \frac{4}{27} \times \frac{c^3}{g} F \gamma = \frac{8 n_1}{81} \frac{c^3}{g} F \gamma.$$

If, now we put $F c = Q$, we have

$$L = \frac{8 n_1}{81} \frac{c^2}{g} Q \gamma = \frac{16 n_1}{81} \frac{c^2}{2 g} Q \gamma,$$

and hence the efficiency is

$$\eta = \frac{16 n_1}{81}.$$

For example, for

$$n_1 = \tfrac{3}{2},\ \eta = \frac{24}{81} = \frac{8}{27} = 0 \cdot 296.$$

The above formula (I.) cannot, however, be applied when the number of the floats is considerable, for it assumes that $A N < A B$, or

$$\frac{c - v}{v} A O < A B,$$

or

$$\frac{c - v}{v} < \frac{\frac{c}{v} A D}{n_1 A D},$$

or

$$n_1 < \frac{c}{c - v}.$$

If, now, $v = \frac{1}{3} c$, for which velocity we have the maximum delivery, then $n_1 < \frac{3}{2}$. If, however, $v = \frac{1}{2} c$, we have $n_1 < 2$, etc. Therefore, when two or more floats are immersed in the water at once, the above does not hold, and we have the formula for the undershot-wheel already given, viz.,

$$\text{II. } L = \left(1 - \frac{c^2}{3 n_1^2 (c - v)^2}\right) \frac{(c - v) v c}{g} F \gamma.$$

The number $n_1$ of immersed floats can be easily found from the whole number $n$, the radius $r$ and the depth of the immersed portion $EF = e_1$. Thus

$$\frac{n_1}{n} = \frac{AO}{2\pi r}.$$

But

$$AO = 2\,AE = 2\sqrt{2\,r\,e_1},$$

hence

$$\frac{n_1}{n} = \frac{\sqrt{2\,r\,e_1}}{\pi r} = 0{\cdot}45\sqrt{\frac{e_1}{r}}.$$

EXAMPLE.—What is the delivery of a paddle-wheel 15 ft. high with 8 floats, each 12 ft. by 1 ft. in the water, the velocity of the water being 5 ft.?

We have here

$$\frac{n_1}{n} = 0{\cdot}45\sqrt{\frac{1}{7{\cdot}5}} = 0{\cdot}45 \times 0{\cdot}365 = 0{\cdot}164,$$

hence

$$n_1 = 0{\cdot}164 \times 8 = 1{\cdot}3.$$

Therefore the formula

$$L = \tfrac{2}{3}\,n_1\,\frac{(c-v)^2\,v\,F\gamma}{g}$$

applies.

If we allow the wheel to revolve with a velocity of 2 ft., we have then for $L$,

$$L = \tfrac{2}{3} \times 1{\cdot}3\,\frac{3^2 \times 2}{g} \times 12 \times 1 \times 62{\cdot}5 = 0{\cdot}032 \times 1{\cdot}3 \times 9 \times 988 = 370 \text{ ft. lbs.}$$

If we give the wheel 16 floats, in order to obtain a greater delivery, we have $n_1 = 2{\cdot}6$, and hence must apply formula II. Thus,

$$L = \frac{(5-2) \times 5 \times 2}{g}\left(1 - \frac{5^2}{3 \times 2{\cdot}6^2 \times 3^2}\right) \times 12 \times 1 \times 62{\cdot}5$$

$$= 0{\cdot}032 \times 0{\cdot}863 \times 22230 = 614 \text{ ft. lbs.}$$

§ **220.—Experiments upon Paddle-Wheels.**—Experiments upon the delivery of water-wheels in open current have been made by Deparcieux, Bossut, and Poncelet. The most extended are those of Bossut made upon a model. This wheel had a height of 0·975 metre (3 ft.), and contained 24 floats of 0·135 metre (6½ inches) long, dipping 0·108 metre (4½ inches) into the water, which had a velocity of 1·854 metre (6 ft.) per second. As the result of the experiments, we have for the coefficient by which the expression

$$L = \frac{(c - v)^2 v}{g} F \gamma$$

must be multiplied in order to obtain the effective delivery, $\chi = 1 \cdot 37$ to $1 \cdot 79$, and, on the other hand, for the coefficient of the expression

$$L = \frac{(c - v) v c}{g} F \gamma,$$

we have

$$\chi = 0 \cdot 847 \quad \text{to} \quad \chi = 0 \cdot 706$$

(see D'Aubuisson's Hydraulique, Art. 352). The limiting values of these last coefficients are somewhat closer than the first. Since, however, the wheel had 24 floats, this was to be expected, as for this case formula II. of the preceding article applies. In general, we should make the number of floats so great that at least two shall always be in the water, and hence can use the last formula above with the coefficient 0·8. Therefore

$$L = 0 \cdot 8 \frac{(c - v) c v}{g} F \gamma = 1 \cdot 55 (c - v) c v F \text{ ft. lbs.}$$

The observations of Poncelet upon three wheels in the Rhone coincide with the above also. These wheels had floats 2½ to 2⅔ metres (8 to 10 ft.) long, ⅔ to ¾ metre (2¼ to 2¾ ft.) deep in the water, which last had a velocity of 1⅕ to 2 metres (4 to 6½ ft.) per second. Poncelet also cites the results of Boistard and Christian, which also agree well with the above.

According to the researches of Bossut, and in close correspondence with theory, the greatest delivery occurs when the velocity $v = 0 \cdot 4 c$. This same result was also found by Poncelet. If in the above formula, then, we make $v = 0 \cdot 4\, c$, we have for the effective work,

$$L = 0 \cdot 8 \frac{0 \cdot 6 \times 0 \cdot 4\, c^3}{g} F\gamma = 0 \cdot 192 \frac{c^3}{g} F\gamma = 0 \cdot 384 \frac{c^2}{2g} Q\gamma,$$

and therefore for the efficiency $\eta = 0 \cdot 384$.

The investigations of Deparcieux were especially directed to the determination of the best position of the floats. From them, as also from those of Bossut, it follows that an inclination of 60° against the stream is the best.

REMARK.—It has been long questioned which of the two expressions,

$$L = \frac{\chi (c - v)^2 v}{g} F\gamma,$$

or

$$L = \frac{\chi_1 (c - v) c v}{g} F\gamma,$$

is the more exact. The first is known as Parent's and the second as Borda's formula. Although for a wheel in open current, not all the water which strikes the floats takes, after impact, the velocity of the wheel, since the water can escape at the circumference, still for floats of considerable area at least the greatest part of the water takes after impact the velocity of the wheel itself, and hence the better agreement of experiment with Borda's formula. The formula of Gerstner developed in the preceding article agrees in form with that of Parent, for this last is, without coefficient,

$$L = \frac{(c - v)^2 v}{2g} F\gamma,$$

and is developed under the assumption that the impact is measured by the height corresponding to the relative velocity $(c - v)$. (Compare Vol. I., Art. 511, where the force of impact is given at $1 \cdot 86 \frac{c^2}{2g} F\gamma$, when $v = 0$.)

EXAMPLE.—For the ship-mill wheel already considered in the preceding article, we have $c = 5$, $v = 2$, $F = 12 \times 1 = 12$, and hence the effective delivery is, according to Poncelet,

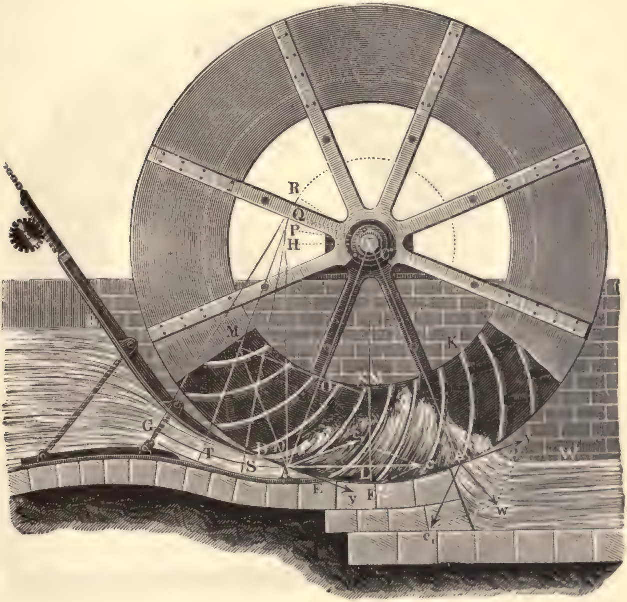
$$L = 1 \cdot 55 \times 3 \times 2 \times 5 \times 12 = 608 \text{ ft. lbs.},$$

while we have already found by the theoretical formulæ, for 8 floats, 395 ft. lbs., and for 16 floats 656 ft. lbs.

§ **221.—Poncelet's Wheels.**—When the floats of undershot-wheels are curved in such a manner that the entering jet of water is allowed to flow along their concave sides and press against them without causing any shock, a greater effect is obtained than when the water strikes more or less perpendicularly against plane floats. Such wheels with curved floats are called after their inventor, Poncelet's wheels. They are especially useful with small falls (under 6 ft.), since they are more effective than undershot-wheels with or without a breast. With greater falls, however, they are inferior in efficiency to middleshot-wheels; their construction is also in this case more difficult, as we shall see further on, so that their use is not desirable with falls of over 6 ft. Poncelet treats these wheels at length, in the special work, "Mémoire sur les roues hydrauliques à aubes courbes, mues par-dessous." Metz, 1827. Their arrangement can be seen in Fig. 441, which shows the lower half of a wheel of this sort. In $C$ we see the axis, and in $A\,K$, $A_1\,K_1$, etc., the floats; $B\,D$ is the inclined sluice, and $T\,A$ the entering stream of water ascending and descending on the floats $A\,K$ and $A_1\,K_1$, and $W$ is the surface of the tail water. In order that almost all the water may have an effect, we must give the wheel only a very slight clearance in the wheel race, and to prevent partial contraction, the lower corner of the sluice must be rounded off underneath; moreover, in order that as little energy as possible may be lost through friction of the water in the supply passage, the opening is moved very close to the wheel, and the sluice is inclined to the horizon; the head race also receives an inclination of $\frac{1}{10}$ to $\frac{1}{15}$, in order to compensate for the friction of the water in it.

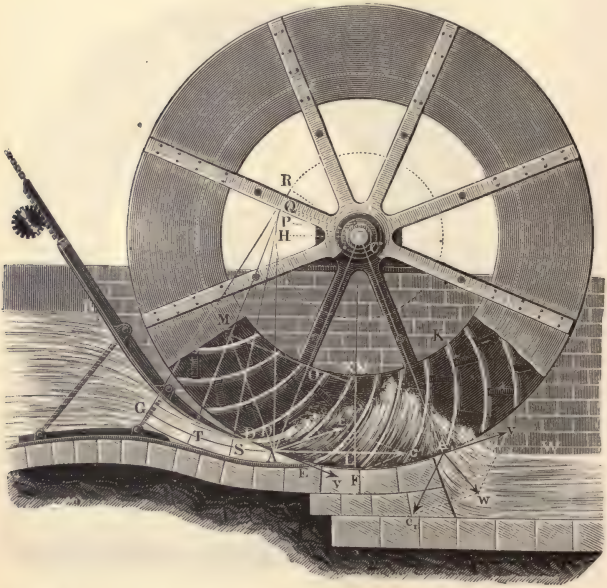
The wheel is generally surrounded by a circular breast which extends over the arc included between the extremities of two floats at least; and in order that the wheel may not be drowned in the tail water, a fall of half a foot is made behind this breast, and for the same object the tail race is made wider. Poncelet's wheels are built from 10 to 20 ft. high, and have from 32 to 48 floats of plate iron or of wood. Wooden floats are composed of staves like a barrel, and must be sharpened at their outer ends or furnished with an edge of plate iron. But floats of plate iron are much more suitable. The use of iron instead of wood is especially to be recommended with Poncelet's wheels, because the efficiency of these wheels depends essentially upon a good construction. The sluice opening is generally made, at the most, 1 ft. high, but with greater falls of 5 to 6 ft., only ½ ft. and even less.

FIG. 441.



§ **222.—Theory of Poncelet's Wheels.**—To obtain as great an effect as possible with one of Poncelet's wheels, it is necessary that the water should enter the wheel without any shock. If $\overline{Ac} = c$, Fig. 442, is the velocity of entrance of the water, and $\overline{Av} = v$ that of the circumference of the

FIG. 442.



wheel, from the parallelogram $A\,v\,c\,c_1$, which has for its side $\overline{Av} = v$, and for its diagonal $\overline{Ac} = c$, we obtain in the side $\overline{Ac_1} = c_1$, the magnitude and direction of the velocity of the water relative to the wheel; if, therefore, we make the float $A\,K$ tangent to $A\,c_1$, the water will begin to ascend this float with the velocity $c_1$, and without causing any shock. Putting the angle $c\,A\,v$, by which the direction of the water arriving at the wheel deviates from the circumference of the latter or from the tangent $A\,v$, equal to $\alpha$, we have the relative velocity of the water as it begins to ascend the float,

$$c_1 = \sqrt{c^2 + v^2 - 2\, c\, v \cos. \alpha}\,;$$

and for the angle $v\, A\, c_1 = \beta$, by which the direction of the float deviates from the circumference or tangent $A\, v$,

$$\sin. \beta = \frac{c \sin. \alpha}{c_1}.$$

In order that the water may enter the wheel at an angle $\alpha$ not only in the centre of the stream, but throughout its entire depth, and therefore at $D$ and $E$ also, it must be brought to the wheel in a path $G\, A$, which has the form of the evolute of a circle. The circle of which $G\, A$ is the evolute must have the same centre $C$ as the wheel, and the generating line must in the beginning stand at right angles to $A\, A_1$ or to the motion of the stream or its entrance into the wheel. If we set off distances equal to half the depth of the stream from this evolute $G\, A_1$, the evolute so formed will cut the circumference of the wheel in $D$ and $E$ at the same angle as the first at $A$. To construct the evolute corresponding to the axis of the entering stream, set off in the involute the arbitrary portions $H\, P$, $P\, Q$, etc., and draw tangents through the points $H$, $P$, etc., so determined. and make these tangents equal to the first tangent $A\, H$ plus the intervening arc $H\, P$, $H\, Q$, etc.

The water ascends the float like a solid body, with decreasing velocity, while at the same time it has the velocity of revolution $v$ of the float. Having attained a certain height, it has lost all its relative velocity, and now falls down along the float with accelerated motion, so that at last it arrives again at the outer end $A_1$ with the same velocity $c_1$ with which it began to ascend. If, by virtue of the parallelogram of velocities, we now find the component of $\overline{A_1\, c_1} = c_1$, the relative velocity of the water which issues at $A_1$, and of $\overline{A_1\, v} = v$, the velocity of the circumference, we obtain in the diagonal $A_1 w = w$ the absolute velocity of the water which flows off from the wheel. This velocity is

$$w = \sqrt{c_1^2 + v^2 - 2\, c_1\, v \cos. \beta},$$

and therefore the mechanical work which the tail water retains and carries off without imparting it to the wheel is

$$L_1 = \frac{w^2}{2g} Q \gamma = \frac{c_1^2 + v^2 - 2\, c_1\, v \cos. \beta}{2g} \cdot Q \gamma.$$

If we subtract this loss from the work

$$\frac{c^2}{2g} \cdot Q \gamma$$

which the water can furnish by virtue of its *vis viva* before its entrance into wheel, we obtain the following expression for the theoretical performance of the wheel,

$$L = \left(\frac{c^2}{2g} - \frac{w^2}{2g}\right) Q \gamma = \left(\frac{c^2 - w^2}{2g}\right) Q \gamma,$$

$$= \left(\frac{c^2 - c_1^2 - v^2 + 2\, c_1\, v \cos. \beta}{2g}\right) Q \gamma,$$

or, since

$$c^2 = c_1^2 + v^2 + 2\, c_1\, v \cos. \beta,$$

we have also

$$L = \frac{2\, c_1\, v \cos. \beta}{g} Q \gamma,$$

and it follows, when we put $c_1 \cos. \beta = c \cos. \alpha - v$, that the following is the power of the wheel,

$$L = \frac{2\, v\, (c \cos. \alpha - v)}{g} Q \gamma.$$

We can now easily see that the power is a maximum for $v = \frac{1}{2} c \cos. \alpha$, and is then

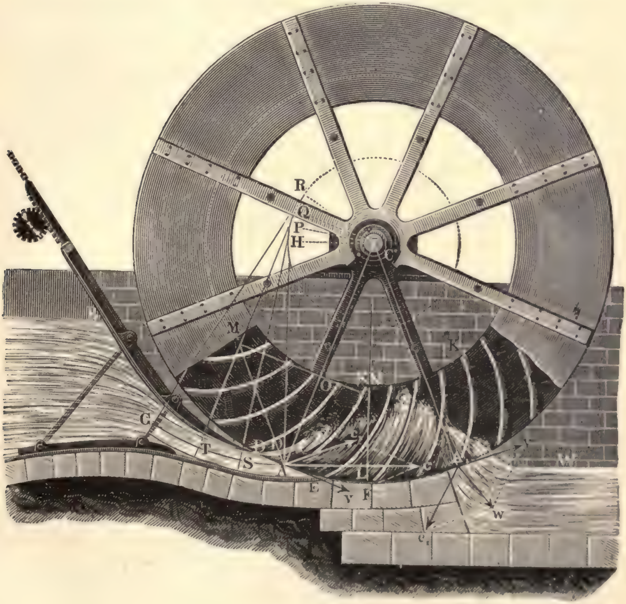
$$L = \frac{c^2 \cos.^2 \alpha}{2g} \cdot Q\gamma,$$

and that the loss of energy is zero, and therefore the total disposable work

$$L = \frac{c^2}{2g} \cdot Q\gamma$$

is gained, when *cos.* $\alpha$ is made equal to unity, and therefore $\alpha = 0$.

FIG. 443.



Although it is impossible to make the angle of entrance $\alpha$ absolutely zero, nevertheless it follows, from the preceding, that $\alpha$ must not be very large (not over 20°) to obtain a good effect, and we can also see from this that the velocity of the circumference must be made only a little less than half the absolute velocity of the entering stream in order to obtain a high degree of efficiency from the wheel.

§ **223.**—The vertical height $LN$ to which the water ascends as it passes along the float would be equal to $\frac{c_1^2}{2g}$, if the wheel were at rest; but as the latter is revolving with the velocity $v$, a centrifugal force is produced which acts nearly in the direction of gravity, and causes an acceleration $p$, which can be put equal to $\frac{v_1^2}{r_1}$, where $r_1$ denotes the mean radius of the wheel, and $v_1$ the mean velocity of the crown, or the velocity of the middle of its depth. (See Vol. I., § 42.) We can therefore put

$$(g+p)\,h_1 = \frac{c_1^2}{2}, \quad \text{or} \quad \left(g + \frac{v_1^2}{r_1}\right) h_1 = \frac{c_1^2}{2};$$

and hence the required height of ascent is

$$h_1 = \frac{c_1^2}{2\left(g + \frac{v_1^2}{r_1}\right)}.$$

In order that the water may not run over the edge at $N$, it is necessary that the crown should be of a certain depth $\overline{FN} = d$, which is determined from the equation

$$d = \overline{LN} + \overline{LF} = h_1 + \overline{CF} - \overline{CL};$$

that is,

$$d = h_1 + r - r \cos. ACF = \frac{c_1^2}{2\left(g + \frac{v_1^2}{r_1}\right)} + r\,(1 - \cos. \lambda),$$

in which $\lambda$ denotes the angle $ACF$ between the point of entrance $A$ and the lowest point of the wheel $F$. But the depth of the stream is always to be added to this, because, under the assumption of a mean velocity, the upper filaments of water would ascend so much higher than the lower ones.

We therefore make the depth of crown

$$d = d_1 + \frac{c_1^2}{2\left(g + \frac{v_1^2}{r_1}\right)} + r\,(1 - cos.\,\lambda).$$

The width of the wheel must be made equal to the width of the stream,

$$e = \frac{Q}{d_1\, c}.$$

If we assume the space $d\,e\,v_1$ to be two to two and a half times as great as $Q$, the quantity of water used per second, we have the equation

$$d\,v_1 = 2\,d_1\,c \text{ to } \tfrac{5}{2}\,d_1\,c,$$

from which we obtain the depth of the stream,

$$d_1 = \tfrac{2}{5}\,\frac{d\,v_1}{c} \text{ to } \tfrac{1}{2}\,\frac{d\,v_1}{c}.$$

Since

$$\frac{v_1}{v} = \frac{r - \frac{1}{2}\,d}{r}, \qquad \text{we have also} \qquad v_1 = v\left(1 - \frac{d}{2\,r}\right),$$

and accordingly

$$d_1 = \tfrac{2}{5}\left(1 - \frac{d}{2\,r}\right)\frac{d\,v}{c} \text{ to } \tfrac{1}{2}\left(1 - \frac{d}{2\,r}\right)\frac{d\,v}{c},$$

or, putting $v = \frac{1}{2}\,c\ cos.\,\alpha$,

$$d_1 = \tfrac{1}{5}\left(1 - \frac{d}{2\,r}\right) d\ cos.\,\alpha \text{ to } \tfrac{1}{4}\left(1 - \frac{d}{2\,r}\right) d\ cos.\,\alpha.$$

According to Morin, $d$ is to be made equal to $\frac{1}{3}\,r$ to $\frac{1}{2}\,r$, and therefore the radius $r$ is only two to three times as large as the depth of the crown.

Another important relation is the determination of the points of entrance and exit, or the length of the water-retaining arc $A\,A_1$, which had best be divided equally at $F$, the lowest point of the wheel. The length of this arc depends upon the time which the water requires to ascend and descend the floats. But to find it, the form and length of the floats must be known. If $t$ is the time in question, we can put

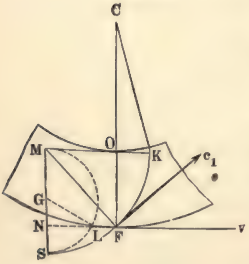
$$A\,A_1 = 2\,\lambda\,r = v\,t,$$

and hence the arc which measures the distance of the points of entrance and exit ($A$ and $A_1$) from the bottom of the wheel $F$, is

$$\lambda = \frac{v\,t}{2\,r}.$$

§ **224.**—In order that, when the water has reached the highest point $K$ on the float, Fig. 444, it may not tumble back on itself, but may glide down along the float, the inner end $K$ must not have any inclination backwards when the float is in its lowest position $F\,K$; but, on the other hand, in order that the float may not be unnecessarily long, the end $K$ must not cut the inner circumference of the wheel at too acute an angle; for these reasons, it is best to have the inner end of the float vertical when the float itself is in its middle position. If a cylindrical form is chosen for the float, we obtain $M$, the centre of its circular section, by drawing $M\,F$ at right angles to $F\,c_1$ and $O\,M$ horizontal. From the depth of wheel or crown $\overline{F\,O} = d$, we obtain the radius of curvature $M\,F = K\,M = R$, since the angle $M\,F\,O = c.\ F\,v = \beta$, and therefore

FIG. 444.



$$R = \frac{d}{cos.\ \beta}.$$

The time occupied by the water in ascending and descending the arc $FK$ is found in the same manner as the time of oscillation of a pendulum, after introducing, in place of the acceleration of gravity, the sum $g + \frac{v_1^2}{a_1}$ of the acceleration of gravity, and the mean acceleration arising from centrifugal force.

This time is accurately found by the following formula, viz.:

$$t = \tfrac{1}{2} \sqrt{\frac{R}{g}} \left(\phi + (\phi + sin.\ \phi) \frac{h}{8R}\right),$$

which was developed in Vol. I., § 322, and which gives the time $t$ it takes a pendulum to pass through the arc $FK$ when $R$ denotes the radius $MF = MK$ of the circular arc formed by the float, $h$ the total height of fall $\overline{MS} = \overline{MF} = R$, and $\phi$ the angle $MGL$ at the centre of the circle $MLS$ described upon $MS = R$ as a diameter—the versed sine of this angle being

$$MN = OF = MF\,cos.\ FMN = R\,cos.\ v\,F\,c_1 = R\ cos.\ \beta.$$

This angle $\phi$ is determined by the following formula:

$$cos.\ \phi = -\frac{NG}{LG} = -\frac{MN - MG}{MG} = -\frac{R\,cos.\ \beta - \frac{1}{2}R}{\frac{1}{2}R}$$

$$= 1 - 2\ cos.\ \beta,$$

or

$$sin.\ \tfrac{1}{2}\phi = \sqrt{cos.\ \beta}.$$

If we now put in place of $g$, on account of centrifugal force, $g + \frac{v_1^2}{r_1}$, we obtain the time of ascent and descent through the arc $FK$:

$$2\,t = \left(\phi + \frac{\phi + sin.\ \phi}{8}\right) \sqrt{\frac{R}{g + \frac{v_1^2}{r_1}}}$$

$$= \left(\frac{9\phi + sin.\,\phi}{8}\right)\sqrt{\frac{R}{g + \frac{v_1^2}{r_1}}},$$

and hence the length of the arc $A\,A_1$ (Fig. 443) is

$$b = 2\,\lambda\,a = 2\,v\,t = v\left(\frac{9\phi + sin.\,\phi}{8}\right)\sqrt{\frac{R}{g + \frac{v_1^2}{r_1}}}.$$

§ **225.—Dimensions of a Poncelet Wheel.**—With the aid of the preceding results, rules must now be established for the arrangement and construction of Poncelet wheels. We can regard as known only the quantity of water $Q$ and the fall $h$, measured from one surface of the water to the other, and hence have to compute the velocities $c$, $c_1$, and $v$, the angles $\alpha$, $\beta$ and $\lambda$, and also the dimensions of the wheel $r$, $d$, $e$, etc.

We have, approximately, the following:

$$v = \tfrac{1}{2}\,c = \tfrac{1}{2}\,\sqrt{2\,g\,h},\ d = \tfrac{1}{4}\,h \quad \text{and} \quad d_1 = \tfrac{1}{4}\,d = \tfrac{1}{16}\,h.$$

In the formula

$$\lambda = \frac{v}{r}\left(\frac{9\phi + sin.\,\phi}{16}\right)\sqrt{\frac{R}{g + \frac{v_1^2}{r_1}}},$$

we can put approximately $\phi = \pi$, and therefore $sin.\,\phi = 0$, and also $v_1 = v = \frac{1}{2}\,c$, $r_1 = r$ and $R = d = \frac{1}{4}\,h$, so that we have

$$\lambda = 1\cdot767\,\frac{\sqrt{2\,g\,h}}{2\,r}\sqrt{\frac{\frac{1}{4}\,h}{g + \frac{c^2}{4\,r}}} = \frac{0\cdot883\,h}{r\sqrt{2 + \frac{h}{r}}},$$

and therefore inversely,

$$r^2 + \tfrac{1}{2}\,h\,r = \tfrac{1}{2}\left(\frac{0\cdot883\ h}{\lambda}\right)^2,$$

and hence the radius

$$r = \frac{h}{4}\left[\sqrt{8\left(\frac{0 \cdot 883}{\lambda}\right)^2 + 1} - 1\right] = \frac{h}{4}\left[\sqrt{\frac{6 \cdot 238}{\lambda^2} + 1} - 1\right].$$

It is best to let the water into the wheel in a horizontal direction, and accordingly we can assume $\alpha = \lambda$, and therefore put

(1) $\alpha = \lambda = 20°$, and hence $arc\ \lambda = 0 \cdot 3491$.

From this we obtain the radius of the wheel,

(2) $r = 1 \cdot 56\, h$.

The velocity with which the water is discharged from the sluice into the wheel is,

$$(3)\ c = \mu \sqrt{2g(h - \tfrac{1}{2} d_1)} = \mu \sqrt{2g \cdot \tfrac{31}{32} h}$$

$$= 0 \cdot 98\, \mu \sqrt{2gh},$$

while the best velocity of the wheel is

(4) $v = \frac{1}{2} c\ cos.\ \alpha$,

and the number of revolutions per minute

(5) $u = \frac{30\, v}{\pi r}$.

$\beta$, the angle of the floats, is also determined by the formula,

$$cot.\ \beta = cot.\ \alpha - \frac{v}{c\ sin.\ \alpha} = \tfrac{1}{2}\ cot.\ \alpha,$$

that is, by

(6) $tan.\ \beta = 2\ tan.\ \alpha$.

For the relative velocity of the water as it begins to ascend the float, we obtain

$$(7)\ c_1 = c\,\frac{\sin.\,\alpha}{\sin.\,\beta} = \frac{v}{\cos.\,\beta},$$

and, if we take approximately

$$\frac{v_1^2}{r_1} = \left(1 - \frac{d}{2\,r}\right)\frac{v^2}{r} = \left(1 - \frac{h}{8\,r}\right)\frac{v^2}{r} = 0{\cdot}9\,\frac{v^2}{r},$$

the depth of the wheel is determined more precisely by the equation

$$d = d_1 + \frac{c_1^2}{2\left(g + 0{\cdot}9\,\frac{v^2}{r}\right)} + r\,(1 - \cos.\,20^\circ)$$

or

$$(8)\ d = \tfrac{2}{3}\,\frac{c_1^2}{g + 0{\cdot}9\,\frac{v^2}{r}} + 0{\cdot}08\,r.$$

In order that the water may not fall over the edge of the float, even when the wheel is moving at a slower rate, a few inches more are added to this depth.

The depth of the stream, more accurately determined, is

$$(9)\ d_1 = \tfrac{1}{4}\left(1 - \frac{d}{2\,r}\right) d\cos.\,\alpha,$$

and the width of wheel,

$$(10)\ e = \frac{Q}{d_1\,c} = \frac{2\,Q}{d\,v_1}.$$

Finally, the radius of the curvature of the floats is

$$(11)\ R = \frac{d}{\cos.\,\beta},$$

and for the auxiliary angle $\phi$ we have

$$(12)\ \sin.\,\tfrac{1}{2}\,\phi = \sqrt{\cos.\,\beta}.$$

With the quantities $v$, $r$, $R$ and $\phi$, we can also more accurately determine

$$(13)\ \lambda = \frac{v}{r}\left(\frac{9\phi + sin.\ \phi}{16}\right)\sqrt{\frac{R}{g \quad 0{\cdot}9\,\frac{v^2}{r}}}.$$

Taking $b$, the mean distance of any two floats from each other, equal to 1 ft., we obtain finally the number of floats,

$$(14)\ n = 2\,\pi\,r_1.$$

EXAMPLE.—A Poncelet wheel is to be designed and computed for a fall $h = 4{\cdot}5$ ft., and a quantity of water $Q = 24$ cubic ft. per second.

Taking $\alpha = \lambda = 20°$, we first obtain the radius $r = 1{\cdot}56\ h = 7{\cdot}0$ ft., and if we put the coefficient of velocity equal to the coefficient of efflux $\mu$, equal to $0{\cdot}90$, we have for the mean velocity of the water at its entrance into the wheel,

$$c = 0{\cdot}98\ \mu\ \sqrt{2\,g\,h} = 0{\cdot}882 \times 8{\cdot}025\ \sqrt{4{\cdot}5} = 15{\cdot}0125 \text{ ft.},$$

and for the best surface velocity of the wheel,

$$v = \tfrac{1}{2}\ c\ cos.\ \alpha = 7{\cdot}506\ cos.\ 20° = 7{\cdot}506 \times 0{\cdot}940 = 7{\cdot}056 \text{ ft.},$$

and the number of revolutions per minute,

$$u = \frac{30\,v}{\pi\,r} = \frac{30 \times 7{\cdot}056}{7\,\pi} = 9{\cdot}63, \text{ or about } 9\tfrac{1}{2}.$$

For the angle $\beta$, we have

$$tan.\ \beta = 2\ tan.\ \alpha = 2\ tan.\ 20° = 2 \times 0{\cdot}364 = 0{\cdot}728,$$

and therefore

$$\beta = 36°\ 3'.$$

or, in round numbers, $\beta = 36°$.

The initial velocity of the water ascending the floats is

$$c_1 = \frac{v}{cos.\ \beta} = \frac{7{\cdot}056}{cos.\ 36°\ 3'} = \frac{7{\cdot}056}{0{\cdot}8085} = 8{\cdot}73,$$

and therefore the necessary depth of crown is

$$d = \tfrac{2}{3}\,\frac{c_1^2}{g + 0{\cdot}9\,\frac{v^2}{r}} + 0{\cdot}08\ r = \tfrac{2}{3}\,\frac{\overline{8{\cdot}73}^2}{32{\cdot}2 + 0{\cdot}9 \times \frac{\overline{7{\cdot}056}^2}{7}} + 0{\cdot}08 \times 7$$

$$= \frac{2}{3}\,\frac{76 \cdot 21}{38 \cdot 60} + 0 \cdot 56 = 1 \cdot 32 + 0 \cdot 56 = 1 \cdot 88 \text{ ft.},$$

for which we might take 1·90 ft.

The depth of the stream is

$$d_1 = \frac{1}{4}\left(1 - \frac{d}{2\,r}\right) d \cos.\,\alpha = \frac{1}{4}\left(1 - \frac{1 \cdot 90}{14}\right) 1 \cdot 90 \cos.\ 20^\circ$$

$$= 0 \cdot 216 \times 1.90 \times 0 \cdot 94 = 0 \cdot 386 \text{ ft.},$$

and the width of the wheel is

$$e = \frac{Q}{d_1\,c} = \frac{24}{0 \cdot 386 \times 15 \cdot 01} = 4 \cdot 14 \text{ ft.}$$

The radius of the floats measures

$$r = \frac{d}{\cos.\ \beta} = \frac{1 \cdot 90}{\cos.\ 36^\circ\ 3'} = 2 \cdot 35 \text{ ft.},$$

and for the corresponding central angle $\phi$, we have

$$\sin.\ \tfrac{1}{2}\phi = \sqrt{\cos.\ \beta} = \sqrt{\cos.\ 36^\circ\ 3'} = 0 \cdot 8992,$$

and therefore

$$\tfrac{1}{2}\,\phi = 64^\circ\ 3' \qquad \text{and} \qquad \phi = 128^\circ\ 6'.$$

We now get with more accuracy

$$\lambda = \frac{v}{r} \cdot \frac{9\,\phi + \sin.\ \phi}{16} \cdot \sqrt{\frac{R}{g + 0.9\,\frac{v^2}{r}}}$$

$$= \frac{7 \cdot 056}{7} \cdot \frac{9 \times 2 \cdot 236 + 0.7869}{16} \cdot \sqrt{\frac{2 \cdot 35}{38 \cdot 60}} = 1 \cdot 008 \times 1 \cdot 3069 \times 0 \cdot 247$$

$$= 0 \cdot 325, \text{ or } = 19^\circ.$$

If we take the distance between every two floats, measured on the outer circumference of the wheel, equal to 1 ft., we obtain the requisite number of floats :

$$n = 2\,\pi\,r = \frac{44 \times 7}{7} = 44,$$

which might be changed to 48, in order to get an easier division.

The quantity of work available is

$$L = Q\,h\,\gamma = 24 \times 4 \cdot 5 \times 62 \cdot 5 = 6750 \text{ ft. lbs.},$$

and the theoretical power of the wheel is

$$L_1 = \frac{c^2}{2g} \cdot \cos.^2 \alpha \, Q \gamma = 0{\cdot}0155 \times \overline{15{\cdot}0125}^2 \cos.^2 20^\circ \times 24 \times 62{\cdot}5$$

$$= 4632 \text{ ft. lbs.},$$

and hence its efficiency is

$$\eta = \frac{L_1}{L} = \frac{4632}{6750} = 0{\cdot}686.$$

§ **226.—Experiments with Poncelet Wheels.**—Poncelet himself experimented upon the performances of these wheels; these experiments are accurately described in the treatise cited above, and their results are recorded. The first experiments were made by Poncelet with a model ½ metre (1·64 ft.) in diameter, or ⅙ of the actual size. It was made entirely of wood, and had 20 curved floats of wood, 2½ millimetres (0·10 inch) thick, 65 millimetres (2·6 inches) wide, and 76 millimetres (3·0 inches) long. He determined the efficiency of this wheel in the same manner as Bossut, Smeaton, and others have done, by the aid of a weight which was raised by a cord wound round the axle of the wheel. The greatest performance occurred, in accordance with theory, when the velocity of the wheel was 0·5 of that of the water, and the efficiency amounted in this case to 0·42 to 0·56; the former occurring with the smaller and the latter with the greater depths of stream, or more complete filling of the buckets. If we regard not the fall, but the height due to the velocity of the water when it arrives at the wheel, as giving the measure of available power, then the efficiency comes out 0·65 to 0·72. Poncelet afterwards experimented with a friction brake upon a wheel of actual size, and arrived thereby at results which differ but little from those quoted above. This wheel was 11 ft. (Paris ft., = 11·73 English ft.) in diameter, and had 30 plate-iron floats 2 millimetres (0·08 inch) thick. The crowns were of wood, as were also the arms and axle, and were 14·42 inches deep and 3·08 inches thick, and their distance apart or the width of the wheel was

28·83 inches. With a mean head of 1·3 metre (4·28 ft.) and a stream 0·2 metre (7·87 inches) deep, and also a velocity ratio of 0·52, the efficiency proved in this case to be 0·52, but it rises to 0·60 when we introduce the height due to velocity in place of the total fall. Poncelet draws from the results of his experiments the following conclusions:

The best velocity ratio $\frac{v}{c}$ is 0·55, but it may be 0·50 to 0·60 without giving an appreciably smaller effect. The efficiency for falls of 2·0 to 2·3 metres (6·5 to 7·5 ft.) is $\eta = 0·5$; for falls of 1·5 to 2·0 metres (5·0 to 6·5 ft.) $\eta = 0·55$; and for falls less than 1·5 metres (5·0 ft.) $\eta = 0·60$. The useful work can therefore be computed from the following formulæ:

In the first case from

$$Pv = 122·3\,(c - v)\,v\,Q \text{ kilog. metres} = 2·33\,(c - v)\,v\,Q \text{ ft. lbs.}$$

In the second case, from

$$Pv = 132·5\,(c - v)\,v\,Q \text{ kilog. metres} = 2·52\,(c - v)\,v\,Q \text{ ft. lbs.}$$

In the third case, from

$$Pv = 142·7\,(c - v)\,v\,Q \text{ kilog. metres} = 2·71\,(c - v)\,v\,Q \text{ ft. lbs.}$$

Poncelet also gives some rules for designing an undershot wheel with curved floats, which he likewise deduces from his observations. The distance apart of any two floats, measured upon the outer circumference, must amount to but 0·2 to 0·25 metre (0·65 to 0·82 ft.), while the radius of the wheel must not be less than 1 metre (3·28 ft.) nor more than 2·5 metres (8·2 ft.); the axis of the stream must meet the circumference of the wheel at an angle of 24° to 30°, and besides be inclined about 3° to the horizon. Moreover, the fall at the tail race should be sufficiently great for the water to fall out of the wheel without any hindrance, and the clearance of the wheel in the breast should be but 1 centimetre (0·4 inch). Some of these relations are, however, not essential, and others can be ascertained with more certainty from the formulæ of the foregoing articles. According to the experiments, the efficiency increases with the depth of the stream; but since, other things being equal, the filling of the buckets increases with the latter, we have the general rule, to be used with certain restrictions, that the buckets must be well filled. Moreover, according to Poncelet, the depth of the stream is never to be made less than 0·1 metre (0·3 ft.).

**§ 227.—New Experiments upon Poncelet Wheels.**— Morin also has since made some experiments with Poncelet wheels, using for this purpose three wooden wheels and one iron one, and applying a friction brake. These experiments were made chiefly with a view to test the advantage of a new curved inlet passage proposed by Poncelet, but also to procure more accurate information upon the influence of the proportions of the wheel upon its efficiency, since it had turned out with several constructions that the dimensions of wheels constructed according to Poncelet's rule were too small; and especially, when the velocity of the wheel deviated from its mean value, too small a performance was given because the water was dashed in over the edges of the floats (see "Comptes rendus," 1845, T. XXII., and the "Polytechn. Centralblatt," Bd. VIII., 1846).

The three wooden wheels experimented upon were 1·6 metres (5·25 ft.), 2·4 metres (7·87 ft.), and 3·2 metres (10·5 ft.) high; while the iron wheel was 2·8 metres (9·2 ft.) high; the floats were of plate iron in all the wheels. The first three wheels were 0·4 metre (1·31 ft.) wide, and the last 0·8 metre (2·62 ft.), and all four had a depth of crown of 0·75 metre (2·46 ft.). There proved to be one special trouble with the wooden wheels, however, since their motion was very irregular on account of their small moment of inertia, and hence a great quantity of water was dashed out of the buckets. The smallest wheel was particularly wanting in uniformity of motion, and with a fall of 0·45 to 0·55 metre (1·47 to 1·80 ft.), and when the buckets were at least half filled, this wheel gave a percentage of effect of only 48·5; with greater weight, an efficiency of 0·55 would perhaps have been given. With the wheel intermediate in size for a fall of 0·75 metre (2·46 ft.), the efficiency was found to be 0·60 to 0·62. Experiments were also made with the third wheel, using floats of different widths. It appeared that, with a fall of 0·56 metre (1·84 ft.), a crown 0·43 metre (1·41 ft.) deep was too small, and that with a fall 0·7 metre (2·3 ft.), a crown 0·59 metre (1·94 ft.) deep was still too small. Experiments were also made on this wheel to determine the effect of the form of wheel race proposed by Poncelet (described in § 222), and not only was a greater efficiency obtained with this, but it was also found that the wheel space might fall off to $\frac{3}{2}$ without having the water fall over the edge of the float inside the wheel.

Finally, with regard to the experiments with the iron wheel which had 42 floats, they were made with falls of 1·2 to 1·4 metres (3·94 to 4·59 ft.), in which case the wheel ran freely, and also with falls of 0·9 metre (2·95 ft.), when the wheel was 0·36 metre (1·18 ft.) deep in the water. When the sluice was raised 0·15 metre (0·49 ft.), 0·2 metre (0·66 ft.), 0·25 metre (0·82 ft.), and 0·277 metre (0.91 ft.), the efficiency had the maximum values, 0·52, 0·57, 0·60, and 0·62; and when the number of revolutions varied within the limits 12 to 21, 13 to 21, 11 to 20, and 12 to 19, the efficiencies differed only $\frac{1}{13}$, $\frac{1}{14}$, $\frac{1}{12}$, and $\frac{1}{9}$ from their maximum values. From the results of these experiments, it follows that the power of a wheel having a curved wheel race can be expressed by the formula

$$Pv = 0{\cdot}871 \left(\frac{c^2 - v^2}{2g}\right) Q\gamma,$$

and also that the best velocity ratio is $\frac{v}{c} = 0{\cdot}50$ to $0{\cdot}55$; that this effect is given by the water, whether the surface of the tail water stands 0·12 metre (0·39 ft.) below or 0·20 to 0·25 metre (0·66 to 0·82 ft.) above the bottom of the wheel; finally, that the efficiency falls to 0·46 when the wheel is drowned to the extent of 0·357 metre (1·16 ft.), or has half the depth of its crown submerged. The principal advantage of this new form of wheel race is that a wheel with it has a wider range for its velocity without losing much of its useful effect. Morin also finds that for falls of 0.9 to 1.3 metres (2.95 to 4.27 ft.), it is best to make the depth of crown equal to half the radius of the wheel, and the wheel space twice as great as the space which the water actually requires—*i. e.*, to use the coefficient of the relative filling of the buckets, $\varepsilon = \frac{1}{2}$.

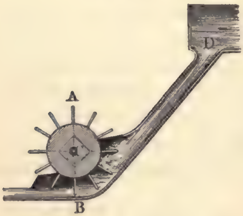
Later experiments have also been made by Marozeau with a wheel having three divisions (see "Bulletin de Mulhouse," 1846, or "Polytechnisches Centralblatt," Jahrgang 1848). This wheel was 4.4 metres (14.43 ft.) high, 3 × 0.67 metres = 2.0 metres (6.56 ft.) wide in the clear, and had a depth of crown of 0.75 metre (2.46 ft.), and received 500 to 1000 litres (17·66 to 35·32 cubic ft.) of water per second, with a fall of 1·5 metres (4·92 ft.). The greatest efficiency was here found to be 66 per cent, and occurred when the water flowed into all three divisions at once. The efficiency was, however, less when the wheel was drowned to the extent of 0·1 metre (0·33 ft.).

Later and very interesting experiments have been made by Captain O. de Lacolonge (in 1847) on a Poncelet wheel in the powder-mill at Angoulème (see "Le Génie Industrielle," par Armengaud Frères, Paris, 1854). This wheel had a radius of 4·8 metres (15·74 ft.), a depth of crown of 1·0 metre (3·28 ft.), and the same width, and made about ten revolutions per minute, giving a performance of 10 horsepowers. With the velocity ratio $\frac{v}{c} = 0 \cdot 579$, in which case the fall was 1·56 metres (5·12 ft.) and the height of opening of the sluice gate was 0·25 metre (0·82 ft.), the efficiency of this wheel amounted to 0·678. The water was brought to the wheel through a race made in the form of the evolute of a circle, and entered the wheel 26½° above the bottom, in such a manner that its relative motion on the float was at first in a horizontal direction. The coefficient for the relative filling of the buckets was very small—for the best effect $\varepsilon = \frac{1}{3}$. The performance above mentioned was somewhat increased (to 0·755) when the wheel was drowned to the extent $\frac{1}{3}\,h$; this circumstance, which points to a better use of the power, has already been observed with other middleshot-wheels (see in the "Polytechnisches Centralblatt," Jahrgang 1851, Hülsse and Brückmann's experiments with the friction brake on a breast-wheel).

§ **228.**—**Small Water-Wheels.**—Other vertical water-wheels have also been sometimes used which cannot be included in any of the systems which have just been treated; for instance, there are some very small wheels which are scarcely one foot high, and which are set in motion by the pressure or impulse of the water. Those which belong mostly to the systems already considered may be included here, but, for special reasons, the others will not be considered until the following chapter.

D'Aubuisson describes in his "Hydraulics" small impulse wheels, like *A C B*, Fig. 445, with large falls of 6 to 7 metres (19·69 to 22·97 ft.), which are frequently used in the Pyrenees. These wheels are only $2\frac{1}{2}$ to 3 metres (8·2 to 9·84 ft.) in diameter, and have 24 floats, somewhat concave. According to D'Aubuisson, their efficiency is said to be $\frac{5}{8}$ of that of an overshot-wheel with the same fall. The power of a wheel of this sort is to be computed from the theory explained above for breast-wheels, since these wheels are properly nothing but breast-wheels with a large head which acts by impulse, and a small head of pressure. The wheel is hung in a breast with closely fitting side walls, in order to prevent as much as possible the waste of the water. Moreover, by using several such wheels, under or close to each other, the water passing from one wheel into another, a still greater efficiency is obtained (see § 217).

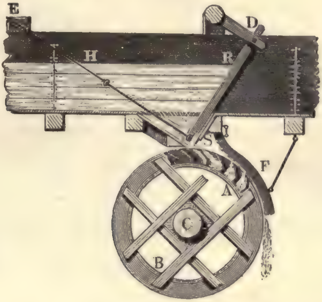
FIG. 445.



These wheels can also be made still smaller and of iron. In the Alps, wheels of this sort are very frequently seen in mills and forges.

An overshot forge-wheel with a large fall which acts by impulse is represented in Fig. 446. *E R D* is the penstock, *S D* the sluice, *A C B* the wheel, and *F* a casing round it, which prevents the premature discharge of the water.

FIG. 446.



Another wheel, Fig. 447, is described in the "Technologiste," September, 1845, and also in the "Polytechnisches Centralblatt," Vol. VII., 1846. While in the foregoing wheels, the water acts chiefly through impulse, this one produces its effects by pressure only. This wheel was built by the Engineer Mary, and its efficiency was found by Belanger with a surface velocity of 1·3 metres (4·26 ft.), to be 75 to 80 per cent, and therefore very high. It has only one crown, made of plate iron, 0·3 metre (0·98 ft.) deep, 0·12 metre (0·39 ft.) thick, and 2·28 metres (7·48 ft.) in diameter; and it is composed of six elliptical floats, made of plate iron and strengthened by ribs. The wheel also hangs in a very closely fitting trough, and iron plates *D E*, also fitting close to the crown, separate with tolerable completeness the head-water *O* from the tail-water *U*, while the crown itself moves in the gap between these plates. The force with which a wheel of this sort revolves is always the product of the distance from one surface of the water to the other, the section of a float, and the density of the water.

FIG. 447.

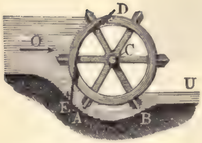
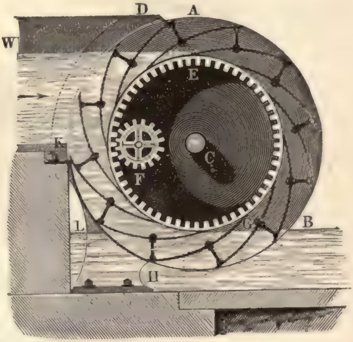


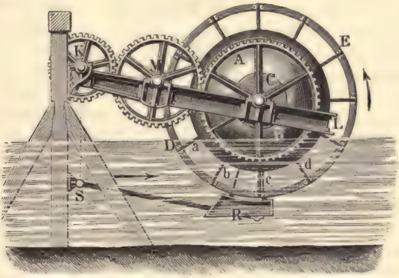
FIG. 448.



Another similar but more perfect wheel is Zupinger's, in Fig. 448. This wheel has but one crown *A B*, and long extended floats of plate iron, which are put either upon only one or both sides of the crown; the wheel is surrounded by an iron casing *D E F G H K* which conducts the water *W* to the wheel, both in front and from the sides, and retains it in the wheel until the lowest float *G H* comes out of the casing. The water, coming up at *W* and falling down in the wheel inside of the casing, flows off along *G H* under the surface of the tail-water *B L*, and thereby gives up its whole potential energy to the wheel. In the construction of such a wheel, care must be taken to see that the inner diameter of the wheel shall be equal to the fall, that the lower opening of the casing shall correspond to the lowest float and shall come below the surface of the tail-water, and that the clearance between the wheel and casing shall be as small as possible. A wheel of this sort is applicable to very small falls and gives a very high efficiency (75 to 80 per cent). See the "Gewerbeblatt für Würtemberg," 1855, and the "Polytechnisches Centralblatt," 1855.

The floating water-wheel of Colladon, of Geneva, is of a peculiar construction. It hangs in the open current like a ship-mill wheel, and consists principally of a cylinder of plate iron, *A B*, Fig. 449, floating upon the water, and having fastened to its outer surface long plate-iron floats *a*, *b*, *c*, *d* . . . , which are joined together by iron rings *D E*. The motion of the wheel is transmitted to a fixed shaft *K* by its axle *c*, which is placed upon two levers like *K L*, movable about *K*; and these shafts are fitted with cog-wheels which are either directly in gear with each other or work together by means of a third wheel *M*, likewise set upon *K L* (compare § 212). In order to increase the effect of the water striking on the floats *b*, *c* . . . , a breast *R* is hung under the wheel, which rises and falls with the wheel according to the state of the water, so that the two are always at the same depth below the surface of the latter. The fixed shaft *K* and the axis *S* of the breast or hanging wheel race are fastened to two pairs of posts. It is evident that the immersion of the body of the wheel produces a diminution of the section of the stream, which causes an increase of the velocity of the impinging water to the advantage of the working of the wheel.

FIG. 449.



**CLOSING REMARK.**—The literature of vertical water-wheels is very extensive, and yet but few works on these motors deserve much consideration, since the majority of them treat of only superficial, and some of even quite erroneous theories upon water-wheels. In Eytelwein's "Hydraulics," water-wheels are only treated generally; a more complete treatment, especially of the theory of undershot-wheels, is found in Gerstner's "Mechanics." Little useful is found in Langsdorf's "Hydraulics" or in his "System der Maschinenkunde." D'Aubuisson, in his "Hydraulique à l'usage des Ingénieurs," discusses the subject at some length, especially that of overshot-wheels. Navier, in his "Applications de la Mécanique," treats of vertical water-wheels only in a general way, but more in detail in the edition prepared by him of the first volume of the "Architecture hydraulique" of Belidor. The "Cours de Mécanique appliquée" of Poncelet, which was published in German under the title "Lehrbuch der Anwendung der Mechanik," contains the theory of water-wheels, treated very briefly, but with considerable accuracy as to theory. The most necessary information upon the performances and rules of construction of water-wheels will be found in Morin's "Aide-mémoire de Mécanique pratique." In the "Treatise on Manufactures and Machinery of Great Britain," by P. Barlow, little is given about the theory but more about the arrangement of water-wheels. Complete descriptions and good drawings of water-wheels are found in Armengaud's "Traité pratique des moteurs hydraulique et à vapeur," and also in the recent volumes of his "Publication industrielle." The "Maschinenkunde," etc., of Sebastian Haindl, also contains good drawings and drawings of water-wheels. The best work on vertical water-wheels is Redtenbacher's "Theorie und Bau der Wasserräder," which was published at Mannheim in 1846, and has 6 small and 23 large lithographic plates. Poncelet's and Morin's memoirs on the efficiencies of vertical water-wheels (see § 22 and § 197 above) form an important element in the literature of vertical water-wheels. Small forge-wheels are treated in detail in Tunner's "Darstellung der Stabeisen- und Rohstahl-Bereitung," Grätz, 1845. Morin's "Leçons de Mécanique pratique," Part II., also treats of water-wheels. Likewise Vol. II. of Redtenbacher's "Maschinenbau," Mannheim, 1863, and Vol. I. of Rühlmann's "Allgemeine Maschinenlehre." A water-wheel with oblique floats, made by Delnest, is described in Dingler's "Polytechnisches Journal," Bd. 173.

# CHAPTER V.

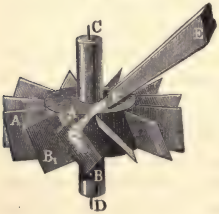
## HORIZONTAL WATER-WHEELS.

§ **229.—Turbines.**—In horizontal water-wheels, or those which revolve about a vertical axis, the water acts by its impulse or by its pressure or by its reaction, but never directly by its weight. Accordingly, we can distinguish horizontal wheels as impulse wheels, pressure wheels, and reaction wheels. Horizontal wheels are usually called by the general name of turbines; sometimes, however, the name turbine is only given to a particular class of reaction wheels. Those wheels on which the water acts by its impulse are provided with floats, plane or concave, on which the water strikes more or less perpendicularly; pressure wheels, on the other hand, have curved floats along which the water merely glides; and, finally, reaction wheels consist of an arrangement of pipes from which the water issues more or less tangentially. Pressure wheels and reaction wheels are very similar in their construction, but they differ essentially from each other, since in pressure wheels the buckets, or passages between the floats or vanes, are not completely filled with water, while in reaction wheels the water fills and flows through the whole section of the passages or pipes. While in impulse wheels the water spreads over the vanes in all directions, in pressure and reaction wheels it flows off only on one side. According to the different directions in which the water flows in the passages of this last class of wheels, we have two principal systems of pressure and reaction wheels; the relative motion of the water in the passages is either horizontal or it is a motion inclined to the horizon, generally taking place in a vertical plane. In the first system, we must, again, distinguish whether the water flows from within out or from without towards the centre; in the second, whether it passes down from above or up from below. The motion is generally either from the centre outwards only, or from above downwards; in the first case, the centrifugal force, and in the second gravity, assists the motion.

Horizontal wheels in which the water flows down from above are also called Danaids.

§ **230.—Impact Wheels.**—The simplest and most imperfect of the horizontal wheels are the so-called impact wheels or impact turbines, such as shown in Fig. 450. They consist of 16 or 20 rectangular blades $A\,B\,A_1\,B_1$, etc., fastened to the wheel at an inclination of 50° to 70° with the horizon. The water is brought on through a race $E\,F$ of 40° to 20° inclination, so that it strikes at about right angles upon the blades. These wheels are used in falls from 10 to 20 ft., where a large number of revolutions is necessary, as in grain mills, where the moving millstone is fastened to the axle of the wheel, so that intermediate gearing is unnecessary. These wheels are found especially in southern Europe, north Africa, in the Alps, Pyrenees, and in Algiers. They have about 5 ft. diameter, and the blades are 15 inches high and 8 to 10 inches long (measured radially).

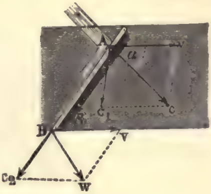
FIG. 450.



The delivery of these wheels is determined by the theory of impact as follows: The velocity $A\,c = c$, Fig. 451, of the impinging water, and $A\,v = v$ of the float, give the velocity $A\,c_1 = c_1$ for the velocity along the float. The water leaving the float with the velocity $B\,c_2 = c_2$, has the absolute velocity $w$, the diagonal of the parallelogram upon

$c_2$ and $v$. If the angle of the direction of motion makes the angle $v\,A\,C$ with that of the stream, we have for the velocity $c_1$ of the water upon the float,

FIG. 451.



$$c_1^2 = c^2 + v^2 - 2\,c\,v\,cos.\,\alpha,$$

and if $\beta = A\,B\,v =$ the angle of the float with the circumference, we have for the absolute velocity $w$ of the departing water,

$$w^2 = c_2^2 + v^2 - 2\,c_2\,v\,cos.\,\beta,$$

or when $c_2 = c_1$,

$$w^2 = c^2 + 2\,v^2 - 2\,v\,(c\,cos.\,\alpha + c_1\,cos.\,\beta)$$

$$= c^2 + 2\,v\,(v - (c\,cos.\,\alpha + c_1\,cos.\,\beta)\,).$$

We can now, $w$ being thus known, determine the work imparted to the wheel. Thus

$$L = \left(\frac{c^2 - w^2}{2\,g}\right) Q\,\gamma = \frac{v\,(c\,cos.\,\alpha + c_1\,cos.\,\beta - v)}{g}\,Q\,\gamma,$$

or inserting

$$c_1 = \sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha},$$

$$L = \frac{(c\,cos.\,\alpha + \sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha}.cos.\,\beta - v)\,v}{g}\,Q\,\gamma.$$

In order that the water after impact shall run down the float, it is necessary that the angle $B\,A\,c_1$ shall be less than 90°. If we indicate $A\,c_1\,c$ by $\theta$, then $B\,A\,c_1 = \theta - \beta$, and hence $\theta - \beta < 90°$, or $\theta < 90° + \beta$; therefore

$$tan.\,\theta < tan\,(90° + \beta),$$

or $cot.\,\beta < -\,tan.\,\theta$ or $tan.\,\beta > -\,cot.\,\theta$. But

$$cot.\, \theta = \frac{c\, cos.\, \alpha - v}{c\, sin.\, \alpha},$$

hence

$$tan.\, \beta > \frac{v - c\, cos.\, \alpha}{c\, sin.\, \alpha}.$$

In order to obtain the greatest delivery, we must give the stream nearly the same direction as the float moves, or make $\alpha = 0$. We have then, since $cos.\, \alpha = 1$,

$$L = \left(\frac{c + (c - v)\, cos.\, \beta - v}{g}\right) v\, Q\, \gamma = \frac{(1 + cos.\, \beta)\,(c - v)\, v}{g}\, Q\, \gamma.$$

But for a small value of $\alpha$,

$$\frac{v - c\, cos.\, \alpha}{c\, sin.\, \alpha}$$

becomes very great; hence $tan.\, \beta = \infty$ and $\beta = 90°$, and the corresponding work is

$$L = \frac{(c - v)\, v}{g}\, Q\, \gamma.$$

This is a maximum for

$$v - \frac{c}{2},$$

and we have for this maximum,

$$L = \tfrac{1}{2}\, \frac{c^2}{2\, g}\, Q\, \gamma,$$

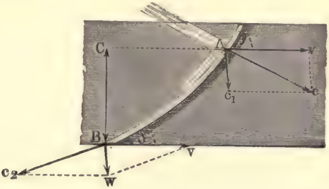
*i. e.*, half the work of the entering water.

**§ 231.—Impact Wheels with Curved Floats.**—Since the formula

$$L = \frac{(c\, cos.\, \alpha + \sqrt{c^2 + v^2 - 2\, c\, v\, cos.\, \alpha}\, .\, cos.\, \beta - v)\, v}{g}\, Q\, \gamma$$

is greater the greater *cos.* $\beta$, and therefore the smaller the exit angle $\beta$, we can with advantage use curved floats, such as *A B*, Fig. 452. The inclination $\beta$ at the point of entrance *A* is then different from $\delta$ at *B* the point of exit, and therefore the above condition

FIG. 452.



$$tan.\,\beta > \frac{v - c\,cos.\,\alpha}{c\,sin\,\alpha}$$

may be satisfied without affecting the formula

$$L = \frac{(c\,cos.\,\alpha + \sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha}\,.\,cos.\,\delta - v)\,v}{g}\,Q\,\gamma.$$

If, therefore, we make the exit angle very small, $cos\,\delta = 1$, and hence

$$L = \frac{(c\,cos.\,\alpha + \sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha} - v)\,v}{g}\,Q\,\gamma.$$

This is a maximum when

$$c\,cos.\alpha - 2\,v + \sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha} - \frac{v\,(c\,cos.\,\alpha - v)}{\sqrt{c^2 + v^2 - 2\,c\,v\,cos.\,\alpha}} = 0,$$

or

$$v = \frac{c}{2\,cos\,\alpha}.$$

The maximum value of *L* is then

$$L = \frac{c\,cos.\,\alpha}{g}\,\frac{c}{2\,cos.\,\alpha}\,Q\,\gamma = \frac{c^2}{2\,g}\,Q\,\gamma,$$

and hence, by this disposition the entire work inherent in the water is gained.

If the height $C B = h_2$ through which the water falls on the float is considerable, we have

$$c_2 = \sqrt{2 g h_2 + c_1^2}.$$

If, again, $\delta$ is very small, we have the absolute velocity

$$w = c_2 - v = \sqrt{2 g h_2 + c_1^2} - v.$$

In order that this may be zero, so that the entire work of the water may be gained, we have

$$v^2 = 2 g h_2 + c_1^2$$

$$= 2 g h_2 + c^2 + v^2 - 2 c v \cos. \alpha,$$

or

$$v = \frac{2 g h_2 + c^2}{2 c \cos. \alpha}.$$

If

$$h_1 = \frac{c^2}{2 g}$$

is the fall necessary for the entrance velocity, we have the entire fall

$$h = h_1 + h_2 = \frac{c^2}{2 g} + h_2,$$

and hence for the best velocity of the wheel for which the whole work of the water is gained,

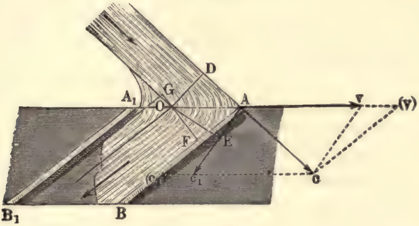
$$v = \frac{g h}{c \cos. \alpha}.$$

If $d$ is the thickness $O D$, Fig. 453, of the stream, and $\theta$ the angle $O A E = A v c$, which the direction of the entering water $A c_1$ makes with the direction of motion $A v$ of the wheel, we have

$$O E = d_1 = \frac{c d}{c_1},$$

since through $OE$ and $OD$ in the same time the same amount of water passes. Since the float $AB$ converts the

FIG. 453.



direction of motion $AE$ into $AF$, but does not change the velocity, we have also

$$FG = OE = d_1 = \frac{c\,d}{c_1}$$

for the thickness upon $AB$. In order that the water enter without hindrance, it is necessary that the normal distance between two floats shall be at least equal to

$$FG = \frac{c\,d}{c_1},$$

so that the approaching water may only occupy a portion $AD$ of the entire opening $AA_1$ between the floats $AB$ and $A_1B_1$. If the entrance velocity ($c_1$) coincides in direction with the float, $\theta = \beta$, and

$$\frac{c}{c_1} = \frac{sin.\,\beta}{sin.\,\alpha},$$

hence

$$d_1 = \frac{d\,sin.\,\beta}{sin.\,\alpha} = OF.$$

In this case, the stream $OAB$ closes directly upon the base $AO$, and the float $A_1B_1$ may be brought nearer to $AB$. Since this case happens only for a certain velocity,

$$v = \frac{c \sin. (\alpha + \beta)}{\sin. \beta},$$

we must have $d_1$ somewhat greater than $\frac{d \sin. \beta}{\sin. \alpha}$, in order that for smaller velocities of rotation $v$, the water may enter without hindrance.

§ **232.**—Experiments have been made by Piobert and Tardy upon impact wheels with curved floats, called by the French *rouets volants* ("Expériences sur les roues hydrauliques à axe vertical," Paris, 1840). The results of these experiments upon a wheel of 5 ft. diameter, 8 inches high, having 20 curved floats, Fig. 454, for a fall of 14 ft. from level of head race to bottom of wheel, and for a feed of 10·6 cubic ft. per second, were as follows:

FIG. 454.

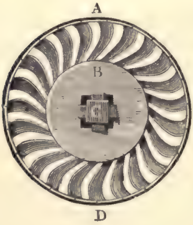
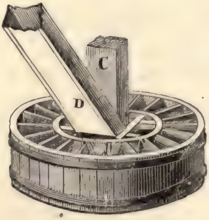


FIG. 455.



For $\frac{v}{c} = 0{\cdot}72$, $\eta = 0{\cdot}16$, for $\frac{v}{c} = 0{\cdot}66$, $\eta = 0{\cdot}31$, and for $\frac{v}{c} = 0{\cdot}56$, $\eta = 0{\cdot}40$. The wheels treated in the preceding paragraph, in which the water acts principally by pressure, while flowing over curved floats, are called Borda's Turbines. The construction of such a turbine is shown in Fig. 455. The curved floats are of beechwood and set between circular sides formed of staves, the outer being hooped with iron. In Fig. 455, *A B* is the curved float, *C* the axle, and *D* the sluice inclined at 45°. The diameter of the wheel was 1½ metres (4·92 ft.), the floats were 0·36 metre (14·2 inches) long, and 0·44 metre (17·3 inches) high. The wheel used a fall of 5 metres (16·4 ft.) and made 40 revolutions per minute. No reliable observations have yet been made upon the efficiency of these turbines. Borda gives for the ratio of the effective delivery to the theoretical 0·75.

Poncelet remarks with justice, that it is well to give these wheels a great height and great diameter, and make the floats rather short, or make the two cylindrical bounding surfaces or drums near each other. By the greater height of wheel, we obtain a smaller velocity fall, and hence less velocity for water and wheel; by a greater diameter, we obtain a less number of revolutions, and since for a larger wheel for equal capacity of water the distance between crowns is less, we have less deviation in the velocity of the adjacent water elements.

EXAMPLE.—What feed is necessary for a Borda turbine using a fall of 15 ft., in order to give a delivery of 3 horse-power?

If we make the wheel 1¾ ft. high, we have for the theoretical velocity of entrance

$$c = 8 \cdot 025 \sqrt{15 - 1 \cdot 75} = 8 \cdot 025 \sqrt{13 \cdot 25} = 28 \cdot 75 \text{ ft.}$$

If we bring the water in at an inclination of 30° to the horizon, we have for the best velocity of rotation,

$$v = \frac{g\,h}{c\, cos.\, \alpha} = \frac{32 \cdot 2 \times 15}{28 \cdot 75\, cos.\, 30^\circ} = 18 \cdot 83 \text{ ft.}$$

The velocity $c_1$ with which it runs down the floats is

$$c_1 = \sqrt{c^2 + v^2 - 2\,c\,v\, cos.\, \alpha} = \sqrt{c^2 + v^2 - 2\,g\,h} = \sqrt{v^2 - 2\,g\,h_2}$$

$$= \sqrt{18 \cdot 83^2 - 2 \times 32 \cdot 2 \times 1.75} = \sqrt{245} = 15 \cdot 65 \text{ ft.}$$

For the angle $\beta$ under which the top of float is inclined to horizon, we have, in order that the water may enter without impact,

$$\frac{sin.\,\beta}{sin.\,\alpha} = \frac{c}{c_1},$$

or

$$sin.\,\beta = \frac{28 \cdot 75}{15 \cdot 65}\,sin.\,30^\circ = 0 \cdot 9185,$$

or

$$\beta = 66\tfrac{3}{4}^\circ.$$

If we incline the bottom of float to horizon by an angle $\delta = 25^\circ$, we have for the absolute velocity of the escaping water,

$$w = 2\,v\,sin.\,\frac{\delta}{2} = 2 \times 18 \cdot 83\,sin.\,12\tfrac{1}{2}^\circ = 8 \cdot 15\text{ ft.},$$

and hence the delivery of the wheel is

$$L = \tfrac{3}{4}\left(h - \frac{w^2}{2\,g}\right) Q\,\gamma = \tfrac{3}{4}\left(15 - \frac{8 \cdot 15^2}{2g}\right) 62 \cdot 5\,Q = 46 \cdot 87\,(15 - 1 \cdot 063)\,Q = 645\,Q.$$

In order that this shall give 3 horse-powers = 1650 ft. lbs. per second, we have

$$Q = \frac{1650}{645} = 2 \cdot 57\text{ cubic ft.}$$

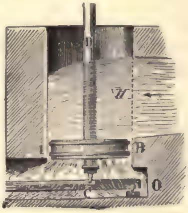
If we give the wheel a mean radius (to middle of float) of 1½ ft., and make the water space $l = \tfrac{1}{4}$ ft. wide, we have for the area of cross-section of the exit openings at bottom of wheel,

$$F = 2\,\pi\,r\,l\,sin.\,\delta = \pi \times 3 \times \tfrac{1}{4}\,sin.\,25^\circ = 2 \cdot 356 \times 0 \cdot 4226 = 1 \cdot 0\text{ sq. ft.},$$

which is abundantly sufficient to pass per second 2·57 cubic ft. with 18·83 ft. velocity.

§ **233.**—**Tub-Wheels.**—To the turbines in which the water flows down upon curved floats, belong tub-wheels (Fr. roues en cuves; Ger. Kufenräder), which are still of frequent occurrence in the south of France, and have been described by Belidor in his "Architecture hydraulique." D'Aubuisson also treats of these wheels very thoroughly in his "Hydraulique." Finally, Piobert and Tardy, in the treatise already cited (Art. 231), have given the results of experiments which are by no means favorable. These wheels (Fig. 456) do not differ in form from the impulse wheel already described (Fig. 454). They are, however, only about 3 ft. in diameter and have only nine curved floats. The axle *CD* rests at *C* upon a lever *CO*, by which the wheel may be raised or lowered as the millstone (not shown in the figure) may require. The wheel is placed near the bottom of a cylindrical shaft *A W B* about 6 ft. high and 3·3 ft. in diameter, and the water flows through a race tangent to the wheel, from 9 to 12 ft. long, having at first a breadth of about 2¼ ft., and diminishing down to 0·75 of a foot. The water flows in with great velocity, acquires in the wheel pit a rotary motion, and acts both by impact and pressure upon the floats as it flows through the wheel. A large part of the water acts, however, incompletely or not at all, either by escaping through the clearance between the wheel and shaft, or because, in its passage through the very wide float apertures, it has not sufficient opportunity to produce its full effect. For such reasons, the efficiency of these wheels is very small. In those of the best construction in the hospital mill at Toulouse, Piobert and Tardy found the efficiency at most only 0·27, for a fall of 3·8 ft., a supply of about 16 cubic ft. and 100 revolutions per minute. If, under otherwise similar circumstances, the number of revolutions ($u$) was 120, the efficiency was found only 0·22, and for $u = 133$ the efficiency was only 0·15. The wheels in the so-called "Basacle Mills," by reason of their poor condition, gave at most only 0·18.

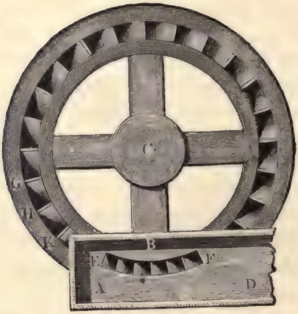
Fig. 456.



D'Aubuisson informs us, that in the best and latest methods of construction, the wheel is not placed within the shaft, but directly below it, and is therefore of somewhat larger diameter; that also the pyramidal race is considerably shortened, and that by such means the efficiency has been increased by $\frac{1}{8}$. If, then, according to D'Aubuisson, we estimate the efficiency of these wheels even as high as 0·25, we still have a much less efficiency than for the impact wheels already described.

§ 234.—**Burdin's Turbine.**—Burdin's turbines, or *turbines à évacuation alternative*, as Burdin calls them, are the best of the impact wheels with curved floats. They differ essentially from the simple Borda turbine only in admitting the water at several points at once, and in distributing the outlet orifices in three concentric circles. The last arrangement is in order to diminish the resistance of the issuing water. The first wheel of this kind was erected by Burdin in the mill at Pont-Gibaud, and is described in the "Annales des Mines," III. Serie, T. III. Fig. 457 shows a plan of the wheel. *A B D* is the feed box into which the race discharges, and which has a row of apertures *E F* furnished with guides, by which the water is made to enter the wheel in a proper direction. The wheel consists of a series of canals whose apertures form a ring *G H K*. . . ., which moves exactly under the feed apertures, so that the water may flow without hindrance from one into the other. These canals are at first perpendicular, but curve round

FIG. 457.

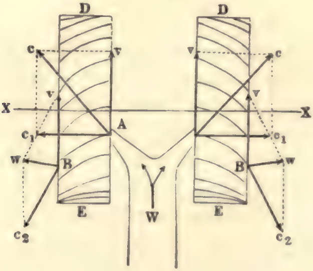


until nearly horizontal and tangential to the wheel, the bottom apertures being disposed in three concentric rings. Thus only the third part of all the bottom or exit apertures are exactly below those shown in the figure where the water enters. Another third form a concentric ring within the first on one side, and still another third a concentric ring without, or on the other side of the centre ring.

Experiments made upon the Burdin turbine at Pont-Gibaud gave for a feed $Q$ of 3·3 cubic ft. per minute and a fall $h$ of 10·6 ft. an efficiency $\eta = 0{\cdot}67$. The impact turbine previously used for the same purpose required for the same effect three times as much water. The diameter of this wheel was 4·6 ft., the height 1·3 ft., and the number of floats 36.

We may also construct *vertical* water-wheels upon the principle of Burdin's turbine, as shown in Fig. 458. Here

FIG. 458.



the water is let on by a pipe $WA$, not far from the lowest point of the wheel. If $c$ is the velocity of efflux from the pipe, $v$ the wheel velocity, and $\alpha$ the angle $c A v$ which the direction of the entering water makes with the circumference of the wheel, we have for the relative velocity $c_1 = c_2$ of the water in the wheel,

$$c_1^2 = c_2^2 = c^2 + v^2 - 2\, c\, v\, cos.\, \alpha.$$

If, now, the water departs with but little relative velocity, we have $c_2 = v$, or $c_2^2 = v^2$, and hence

$$2\,c\,v\cos.\alpha = c^2,$$

or

$$v = \frac{c}{2\cos.\alpha}.$$

If we allow the wheel to revolve with this velocity and make the angle of exit $\delta = 180° - c_2 B v$ as small as possible, the absolute velocity $w$ becomes so small that the work of the departing water

$$\frac{w^2}{2g} Q\gamma$$

may be regarded as null, and consequently the theoretical effect of the wheel may be put

$$L = Q h \gamma.$$

In order that the water may flow without hindrance through the wheel canals $A B$, it is necessary that the cross-section of the outlet orifice $B$ may not be less than that of the inlet orifice $A$, and the exit angle $\delta$ must then at least be equal to the entrance angle $\alpha$. A single wheel $D E$ of this construction has, in consequence of the deviation of the force direction from the plane of rotation, a tendency to rotate about an axis in this plane, and in order to prevent this, we may put upon the same axis $X X$ two similar wheels $D E$ and $D E$, which receive the water from the pipe $W$ upon opposite sides.

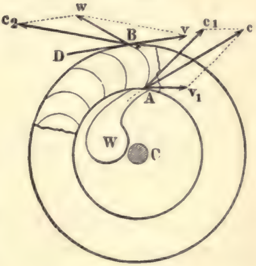
§ **235.—Tangential Wheels.**—In the turbines thus far noticed, the water moves nearly or entirely in a cylindrical surface, and hence no water element changes its distance from the axis of revolution, or at least the deviation is very slight. We have now to consider wheels where the water, besides a motion of revolution, and, if desired, also a vertical motion, has also a more or less *radial* inward or outward flow, with reference to the axis. The action of such wheels, therefore, is directly influenced by the *centrifugal force* of the water. We may therefore call such wheels *centrifugal turbines*. They are also known as tangential wheels.

The theory of these turbines rests upon that of the mechanical work of the centrifugal force as set forth in Vol. I., Arts. 303 and 304. If a body or element of water moves along a float $A\,B$, Fig. 459, while the wheel itself revolves with a certain angular velocity $\omega$, then if $G$ is the weight of the body, the increase of work due to the centrifugal force is

$$L = \left(\frac{v^2 - v_1^2}{2g}\right) G,$$

where $G$ is the weight of the body and $v_1$ and $v$ are the velocities at the point of entrance and exit. This gain of work becomes a loss when the body moves from without inwards, or when $A$ is the point of entrance and $B$ that of exit (Fig. 461). If, therefore, the relative entrance velocity of the water is $c_1$, the relative exit velocity at $B$ or $c_2$ is for both cases given by

FIG. 459.



$$\frac{c_2^2}{2g} = \frac{c_1^2}{2g} + \frac{v^2 - v_1^2}{2g},$$

or

$$c_2^2 = c_1^2 + v^2 - v_1^2.$$

In order that the water may enter without hindrance and without impact at $A$, it is necessary that the absolute velocity $c$ shall be decomposed into two velocities $v_1$ and $c_1$, of which the first is the velocity of rotation at the point of entrance, and the second is tangent to the float at $A$. If, now, $\alpha$ is the angle $c\,A\,v_1$, and $\beta$ the angle $c_1\,A\,v_1$, we have for the relative velocity of entrance

$$c_1^2 = c^2 + v_1^2 - 2\,c\,v_1\,cos.\,\alpha$$

and

$$\frac{sin.\,\beta}{sin.\,\alpha} = \frac{c}{c_1}.$$

FIG. 460.

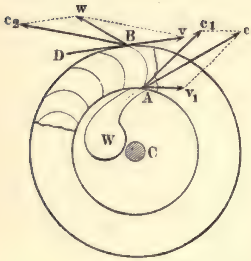
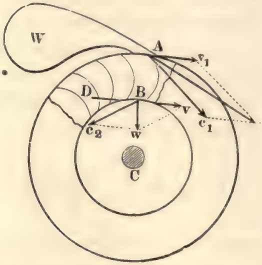


FIG. 461.



If, then, the values of $c, v$, and $\alpha$ are known, we can find $c_1$ and $\beta$ by the equations

$$c_1 = \sqrt{c^2 + v_1^2 - 2\,c\,v_1\,cos.\,\alpha}$$

and

$$sin.\,\beta = \frac{c\,sin.\,\alpha}{c_1};$$

or, since

$$c\,cos.\,\alpha - v_1 = c_1\,cos.\,\beta = c_1\frac{cos.\,\beta}{sin.\,\beta}sin.\,\beta = c_1\frac{cos.\,\beta}{sin.\,\beta} \times \frac{c\,sin.\,\alpha}{c_1}$$

$$= c\,cot.\,\beta\,sin.\,\alpha,$$

we have

$$c_1 = \frac{c\,sin.\,\alpha}{sin.\,\beta}$$

and

$$cot.\,\beta = cot.\,\alpha - \frac{v_1}{c\,sin.\,\alpha}.$$

For the relative exit velocity, we have, as above,

$$c_2^2 = c_1^2 + v^2 - v_1^2 = c^2 + v_1^2 - 2\,c\,v_1 \cos.\alpha + v^2 - v_1^2$$
$$= c^2 + v^2 - 2\,c\,v_1 \cos.\ \alpha.$$

If $\delta$ is the angle $D\,B\,c_2$, we have for the absolute velocity $w$,

$$w^2 = c_2^2 + v^2 - 2\,c_2\,v \cos.\ \delta.$$

In order that the work of the departing water $\frac{w^2}{2g}Q\gamma$ may be as small as possible, and hence the work imparted to the wheel as great as possible, we must have $w$ as small as possible, and hence $c_2 = v$ and $\delta = 0$, or differing as little from zero as may be. If $\delta = 0$, or if the float is tangent to the outer circumference, we have $w = c_2 - v = 0$, and therefore the loss of work is zero. In order to give, however, at $B$ a sufficient cross-section to the departing water, we must give $\delta$ a small value (20°), and if then $c_2 = v$, we have

$$w^2 = c_2^2 + v^2 - 2\,c_2\,v \cos.\delta = 2\,v^2 - 2\,v^2 \cos.\ \delta,$$

or

$$w^2 = 2\,v^2\,(1 - \cos.\ \delta) = 2\,v^2 \times 2 \sin.^2 \frac{\delta}{2} = 4\,v^2 \sin.^2 \frac{\delta}{2};$$

hence

$$w = 2\,v \sin.\ \frac{\delta}{2},$$

and the loss of work is then

$$\frac{w^2}{2g}Q\gamma = \frac{\left(2\,v \sin.\ \frac{\delta}{2}\right)^2}{2g}Q\gamma.$$

If in the equation

$$c_2^2 = c^2 + v^2 - 2\,c\,v_1 \cos.\ \alpha,$$

we put $v = c_2$, we have simply

$$2\,v_1 \cos.\ \alpha = c,$$

and hence the required velocity of rotation of the wheel is

$$v_1 = \frac{c}{2\,cos.\,\alpha};$$

or since

$$c = \sqrt{2gh},$$

where $h$ is the fall necessary to produce the velocity $c$,

$$v_1 = \frac{\sqrt{2gh}}{2\,cos.\,\alpha}.$$

If $r$ and $r_1$ are the radii $CB$ and $CA$,

$$v = \frac{r}{r_1}\,v_1 = \frac{r}{r_1}\,\frac{c}{2\,cos.\,\alpha} = \frac{r}{r_1}\,\frac{\sqrt{2gh}}{2\,cos.\,\alpha}.$$

If, now, we put this value for $v$ in the expression

$$\frac{w^2}{2g}Q\gamma = \frac{\left(2\,v\,sin.\,\frac{\delta}{2}\right)^2}{2g}Q\gamma,$$

we have

$$\frac{w^2}{2g}Q\gamma = \left(\frac{r}{r_1}\cdot\frac{sin.\,\frac{1}{2}\delta}{cos.\,\alpha}\right)^2 Qh\gamma,$$

and hence the theoretical work of the wheel is

$$L = Qh\gamma - \frac{w^2}{2g}Q\gamma = \left[1 - \left(\frac{r}{r_1}\,\frac{sin.\,\frac{1}{2}\delta}{cos.\,\alpha}\right)^2\right]Qh\gamma.$$

§ **236.**—**Effect of Friction.**—In consequence of the friction of the water in the guide race and in the wheel, the above theoretical work imparted to the wheel experiences two losses which increase as the square of the exit velocities $c$ and $c_2$, and are together

$$= \left(\zeta\,\frac{c^2}{2g} + \zeta_1\,\frac{c_2^{\,2}}{2g}\right)Q\gamma,$$

where $\zeta$ and $\zeta_1$ are the experimentally determined coefficients of resistance. If we put

$$\frac{c^2}{2g} = h \quad \text{and} \quad \frac{c_2^{\,2}}{2g} = \frac{v^2}{2g} = \left(\frac{r}{r_1}\right)^2 \frac{v_1^{\,2}}{2g} = \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\cos.\alpha}\right)^2 h,$$

we have for the delivery of the wheel

$$L = \left[1 - \zeta - \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\cos.\alpha}\right)^2 - \left(\frac{r}{r_1}\right)^2 \left(\frac{\sin.\frac{1}{2}\delta}{\cos.\alpha}\right)^2\right] Q h \gamma.$$

We may take $\zeta = \zeta_1 = 0{\cdot}05$ to $0{\cdot}10$.

Moreover, we have more exactly

$$(1 + \zeta)\, c^2 = 2 g h$$

and

$$(1 + \zeta_1)\, c_2^{\,2} = c^2 + v^2 - 2 c v_1 \cos.\alpha,$$

whence

$$c = \sqrt{\frac{2 g h}{1 + \zeta}}$$

and

$$\zeta_1 c_2^{\,2} = c^2 - 2 c v_1 \cos.\alpha.$$

If, now, we put

$$\zeta_1 c_2^{\,2} = \zeta_1 v^2 = \zeta_1 \left(\frac{r}{r_1}\right)^2 v_1^{\,2} = \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{c}{2\cos.\alpha}\right)^2 = c^2 - 2 c v_1 \cos.\alpha,$$

we have for the best velocity of revolution

$$v_1 = \left[1 - \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\cos.\alpha}\right)^2\right] \frac{c}{2\cos.\alpha}.$$

Since, in the above expression for the work, $\cos.\alpha$ occurs in the denominator, we should make the angle of approach $\alpha$ as small as possible, and therefore bring the water in tangential to the wheel. We call these wheels, therefore, *tangential wheels*. The water may flow in from without or from within, and we have therefore inward (Fig. 461) or outward flow (Fig. 460) tangential wheels.

### VALUES OF $F$, $F_1$, $F_2$, AND $\beta$.

If the amount of feed $Q$ is known, we can, by the aid of the velocities $c$, $c_1$ and $c_2$, now determine the necessary cross-section $F$ of end of guide race, as well as the cross-section $F_1$ of the entrance aperture and $F_2$ of the exit aperture. Thus

$$Q = Fc = F_1 c_1 = F_2 c_2,$$

or

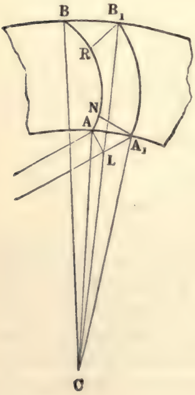
$$c = \frac{Q}{F}\, c_1 = \frac{Q}{F_1} \quad \text{and} \quad c_2 = \frac{Q}{F_2}.$$

By comparison of the two first velocities, we have

$$\frac{c_1}{c} = \frac{F}{F_1} = \frac{A\,L}{A_1\,N},$$

where $A\,L$ and $A_1 N$, Fig. 462, are the thicknesses of the water stream before and after entrance. If, now, $A\,A_1$ is the arc of the circumference which the stream of water occupies, we have

FIG. 462.



$$\frac{A\,L}{A_1\,N} = \frac{A\,A_1\,sin.\,A\,A_1\,L}{A\,A_1\,sin.\,A_1\,A\,N} = \frac{sin.\,\alpha}{sin.\,\beta},$$

hence

$$\frac{c_1}{c} = \frac{sin.\,\alpha}{sin.\,\beta},$$

precisely as above.

Since $c$ and $\alpha$ are given and $v_1$ may be found, we have $\beta$ to be determined by the expression

$$cot.\,\beta = cot.\,\alpha - \frac{v_1}{c\,sin.\,\alpha}.$$

Without reference to friction,

$$v_1 = \frac{c}{2\,cos.\,\alpha},$$

hence

$$cot.\,\beta = cot.\,\alpha - \frac{1}{2\,sin.\,\alpha\,cos.\,\alpha} = \frac{2\,cos.^2\,\alpha - 1}{2\,sin.\,\alpha\,cos.\,\alpha}$$

$$= \frac{cos.\,2\,\alpha}{sin.\,2\,\alpha} = cot.\,2\,\alpha,$$

and therefore $\beta = 2\,\alpha$.

Finally, from $F c = F_2 c_2$, we have

$$\frac{c_2}{c} = \frac{F}{F_2} = \frac{A\,L}{B_1\,R} = \frac{A\,A_1\,sin.\,\alpha}{B\,B_1\,sin.\,\delta}.$$

Now,

$$\frac{A\,A_1}{B\,B_1} = \frac{C\,A}{C\,B} = \frac{r_1}{r},$$

hence

$$\frac{c_2}{c} = \frac{r_1\,sin.\,\alpha}{r\,sin.\,\delta},$$

and

$$sin.\,\delta = \frac{r_1}{r}\,\frac{c}{c_2}\,sin.\,\alpha,$$

or

$$sin.\,\delta = \frac{r_1}{r}\,\frac{c}{v}\,sin.\,\alpha = \left(\frac{r_1}{r}\right)^2 \frac{c}{v_1}\,sin.\,\alpha,$$

or, taking

$$v_1 = \frac{c}{2\,cos.\,\alpha},$$

approximately,

$$sin.\,\delta = \left(\frac{r_1}{r}\right)^2 sin.\,2\,\alpha.$$

For outward-flow tangential wheels, Fig. 460, $r$ is greater than $r_1$, and $\frac{r_1}{r}$ is a fraction less than unity. For inward flow, $r_1$ is greater than $r$, and $\frac{r_1}{r}$ is greater than unity. For these latter wheels, then, under similar circumstances, the exit angle $\delta$ is greater than for outward-flow turbines. If we put approximately

$$sin.\ \tfrac{1}{2}\,\delta = \tfrac{1}{2}\,sin.\ \delta = \left(\frac{r_1}{r}\right)^2 sin.\ \alpha\ cos.\ \alpha$$

in the formula already found for the delivery, viz.,

$$L = \left[1 - \zeta - \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\ cos.\ \alpha}\right)^2 - \left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\ \frac{1}{2}\,\delta}{cos.\ \alpha}\right)^2\right] Q\,h\,\gamma,$$

we have approximately,

$$L = \left[1 - \zeta - \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\ cos.\ \alpha}\right)^2 - \left(\frac{r_1}{r}\right)^2 (sin.\ \alpha)^2\right] Q\,h\,\gamma.$$

Since in this expression, the member

$$\zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\ cos.\ \alpha}\right)^2$$

is greater for outward flow than for inward, and, on the other hand,

$$\left(\frac{r_1}{r}\right)^2 (sin.\ \alpha)^2$$

is greater for inward flow, we cannot in general give the preference to one of these wheels over the other.

Example.—Required for a fall $h = 150$ ft. and a feed $Q = \frac{3}{4}$ cubic ft. per minute, to construct and calculate a tangential wheel with *inward flow.*

The velocity of exit of water from the guide race is

$$c = 0{\cdot}95\sqrt{2\,g\,h} = 0{\cdot}95\sqrt{64{\cdot}4 \times 150} = 0{\cdot}95\sqrt{9375} = 92 \text{ ft.}$$

If we take

$$\frac{r_1}{r} = \tfrac{4}{3},\ \zeta_1 = 0{\cdot}10 \text{ and } \alpha = 10°,$$

we have for the outer velocity at circumference

$$v_1 = \frac{c}{2\ cos.\ \alpha}\left[1 - \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{(2\ cos.\ \alpha)^2}\right]$$

$$= \frac{92}{2\ cos.\ 10}\left(1 - 0{\cdot}1\,(\tfrac{3}{4})^2 \frac{1}{(2\,cos.\ 10°)^2}\right) = 46{\cdot}0 \text{ ft.}$$

The work at disposal is

$$L = Q\,\gamma\,h = \tfrac{3}{4} \times 150 \times 62\cdot 50 = 6947 \text{ ft. lbs.}$$

We have also $1 - \zeta = 1 - 0\cdot 10 = 0\cdot 9$,

$$\zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{(2\,cos.\,\alpha)^2} = 0\cdot 015,$$

$$sin.\,\delta = \left(\frac{r_1}{r}\right)^2 sin.\,2\,\alpha = (\tfrac{4}{3})^2\,sin.\,20^\circ = \tfrac{16}{9} \times 0\cdot 3420 = 0\cdot 608, \text{ or } \delta = 37\tfrac{1}{2}^\circ, \text{ and}$$

$$\beta = 2\,\alpha = 20^\circ, \text{ and}$$

$$\left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\,\frac{1}{2}\,\delta}{cos.\,\alpha}\right)^2 = (\tfrac{3}{4})^2 \left(\frac{sin.\,18^\circ\,45'}{cos.\,10^\circ}\right)^2 = \tfrac{9}{16} \times 0\cdot 1065 = 0\cdot 060,$$

hence the effective delivery is

$$L = \left[1 - \zeta - \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{(2\,cos.\,\alpha)^2} - \left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\,\frac{1}{2}\,\delta}{cos.\,\alpha}\right)^2\right] Q\,h\,\gamma$$

$$= (0\cdot 900 - 0\cdot 015 - 0\cdot 060)\,Q\,h\,\gamma = 0\cdot 825\,Q\,h\,\gamma = 5722 \text{ ft. lbs.}$$

For *outward flow*, we take reversely $\alpha = 20^\circ$ and $\beta = 40^\circ$, and hence the inner velocity

$$v_1 = \frac{c}{2\,cos.\,\alpha}\left[1 - \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{(2\,cos.\,\alpha)^2}\right] = \frac{46}{0\cdot 940}\left[1 - 0\cdot 1\,(\tfrac{4}{3})^2 \frac{1}{4 \times 0\cdot 884}\right]$$

$$= 48\cdot 9\left(1 - \frac{0\cdot 4}{7\cdot 956}\right) = 48\cdot 9 \times 0\cdot 95 = 46\cdot 5 \text{ ft.}$$

For the exit angle $\delta$,

$$sin.\,\delta = \left(\frac{r_1}{r}\right)^2 sin.\,2\,\alpha = \tfrac{9}{16}\,sin.\,40^\circ = 0\cdot 3616, \text{ or } \delta = 21^\circ\,12'.$$

We have also

$$\zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{1}{2\,cos.\,\alpha}\right)^2 = 0\cdot 050,\ \left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\,\frac{1}{2}\,\delta}{cos.\,\alpha}\right)^2 = (\tfrac{4}{3})^2 \left(\frac{sin.\,10^\circ\,36'}{cos.\,20^\circ}\right)^2 = 0\cdot 055.$$

Hence the effective delivery is

$$L_1 = \left[1 - \zeta - \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{(2\, cos.\ \alpha)^2} = \left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\ \frac{1}{2}\delta}{cos.\ \alpha}\right)^2\right] Q\, h\, \gamma$$

$$= (0 \cdot 900 - 0 \cdot 050 - 0 \cdot 055)\, Q\, h\, \gamma = 0 \cdot 795\, Q\, h\, \gamma = 5523 \text{ ft. lbs.}$$

For both wheels, we have for the aperture of the guide race,

$$F = \frac{Q}{c} = \frac{0 \cdot 75}{92} = 0 \cdot 00815 \text{ square ft.} = 1 \cdot 17 \text{ square inches.}$$

If the depth of this aperture is taken at $d = \frac{1}{4}$ inch, we have for its height

$$e = \frac{F}{d} = \frac{1 \cdot 17}{\frac{1}{4}} = 4 \cdot 68 \text{ inches,}$$

for which in practice we may take 5 inches. The depth of the wheel should be but little greater, or 5½ inches. If we take the outer radius at 3 ft., we have for the number of revolutions per minute,

(1) for inward flow,

$$u = \frac{30\, v_1}{\pi\, r} = \frac{30 \times 46}{3\, \pi} = \frac{460}{3 \cdot 14} = 146,$$

and (2) for outward flow,

$$u = \frac{30\, v_1}{\pi\, r_1},$$

or since $r_1 = \frac{3}{4} r = 2 \cdot 25$ ft.,

$$u = \frac{30 \times 46 \cdot 5}{2 \cdot 25\, \pi} = \frac{620}{\pi} = 197.$$

§ **237.**—Inward-flow tangential wheels were first constructed by Zupinger in the factory of Escher, Wyss & Co., in Zurich.

The first idea of an inward-flow tangential wheel is due to Poncelet ("Cours de mécanique appliquée aux machines").

In the Figs. 463 and 464, we have an elevation and plan of such a wheel. *A* is the reservoir, *B* the conduit pipe, and *C* the guide-blade apparatus, consisting of three canals, by which the water is brought in nearly tangential to the wheel. To regulate the feed, we have a sluice board *D*, movable by means of a toothed wheel *E*. In the situation shown in the figure, one guide canal is shut off entirely, and the water flows through only two. The wheel con-

FIG. 463.

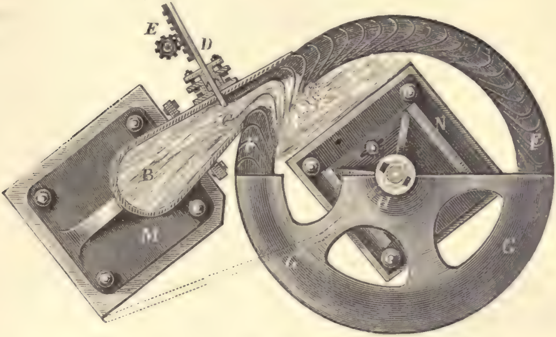
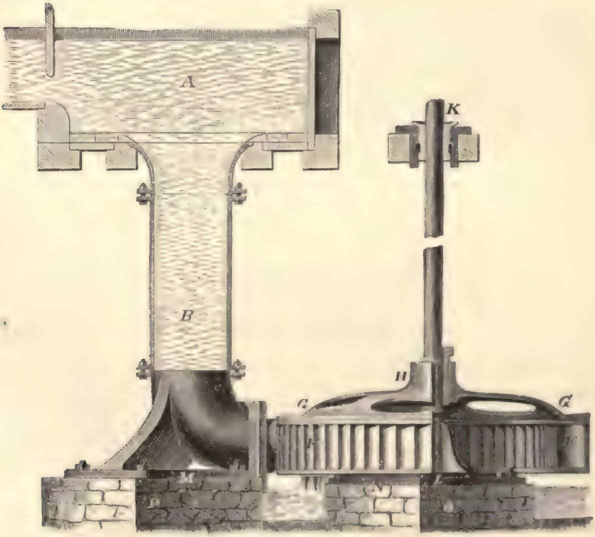
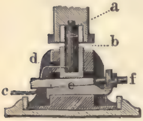


FIG. 464.



sists of sixty floats, and is united with the axle *K L* by means of a plate *G G* and clamp *H*. The axle passes through a box *K* above, and is let into a socket below, which is shown in Fig. 465. Here the pivot *b* is let into the axle at *a*, *c d* is a pipe for the supply of oil, and *e* a wedge, adjusted by the screw *f*, by means of which the pivot can be raised or lowered. The wheel, represented in the figure in $\frac{1}{30}$ of its natural size, with a fall of 20 ft. (shortened in the figure), and a feed of 7 cubic ft. per second, making 65 revolutions per minute, gave an efficiency of 0·72.

FIG. 465.

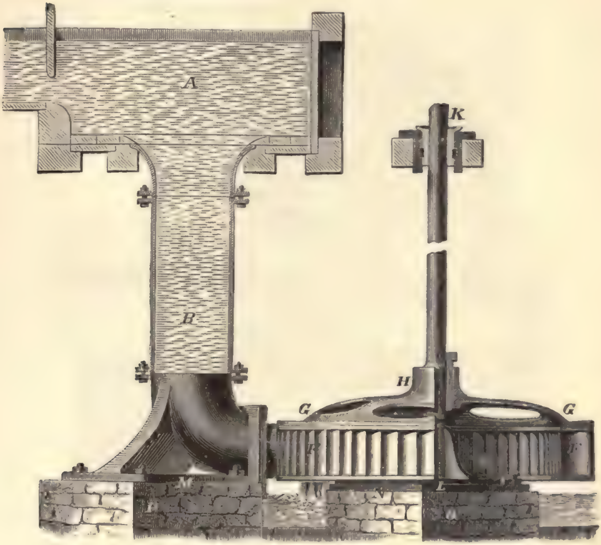


In the "Polytechnischen Centralblatte," 1847 and 1849, are given the results of experiments upon two pairs of such wheels. The first pair had a feed of 7 cubic ft. per second and a fall of 76 ft., the outer diameter of each wheel was 24 and the inner 16 inches, the depth of wheel was only 3 inches, and the number of floats 48. The water was brought on by a pipe of boiler iron 76 ft. long and 18 inches wide. From it proceeded two horizontal branches, the one to one wheel and the other to the other. Each aperture was closed by a sluice worked by an endless screw, and the water entered through a guide-blade arrangement forming three canals, nearly tangential to the wheel. The results of experiments made by Prof. Hülsze upon one of these wheels, making 270 revolutions per minute, gave for the efficiency

0·75 for sluice entirely open,
0·60 for sluice $\frac{3}{4}$ open,
0·46 for sluice $\frac{1}{2}$ open.

These wheels were in use in a cotton mill in Tanneberg, near Annaberg. The second pair were in use in Birkigt, near Tetschen, in a grist mill. The fall was $20\frac{1}{4}$ ft. only, each wheel had 75 floats, 5 ft. outer diameter, 5 inches crown breadth, and $11\frac{1}{2}$ inches deep. The water was brought on in a similar manner to the first case, as shown in Fig. 466. From experiments made by Prof. Brückmann, for 61 revolutions per minute the maximum efficiency was attained and was 0·70, and this sank to 0·65 when the number of revolutions sank to 50 or rose to 70, or when the feed was diminished one half.

FIG. 466.



§ **238.—Vertical Tangential Wheels.**—The principle of tangential wheels may be applied to vertical wheels also (see Art. 235). Such wheels, with horizontal axis and outward flow, were first constructed by Schwamkrug. (See "Jahrbuch für den Berg- und Hüttenmann," for the years 1850 and 1853.) In Fig. 467, we have such a wheel. The wheel *R R* is joined to the horizontal axis *C* by a single system of arms and rosette, the power being taken from the axle by a toothed wheel, etc. The water enters near the lower part of the wheel through the pipe *L L* which discharges into a chamber containing guides. This is shown in Fig. 468. Here we have the pipe discharging at $L$ into the chamber $K E'$. This last is divided by a tongue into two portions, each of which is provided with valves $D E$, $D_1 E_1$, turning about $D$ and $D_1$, by which the flow through each orifice can be regulated. These valves can be moved by the arms $a a_1$, shown in Fig. 467, which are outside of the chamber, and so fastened to each other and to the axes $D$ and $D_1$, that, by means of the arm $b$ and rod $Z S$, they may be moved together.

FIG. 467.

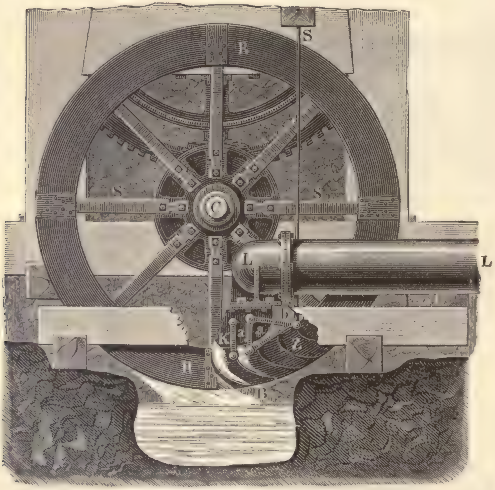
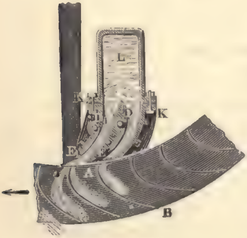


FIG. 468.



Turbines with horizontal axis have the advantage over those with vertical axis of better means for the introduction of the water. The above wheel, experimented upon by the constructor, had an outer diameter of 7⅝ ft., inner diameter 6 ft., 4 inches width, and 45 floats. The fall was 103½ ft., the quantity of water 38·7 to 133·6 cubic ft. per minute, and the efficiency was, for 112 to 148 revolutions per minute, $\eta = 0 \cdot 58$ to $0 \cdot 79$.

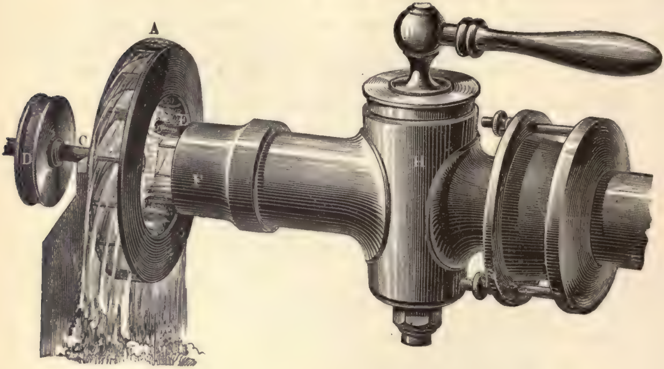
More concerning this turbine will be found in the "Polytechn. Centralblatt," Jahrgang 1849, Nos. 8 and 9, as also in the "Jahrbuch für den sächs. Berg- und Hüttenmann." Another turbine of this kind, used for working the man ladder in the mine "Churprinz Friedrich August Erbstolln" at Freiberg, is described by Schwamkrug in the "Jahrbuch für den Berg- und Hüttenmann," 1853. It has a fall of 145 ft., a supply of 550 cubic ft. per minute, a width of 13 inches, and inner diameter of 8 ft.

REMARK.—In France also, tangential turbines with interior feed have been recently constructed. At the Exhibition of 1855 in Paris, several such wheels, entirely of plate iron, were exhibited.

Since these turbines make a large number of revolutions per minute, considerable intermediate mechanism is necessary in order to transmit the power with the required speed.

§ **239.—Jet Turbines.**—Instead of admitting the water at one side, we can bring it in in the direction of the axle, and turn it off radially in all directions. As such a turbine is set in motion by an isolated stream, it is not improperly called a *jet turbine.* Such a wheel may be either horizontal or vertical. In Fig. 469, we have such a wheel with horizontal axis *C D.* The jet of water enters at *E*, is led through a pipe *H* furnished with a cock, by which the motion can be regulated. Such a machine is especially suited for the performance of small amounts of work, such as replacing man-power by the use of city water. The theory of tangential wheels just developed, finds here only partial application. Since here the water enters the canals between the floats in a radial direction, $\alpha = 90°$,

FIG. 469.



and hence

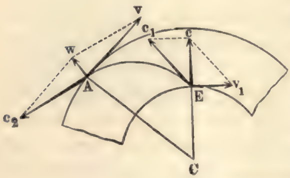
$$v_1 = \frac{c}{2 \cos. \alpha}$$

gives for the best velocity,

$$v_1 = \frac{c}{0} = \infty.$$

For these turbines, therefore, the exit velocity $w$ cannot be zero even if the exit angle $\delta$ could be $= 0$, and hence the maximum of the theoretical effect cannot be attained. If, however, we make the velocity of circumference $v$ as great as the velocity of entrance of the water, the *vis viva* of the departing water is so small that the efficiency of the machine is still considerable. We have here for the relative velocity of the water $c_1$,

FIG. 470.



$$c_1^2 = c^2 + v_1^2 \text{ (Fig. 470)},$$

and

$$\frac{c_2^2}{2g} - \frac{c_1^2}{2g} = \frac{v^2 - v_1^2}{2g},$$

or

$$c_2^2 = c_1^2 + v^2 - v_1^2 = c^2 + v^2.$$

Hence if we make $v = c$, we have

$$c_2^2 = 2\,v^2 \qquad \text{and} \qquad c_2 = \sqrt{2}\,.\,v = 1\cdot414\,v.$$

If, now, we take $\delta = 0$, the velocity

$$w^2 = c_2^2 + v^2 - 2\,c_2\,v\,\cos.\,\delta = c_2^2 + v^2 - 2\,c_2\,v,$$

or

$$w = c_2 - v = 0\cdot414\,v.$$

Hence the loss of work

$$\frac{w^2}{2g} = 0\cdot171\,\frac{v^2}{2g} = 0\cdot171\,h,$$

where $h = \frac{c^2}{2g}$ = the fall necessary to produce the velocity $c$. If we make $v$ greater than $c$, this loss is less. For $v = \frac{3}{2}\,c$,

$$\frac{w^2}{2g} = (1\cdot803 - 1\cdot5)^2\,\frac{c^2}{2g} = (0\cdot303)^2\,\frac{c^2}{2g} = 0\cdot092\,h.$$

But $\delta$ can never be zero, and this loss is therefore greater than above; also friction consumes a portion of work, so that a low efficiency is to be expected. For the exit angle $\delta$, since we must have

$$2\,\pi\,r_1\,c = 2\,\pi\,r\,c_2\,\sin.\,\delta,$$

we have

$$\sin.\,\delta = \frac{r_1}{r}\,\frac{c}{c_2}.$$

For example, for $\frac{r_1}{r} = \frac{1}{2}$ and $v = c$, therefore $c_2 = c\sqrt{2}$, we have

$$\sin.\,\delta = 0\cdot5 \times 0\cdot707 = 0\cdot354 \qquad \text{or} \qquad \delta = 20\tfrac{3}{4}^\circ,$$

and

$$\frac{w^2}{2\,g} = \frac{v^2 + c_2^{\,2} - 2\,v\,c_2\,cos.\,\delta}{2\,g} = \frac{(1 + 2 - 2\,\sqrt{2}\,cos.\,\delta)\,c^2}{2\,g}$$

$$= (3 - 2\cdot828\ cos.\ 20\tfrac{3}{4}^\circ)\,h = 0\cdot356\,h.$$

These turbines give indeed a small efficiency, but possess the advantages of simplicity and compactness. For high falls, they are true liliputian wheels, and have so great a number of revolutions that in most cases gearing must be used to reduce it.

Example.—Required the disposition of a jet turbine, for a fall of $h = 100$ ft., delivery $L = 4$ horse-power = 2200 ft. lbs. per second.

If we let the wheel revolve with a velocity of

$$c = v = \sqrt{2\,g\,h} = \sqrt{6440} = 80 \text{ ft.},$$

we have

$$\frac{w^2}{2\,g} = 0\cdot356\,h = 35\cdot6 \text{ ft.}$$

If we assume that friction consumes still $0\cdot144\,h = 14\cdot4$ ft., we have the effective delivery of the turbine,

$$L = (1 - 0\cdot356 - 0\cdot144)\,Q\,h\,\gamma = 0\cdot5\,Q\,h\,\gamma = 50 \times 62\cdot5\,Q = 3125\,Q.$$

The necessary feed is therefore

$Q = \frac{2200}{3125} = 0\cdot704$ cubic ft., or $60\,Q = 42\cdot2$ cubic ft. per minute.

For the velocity $c = v = 80$ ft., the cross-section of the jet must therefore be

$$F = \frac{Q}{v} = 0\cdot00891 \text{ sq. ft.} = 1\cdot283 \text{ sq. inches.}$$

Since a circular area of this area has a diameter of about 1·3 inches, it is sufficient to make the inner radius of the wheel 1 inch and the outer 2 inches. For the diameter of 4 inches therefore we have the number of revolutions,

$$u = \frac{30\,v}{\pi\,r} = \frac{30 \times 80}{\frac{1}{3} \times 3\cdot141} = \frac{2370}{1\cdot047} = 2263 \text{ per minute.}$$

§ **240.—Danaïdes.**—Allied to tangential turbines are horizontal wheels which have more or less the shape of an inverted cone. Such wheels are called in France *roues à* *poires* or *Danaïdes*. They are described by Belidor in his "Architecture Hydraulique." Fig. 471 represents the general arrangement of such a wheel. It consists essentially of a vertical axis $CD$ with a double conical casing, the space between being divided up by division plates forming conduits running from top to bottom. The water is brought on by a race $G$ above, enters at $A$, and departs at $F$, below and nearer to the axis. In the simplest form of these wheels, the division partitions are plane surfaces running vertically; in other cases, they are spiral or screw-shaped. The wheels described by Belidor are without the outer casing, but are placed in a conical vessel fitting closely to the blades. We shall consider only wheels of the first kind.

FIG. 471.

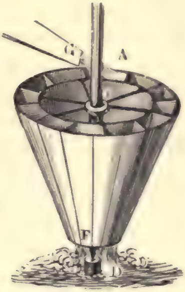
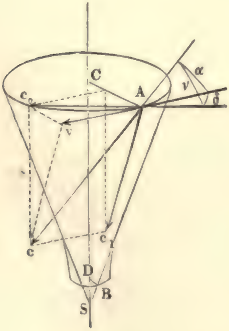


FIG. 472.



In these wheels, the water is acted upon simultaneously by gravity and by centrifugal force. If the water enters with the absolute velocity $A c = c$, Fig. 472, and if the angle $c A v$ between the direction $A c$ and that of the wheel velocity $A v = v$ at $A$ is $\alpha$, we have for the relative velocity $A c_1 = c_1$ with which the water runs down the vertical float,

$$c_1^2 = c^2 - v^2.$$

Since the water falls in the wheel through the height $CD = h_1$, and at the same time approaches nearer the axis, the velocity of which is $v_1 = 0$, we have for the relative velocity $c_2$ of the water at $F$,

$$c_2^2 = c_1^2 + 2\,g\,h_1 - v^2 = c^2 + 2\,g\,h_1 - 2\,v^2,$$

or, denoting the entire fall $c^2 + 2\,g\,h_1$ by $h$,

$$c_2^2 = 2\,g\,h - 2\,v^2.$$

In order, now, that the water may leave with as little velocity as possible, $c_2$ must be zero, and therefore the upper velocity of circumference $v = \sqrt{g\,h}$. The theoretical effect of the wheel is then

$$L = Q\,h\,\gamma.$$

Since the mean values of $c_2$ and $v_1$ cannot be entirely null, the water flows out with the absolute velocity

$$w = \sqrt{c_2^2 + v_1^2},$$

and there is an amount of work lost equal to

$$\frac{w^2}{2\,g} Q\,\gamma = \frac{c_2^2 + v_1^2}{2\,g} Q\,\gamma.$$

If the half angle of convergence $A\,S\,C = \theta$, or the inclination of outer casing to horizon is $90° - \theta$, we have for the angle $c\,A\,v = \alpha$ of the stream to horizon,

$$c \; sin.\, \nu = c_1 \, cos.\, \theta = c \; sin.\, \alpha \; cos.\, \theta,$$

hence

$$sin.\, \nu = sin.\, \alpha \; cos.\, \theta \; . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

and for the angle $v\,A\,c_0 = \delta$,

$$v \; tan.\, \delta = v \; tan.\, \alpha \; sin.\, \theta,$$

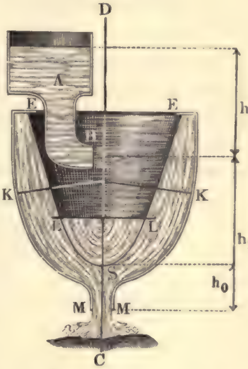
or

$$tan.\, \delta = tan.\, \alpha \; sin.\, \theta \; . \; . \; . \; . \; . \; . \; . \quad (2).$$

If we give the wheel the shape of a plane circular surface, we have a tangential wheel with plane floats, and in the above formulæ have $\theta = 90°$, or $sin.\, \theta = 1$ and $cos.\, \theta = 0$, so that $\nu = 0$ and $\delta = \alpha$, and hence the stream of water must be led on in a horizontal direction. If the wheel is cylindrical, we have $\theta = 0$, and $cos.\, \theta = 1$, $sin.\, \theta = 0$, so that $= \alpha$ and $\delta = 0$, and hence the stream must be tangential to the wheel. The wheel just described is known as *Burdin's Danaïde.*

**§ 241.**—We may also have a Danaïde of the form shown in Fig. 473, consisting of a vessel $EKM$ divided by a vertical partition into two equal portions. In order that the water entering from the feed box $AB$ may suffer no impact, its velocity $c$ must equal the velocity of rotation $v_1$ of the water in the wheel at $B$. If $h_1$ is that part of the entire fall which is necessary for the generation of the entrance velocity, we have then

FIG. 473.



$$c = v_1 = \sqrt{2gh_1}.$$

If also $h_2$ is the pressure height of the water in the wheel, and $v$ the velocity of rotation of the wheel at $M$, we have for the relative velocity of exit $c_2$,

$$c_2^2 = 2gh_2 - v_1^2 \quad v^2,$$

or if the orifice, and therefore $v$, is very small,

$$c_2^2 = 2gh_2 - v_1^2.$$

In order that the velocity $c_2$ may be as small as possible, and the work therefore as great as possible, we must have $v_1^2 = 2gh_2$, and hence

$$v_1 = \sqrt{2gh_2}.$$

Accordingly

$$h_1 = h_2 = \tfrac{1}{2}h;$$

or one half of the entire fall is to be used in introducing the water, and the other half should be the height of the wheel.

The velocity $c_2$ cannot of course be zero in practice, but should be very small (4 to 6 ft.), and the orifice $M M$ should have the area $F = \frac{Q}{c_2}$, corresponding to the radius $r = \sqrt{\frac{Q}{\pi c_2}}$.

Since the water has not only the velocity $c_2$ in a vertical direction, but also a velocity of rotation, whose mean square is, according to the theory of the moment of inertia (Vol. I., Art. 288), $= \frac{1}{2} v^2$, we have for the work inherent in the departing water,

$$L_1 = \left(\frac{c_2^2 + \frac{1}{2} v^2}{2 g}\right) Q \gamma,$$

and the efficiency of the wheel is

$$\eta = 1 - \frac{c_2^2}{2 g h} - \frac{1}{2} \frac{v^2}{2 g h}.$$

Since we may put

$$v_1 = \sqrt{\tfrac{1}{2} \times 2 g h} = \sqrt{g h}$$

and

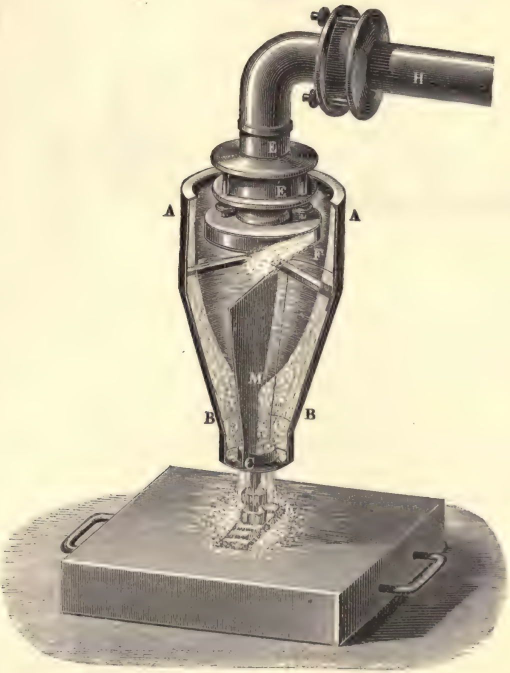
$$v = \frac{r}{r_1} v_1 = \frac{r}{r_1} \sqrt{g h},$$

we have also

$$\eta = 1 - \frac{c_2^2}{2 g} - \frac{1}{4} \left(\frac{r}{r_1}\right)^2.$$

The vertex $S$ of the paraboloid of revolution $E S E$, formed by the free surface of the water in the wheel, is at a distance $S M = h_0 = \frac{c_2^2}{2 g}$ from the orifice; this distance being the lost head. In order to hinder the formation of the funnel *E S E* directly over the orifice *M M*, we may use a partition *L L*.

Fig. 474.



In Fig. 474, we have a model of a Danaïde constructed upon the above principle. The water is led on through the pipe *H E*, and by means of the two orifices *F*, *F*, introduced tangentially into the wheel *A B C*, and passes out through the orifice *C*. By a gear wheel at *B B*, the power is taken off.

**§ 242.—Reaction-Wheels.**—If we set the vessel *H R F*, Fig. 475, upon wheels, the *reaction* of the issuing water will cause the vessel to move in a direction opposite to that in which the water issues. If we have a vessel attached to an axle *C D*, Fig. 476, the reaction of the issuing water will cause it to revolve. If the escaping water is always replaced, so that a constant level is maintained, we can thus produce a continual and permanent revolution. Such a machine we may call a reaction-wheel (Fr. roue à réaction; Ger. Reactionsrad). In England, it is known as Barker's mill, in Germany as Segner's water-wheel.

Fig. 475.

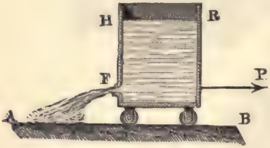


Fig. 476

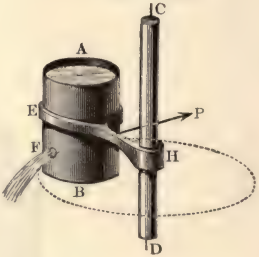


Fig. 477.

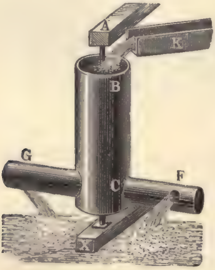
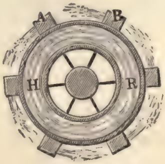


Fig. 478.



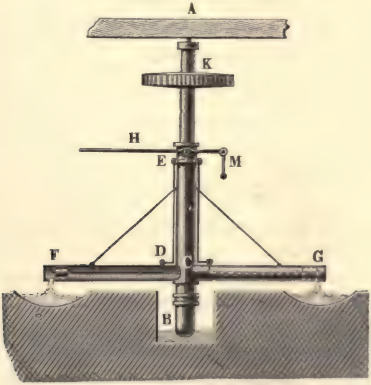
The simplest wheel of this kind is shown in Fig. 477. It consists simply of a tube *B C* kept full of water, and the pipes *C F* and *C G*, from which the water issues through the side orifices *F* and *G*. The water is supplied by the race *K*.

Applied to grist mills, the upper millstone may be set directly upon *A X*. For other purposes, the motion may be taken from *A X* by gearing or belts.

, We may also have reaction-wheels with many arms, as shown in Fig. 478. The vessel *H R* may be cylindrical or conical. In order to introduce the water without impact, Euler has applied a similarly shaped vessel directly above the wheel, and placed in the bottom of this vessel inclined guides, similar to Burdin's later arrangement for his turbine (Art. 234). Burdin has also constructed similar reaction-wheels. To this class belongs also the experimental wheel in Vol. I., Art. 504.

A simple reaction-wheel was seen in action by the author in Vallendar near Ehrenbreitenstein. It was constructed by Althaus, and served as motor for two tan mills.

FIG. 479.



The essential construction of this wheel is shown in Fig. 479. The water is brought on by a pipe which is bent upwards at *B*. The axle *A C*, with the two arms *C F* and *C G*, is hollow below and passes into the end *B* of the entrance pipe. In order that water may not escape, there is at *B* a stuffing box. The rectangular openings *F* and *G* are closed by slides moved by rods and levers *D* fastened to a collar *E*, which can be raised by the lever *H M*. Above we have the toothed wheel *K* for taking off the power. This construction has the advantage that the entire weight of the machine is borne by the water, and there is little friction at bottom. If *G* is the weight of the machine, *h* the pressure head, and $2r$ the width of the ascending pipe, we have for this case,

$$\pi r^2 h \gamma = G,$$

and therefore the necessary radius of pipe is

$$r = \sqrt{\frac{G}{\pi h \gamma}}.$$

The feed was 18 cubic ft. per minute and the fall 94 ft., hence the work at disposal $Q \gamma h = 1762$ ft. lbs. per second. The length of an arm was 12½ ft., and the number of revolutions per minute while in action was 30, therefore the velocity of circumference was 39·3 ft.

REMARK 1.—The first description of a reaction-wheel, as the invention of Barker, is to be found in Desagulier's "Course of Experimental Philosophy," Vol. II., London, 1875. Euler treats thoroughly the theory and construction of these wheels in the "Memoiren der Berliner Akademie," 1750, 1754.

REMARK 2.—The efficiency of the older reaction-wheels was very small. Nordwall found it only ⅛ of that for an overshot-wheel. Schitko (see his "Beiträge zur Bergbaukunde," etc., Wien, 1833) found the greatest efficiency for such a wheel 0·15.

§ **243.—Theory of Reaction-Wheels.**—The action of the reaction-wheel may be discussed theoretically as follows: Let *h* be the fall or distance of centre of the orifices below the water surface in the conduit pipe, and *v* the velocity of rotation of the same. We have, then, for the height corresponding to the pressure at the orifices,

$$h_1 = h + \frac{v^2}{2g},$$

and hence the theoretical velocity of exit is

$$c = \sqrt{2\,g\,h_1} = \sqrt{2\,g\,h + v^2}.$$

If $\phi$ is the coefficient of velocity, we have for the effective velocity of exit,

$$c = \phi\sqrt{2\,g\,h + v^2}.$$

This velocity is, however, not the absolute velocity of the water at exit, for this has also the opposite velocity $v$ in common with the wheel. The absolute velocity of the issuing water is therefore

$$w = c - v = \phi\sqrt{2\,g\,h + v^2} - v,$$

and the corresponding loss of work is

$$L_1 = \frac{w^2}{2g}Q\gamma = \frac{(\phi\sqrt{2\,g\,h + v^2} - v)^2}{2\,g}Q\gamma.$$

If we take $\phi = 1$, we have

$$L_1 = \frac{(\sqrt{2gh + v^2} - v)^2}{2g}Q\gamma = \left(h - \frac{v(\sqrt{2\,g\,h + v^2} - v)}{g}\right)Q\gamma.$$

and if we subtract this from the work at disposal, we have the effective delivery,

$$L = \left(h - \frac{w^2}{2g}\right)Q\gamma = \frac{v(\sqrt{2\,g\,h + v^2} - v)}{g}Q\gamma.$$

This will be greater the greater $v$, or if we put

$$\sqrt{v^2 + 2\,g\,h} = v + \frac{g\,h}{v} - \frac{g^2\,h^2}{2\,v^3} + \ldots,$$

we have

$$L = v\left(\frac{g\,h}{v} - \frac{g^2\,h^2}{2\,v^3} + \ldots\right)\frac{Q\gamma}{g},$$

or for $v = \infty$,

$$L = Q h \gamma,$$

the entire disposable work.

This condition, that the maximum delivery is given by an infinitely great velocity, is an unfavorable one, because the resistances increase rapidly with the velocity, as is easy to see, since even the unloaded wheel has by no means an infinite velocity. These resistances can consume all the work long before such a velocity is attained. Wheels whose theoretical maximum delivery occurs for an infinitely small velocity should have a greater efficiency than those from which an infinitely great velocity is demanded. The velocity of the wheel also cannot be infinitely great, because the water, by reason of the height due to air and water pressure $(33 + h)$, can at most only have the velocity $\sqrt{2 g (33 + h)}$, and hence for a quicker discharge the steady exit would be interrupted.

It remains to ascertain whether the delivery for mean or not very great velocities of rotation deviates considerably from the maximum or disposable work $Q h \gamma$. If we load the wheel to such an extent that the height due to the velocity of rotation is equal to the fall, or

$$\frac{v^2}{2 g} = h \qquad \text{or} \qquad v = \sqrt{2 g h},$$

then we have for the delivery

$$L = \frac{\sqrt{2 g h}\,(\sqrt{4 g h} - \sqrt{2 g h})}{g} Q \gamma = 2 (\sqrt{2} - 1) Q h \gamma$$

$$= 0{\cdot}828\, Q h \gamma.$$

If, however, we make

$$\frac{v^2}{2 g} = 2 h,$$

we have

$$L = \frac{\sqrt{4gh}(\sqrt{6gh} - \sqrt{4gh})}{g} Q\gamma = 4(\sqrt{1 \cdot 5} - 1) Qh\gamma$$

$$= 0 \cdot 899\, Q\, h\, \gamma.$$

If, finally, we make

$$\frac{v^2}{2g} = 4h,$$

we have

$$L = \frac{\sqrt{8gh}(\sqrt{10gh} - \sqrt{8gh})}{g} Q\gamma = 8(\sqrt{1 \cdot 25} - 1) Qh\gamma$$

$$= 0 \cdot 944\, Q h \gamma.$$

We lose then in the first case 17, in the second 10, and in the third only 6 per cent of the disposable work, and see therefore that for moderate falls and a velocity of rotation near that due to the fall, a considerable efficiency is to be expected. The great simplicity of this machine is also a great advantage as compared with other wheels.

REMARK.—The force of reaction is

$$P = \frac{L}{v} = \frac{\sqrt{2gh + v^2} - v}{g} Q\gamma,$$

or for $v = 0$,

$$P = \frac{\sqrt{2gh}}{g} Q\gamma = \frac{c}{g} Q\gamma = 2\frac{c^2}{2g} F\gamma,$$

*i. e.*,, the reaction of a horizontal stream *is equal to the weight of a column of water whose cross-section is that of the stream*, *and whose height is double that due to the velocity*, as already proved in Vol. I., Art. 495.

§ **244.—Effective Delivery of the Reaction-Wheel.**—If in the formula already found for the delivery, viz.,

$$L = \frac{(\sqrt{2gh + v^2} - v)v}{g} Q\gamma,$$

we take account of the resistance to efflux, we have

$$c = \phi \sqrt{2 g h + v^2} = \sqrt{\frac{2 g h + v^2}{1 + \zeta}},$$

and if we put

$$Q = F c = \phi F \sqrt{2 g h + v^2},$$

the above formula becomes

$$L = (\phi \sqrt{2 g h + v^2} - v) \frac{v Q \gamma}{g}$$

$$= (\phi \sqrt{2 g h + v^2} - v) \frac{\phi F v \gamma}{g} \sqrt{2 g h + v^2}$$

where $\phi$ is the coefficient of efflux and $F$ the sum of the areas of the orifices.

If $Q$ is known, we have also

$$L = \left(\frac{Q}{F} - v\right) \frac{v Q \gamma}{g},$$

and hence the efficiency is

$$\eta = \frac{L}{Q h \gamma} = \left(\frac{Q}{F} - v\right) \frac{v}{g h} = (\phi \sqrt{2 g h + v^2} - v) \frac{v}{g h}.$$

This is a maximum when

$$\phi v \sqrt{2 g h + v^2} - v^2$$

is a maximum, or differentiating and putting first differential equal to zero, when

$$\sqrt{2 g h + v^2} + \frac{v^2}{\sqrt{2 g h + v^2}} = \frac{2 v}{\phi},$$

whence we obtain

$$v^4 + 2 g h v^2 = \frac{\phi^2 g^2 h^2}{1 - \phi^2},$$

or

$$v = \sqrt{g h} \sqrt{\frac{1}{\sqrt{1 - \phi^2}} - 1}.$$

This value of $v$, then, gives the maximum efficiency, hence

$$c = \sqrt{2 g h + v^2} = \sqrt{g h} \sqrt{\frac{1}{\sqrt{1 - \phi^2}} + 1},$$

and

$$\eta = \phi \sqrt{\frac{1}{\sqrt{1-\phi^2}} - 1} \, . \sqrt{\frac{1}{\sqrt{1-\phi^2}} + 1} - \frac{1}{\sqrt{1-\phi^2}} + 1,$$

or

$$\eta = 1 - \sqrt{1-\phi^2}.$$

The effective delivery, therefore, is

$$L = \eta \, Q \, h \, \gamma = (1 - \sqrt{1-\phi^2}) \, Q \, h \, \gamma$$

$$= \phi^2 \sqrt{\frac{1}{\sqrt{1-\phi^2}} - 1} \, . \, (F \gamma \sqrt{g \, h^3}),$$

since we can put

$$Q = \phi F c = \phi F \sqrt{g \, h} \, F \sqrt{\frac{1}{\sqrt{1-\phi^2}} + 1}.$$

If we divide the delivery by the velocity $v$, we have the force of reaction,

$$P = (\phi \sqrt{2 g h + v^2} - v) \frac{Q \gamma}{g} = (\phi \sqrt{2 g h + v^2} - v) \sqrt{2 g h + v^2} \frac{\phi F \gamma}{g},$$

and hence when the wheel is still, or

$$v = 0, \; P = \phi^2 \times 2 \, F \, h \, \gamma.$$

The correctness of the preceding theory of the reaction-wheel has been tested experimentally by the author. The experiments were made upon a model of 3·28 ft. diameter, 1·15 sq. in. area of orifice, 1·3 ft. pressure height. By comparison of the effective discharge $Q$ with the theoretical, viz.,

$$F c = F \sqrt{2 \, g \, h + v^2},$$

the coefficient $\phi$ was found 0·9425, and this value of $\phi$ in the formala gives

$$\eta = 1 - \sqrt{1 - 0{\cdot}9425^2} = 1 - \sqrt{0{\cdot}1117} = 0{\cdot}666,$$

which result was also given by experiment. The velocity of rotation for the above maximum efficiency is theoretically,

$$v = \sqrt{g h} \sqrt{\frac{1}{\sqrt{1-\phi^2}} - 1} = \sqrt{\frac{1 - 0{\cdot}334}{0{\cdot}334}} \sqrt{g h} = \sqrt{2 g h},$$

or equal to the velocity due to the height, and this value also was given by experiment.

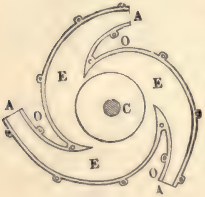
If, finally, we put $\phi = 0\cdot9425$ in the formula

$$P = \phi^2 \times 2\,F h \gamma,$$

we have for the reaction of the water $0\cdot888 \times 2\,F h \gamma$, which also was confirmed by the experiments. When the wheel went with a velocity more than $\sqrt{2\,g\,h}$, the resistance of the air became perceptible, so that the variation between theoretical and effective efficiency increased nearly as $v^3$, and finally the wheel unloaded went round with the velocity $v = 2\sqrt{2\,g\,h}$.

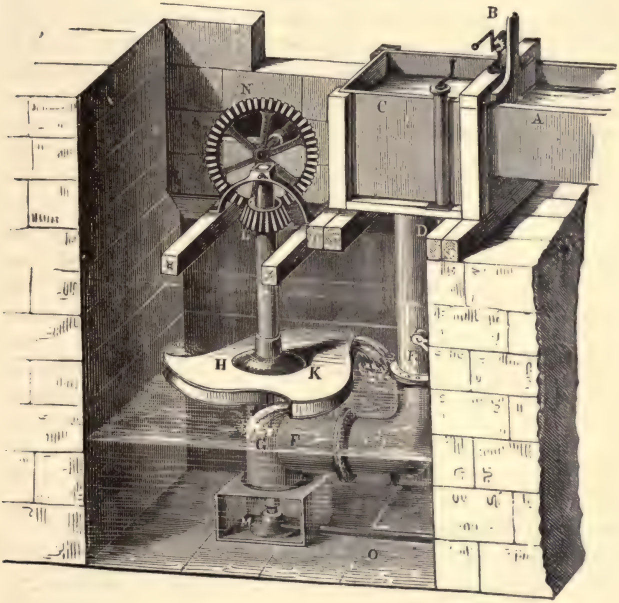
§ **245.—Scottish or Whitelaw's Turbine.**—The Scottish or Whitelaw's turbine consists of a reaction-wheel in which the arms are curved, otherwise it does not differ essentially from the reaction-wheel. Manouri d'Ectot constructed wheels on nearly the same plan as long ago as 1813. (See "Journal des Mines," T. XXXII.) The Scottish turbines constructed by Messrs. Whitelaw and Stirrat, of Paisley, do not differ essentially from the reaction-wheel of Manouri. (See Dingler's "Polytechn. Journal," Vol. 88, and "Polytechn. Centralblatt," Vol. 2, 1843, also the "Description of Whitelaw's and Stirrat's Patent Watermill," 2d ed., London and Birmingham, 1843.) By a special apparatus, the apertures may be contracted in Whitelaw's turbine, and thus the discharge regulated. A horizontal section of such a turbine is shown in Fig. 480. There are three arms; the water enters at *E* and departs at *A*. *O A* is the flap, moving about *O*, which regulates the discharge. The position of these flaps is regulated by an apparatus similar to that described in Fig. 479. The complete arrangement of a Whitelaw turbine is shown in Fig. 481. *A* is the head race; *B* a sluice board, and *C* the reservoir, from which the water passes

FIG. 480.



into the pipe *D E F*. At *E* is a valve by which the water pressure may be regulated. From *F* the water passes into the cylinder *G*, and from there enters the wheel *H K* which

FIG. 481.

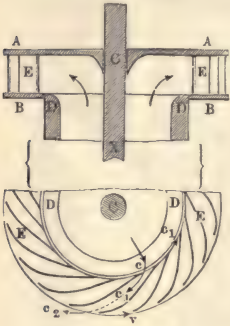


turns upon the axle *L M*. The reaction of the escaping water drives the wheel in the opposite direction from that of the water, and this motion is taken off by the toothed wheels *L* and *N*. The wheel, axle, and entrance pipe are of cast-iron, but the shoe *M* for the axle is shod with brass. Oil for lubricating it runs in through a pipe *O*, which rises above the level of the water in the race. According to Redtenbacher ("Theorie und Bau der Turbinen und Ventilatoren"), we can shield the axle and pivot entirely from the water by enclosing both in a casing.

The theory and geometrical construction of this wheel will be discussed hereafter.

**§ 246.—Combes' Reaction-Wheel.**—Next to the Whitelaw turbine comes Combes' reaction-wheel. In this wheel also the water enters from below, but it differs essentially from the preceding by having a greater number of discharge canals formed of curved floats set between two circular crowns. The essential construction is shown in Fig. 482. *A A* is a plate fastened to the axle *C X* and forming the upper crown, *B B* is the under crown, united with the first by floats *E E*. *D D* is the cylinder through which the water enters along the entire inner circumference, and after passing along the curved floats departs along the whole circumference.

FIG. 482.

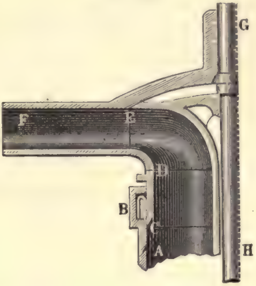


Another essential difference between the Combes and Whitelaw turbine is that a water and air tight joint between the wheel *B* and reservoir *D* is no longer necessary, although such joint is indispensable in Whitelaw's wheel. The reason is as follows: The pressure of the water in a pipe may be in different places very different. Where the water is nearly still the pressure is greatest, and where it runs swiftest this pressure is least. The velocity of the water depends again upon the cross-section, it is inversely as the cross-section. By suitable change of the cross-section, then, we can make the pressure what we will—for instance, equal to the atmospheric pressure. If, now, on the spot where the water pressure equals the atmospheric, we bore a hole, water will neither escape nor air enter. In order, then, that water may not escape through the circular space between the wheel *B* and pipe *D*, which should, however, of course be as small as possible, it is only necessary to give to the cross-sections a proper value.

REMARK 1.—Combes' reaction-wheels are often furnished with guide blades, which give the entering water a certain direction. The turbines constructed in Germany by Wedding and Nagel are similar to the Combes wheel in being fed from below. In construction, however, they more resemble Fourneyron's turbine. Here, also, belong the turbines of Laurent and Deckherr. (See Armengaud's "Publication industrielle," Vol. 6.)

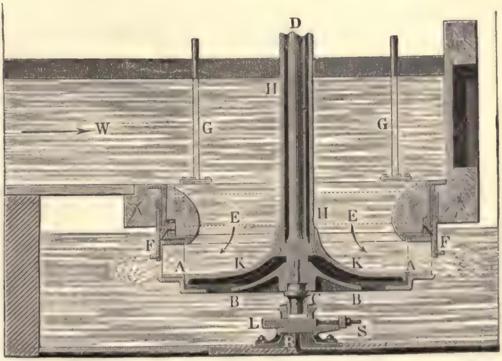
2. Redtenbacher effects the water-tight connection between the reservoir *A B*, Fig. 483, and the wheel *D E F*, by a movable brass ring *C D* which is pressed by the water so tightly against the lower surface *D* of the wheel that no water can escape. The surfaces of contact at *D* must of course be accurately planed. The ring is also fitted with a leather packing *B* stiffened with metal rings.

FIG. 483.



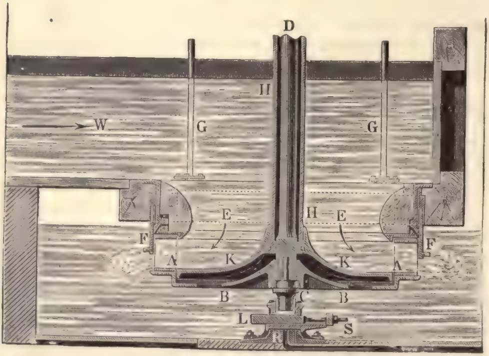
§ **247.—Cadiat's Turbine.**—Next among the class of wheels under consideration comes Cadiat's turbine. It is without guides, like Whitelaw's and Combes', and is, like Fourneyron's turbine, fed from above. A peculiarity of this wheel also is the enveloping circular sluice. In Fig. 484 we have a vertical section of this wheel. *A A* is the

FIG. 484.



wheel proper, and *B B* the disk which unites it with the axle *C D*. The pivot *C* rests in a step, to which we shall call attention hereafter. *E E* is the reservoir, of circular cross-section, connected with the conduit canal *W*, and discharging directly over the upper wheel crown. In order that the water as it flows through *W*, sinks in the reservoir, and flows along *E A* to the wheel, may experience as little contraction and disturbance as possible, the reservoir *E* is enlarged both above and below, as may be seen from the figure. The discharge is regulated by a circular sluice *F F* which encloses the wheel. The raising or lowering of this is effected by four rods and a special mechanism not shown in the figure. In order that the water may not escape between the sluice board and the walls, there is a leather packing upon the inner surface of the sluice.

FIG. 485.



The axis *C D* is also surrounded by a tube *H H* which carries the plate *K K*, which is enclosed by the inner circumference of the lower crown, so that the joint is water-tight and the water cannot penetrate to the plate *B B*. This construction (according to Redtenbacher) is different from that used by Cadiat, but is precisely the same as in Fourneyron's turbine. Cadiat omits the plate with the tube, and balances the pressure of the water upon *B B* by a counter-pressure from below from another reservoir which almost touches the under surface of the wheel *A* and is in communication with the water in *G H*. This arrangement is not as good as Fourneyron's, especially as it is not possible to prevent the escape of the water from this reservoir under hydrostatic pressure, through the circular crack or joint between the wheel and reservoir, be it ever so small. The turbine represented here runs, as is seen, under water.

REMARK.—A complete and exact description of a Cadiat turbine without foot plate, and with water pressure below the wheel plate, is given by Armengaud in the second volume of his "Publication industrielle."

§ **248.—Fourneyron's Turbine.**—Fourneyron's turbine is, in its latest form, when properly constructed, one of the most perfect of the horizontal water-wheels. It revolves either in the air or under water, and may be either high or low pressure. For the low-pressure wheel, the water enters the open reservoir with free surface, as in Fig. 486. For high pressure, the reservoir is boxed up and the water brought in at the side through a pipe, as shown in Fig. 487. The first is for low and the second for high falls. Essentially the wheel consists (see following figure) of two horizontal crowns of iron *A A*, of a cast-iron plate *B B* and an axle *C D*. The turbine of Cadiat is thus far also precisely similar, as shown in Fig. 484. The water entering at *W* passes first to the cylindrical reservoir *E E*. In order that it may not press directly upon the plate *B B* and thus cause an increase of friction, the axle is surrounded by a tube *G H* having at its lower end a plate *F F* which sustains the pressure of the water. This arrangement is also precisely similar in Cadiat's turbine. Fourneyron's turbine differs, however, from Cadiat's in having upon this plate cylindrically curved plates, or *guide blades*, which direct the water in the proper direction to the floats which are

Fig. 486. I.

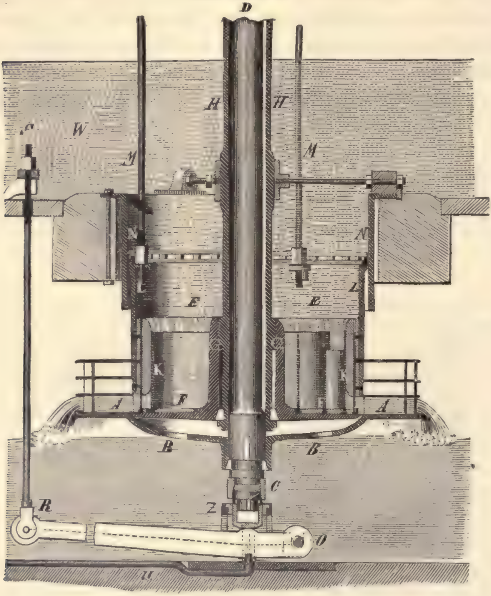
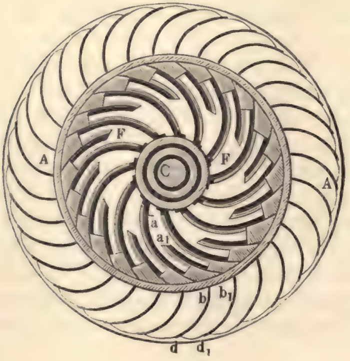
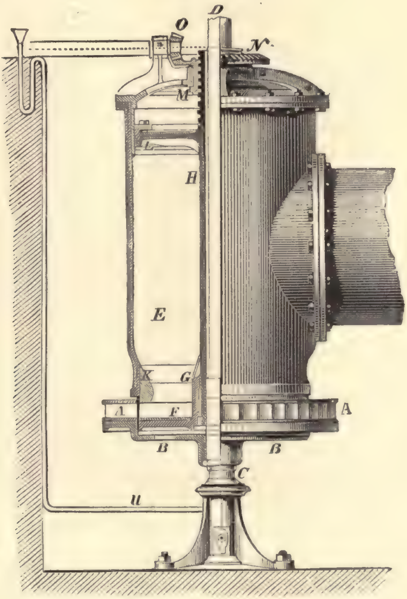


Fig. 486. II.



fixed between the two wheel crowns. These guide blades $a\,b$, $a_1\,b_1$, etc., Fig. 486, with the plate $F\,F$ into which they are set, remain of course immovable. The water passes from them into the floats which revolve with the wheel $A\,B\,B\,A$ and axle in a direction opposite to that in which the water issues.

FIG. 487.



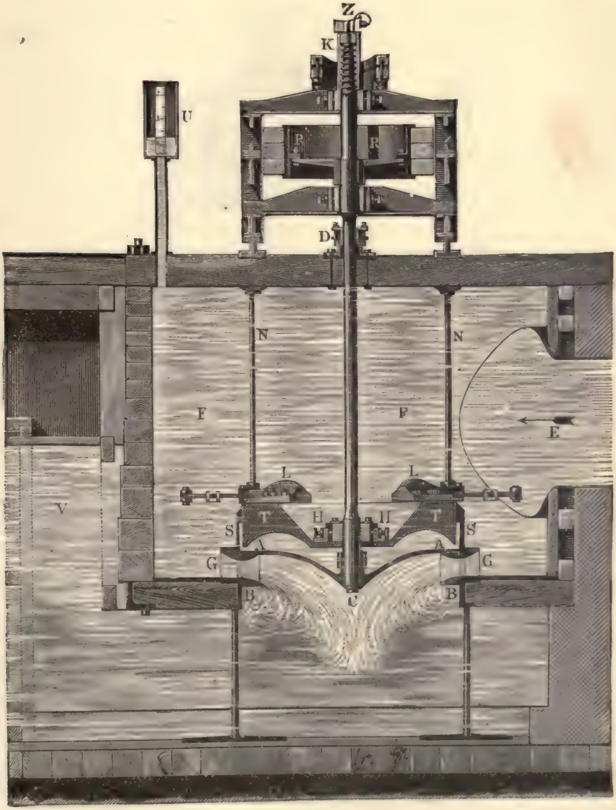
This turbine also differs from Cadiat's in having a cylindrical sluice $K\,L\,L\,K$, consisting of a hollow cylinder, whose outer surface just fits the inner circumference of the upper wheel crown, and which thus fits between the guide blades and the *inside* of the wheel, instead of the outside, as in Cadiat's. These are the only two points of difference between Cadiat's and Fourneyron's wheels, viz., the latter has an internal sluice and guide blades; the former has an external sluice and no guide blades. In other respects, they are entirely similar. In order that no water may escape between the sluice *K L* and the cylinder *N N*, we have above *L L* a leather washer, as in pump pistons. Finally, upon the inner surface of the sluice cylinder, pieces of wood or metal *K K* are attached, which are rounded off gradually and smoothly in order that the water may escape without contraction and with as little loss of *vis viva* as possible. In high-pressure turbines, the sluice rods pass through stuffing boxes in the top of the reservoir, or they may be fastened to the sluice upon the outside. We may also regulate the flow by raising the plate *F F* by means of the hollow pipe *G H*, Fig. 487, worked by the gearing *O* and *N* and the screw thread *M*.

§ **249.**—**The Francis Turbine.**—Instead of having the water enter from within and flow outward, we may have it, as in tangential wheels, enter from *without* and flow *inwards.* Such reaction-wheels with inward flow differ from tangential wheels only in that the water enters along the whole outer circumference, instead of at one spot only, and thus all the wheel canals are filled at once.

Such wheels of inward flow were constructed by Howd, in Geneva, N. Y., and known as Howd or United States wheels. They were for the most part of wood, very simple of construction, but also very faulty. These turbines have been greatly improved upon by Francis (Lowell Hydraulic Experiments), who calls them centre vent wheels. He made the guide blades of curved iron plate, instead of straight pieces of wood, gave to the floats a better form, and introduced many other improvements, which make the wheel one of the best in use. Two such turbines in the Lowell cotton mills possess for 19 ft. fall a delivery of 230 horse-power.

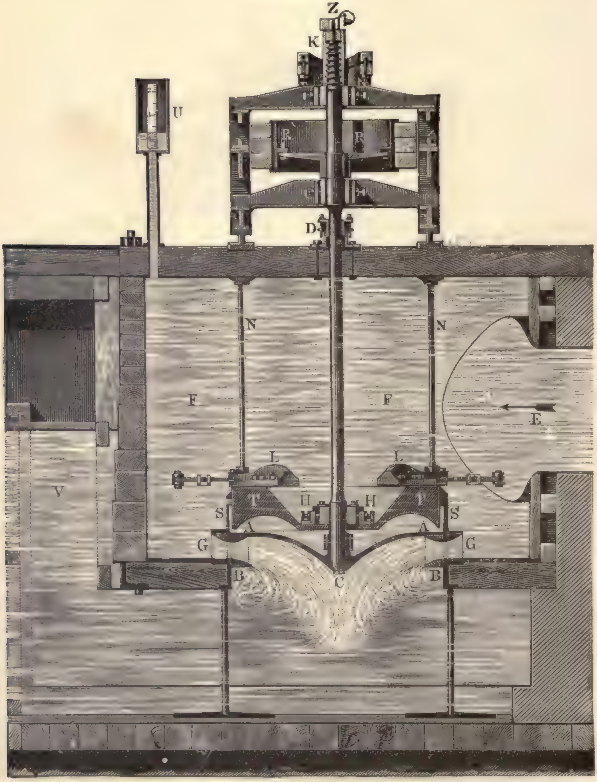
In the Fig. 488, we have a vertical section of such a wheel. *E* is the end of the conduit pipe 8 ft. wide and 130 ft. long, $\frac{3}{8}$ inch thick. This pipe discharges into the closed reservoir *F F*, whose top is some 6 or 7 ft. below the surface of the head-water. The wheel plate *A C A* has a bell-like

FIG. 488.



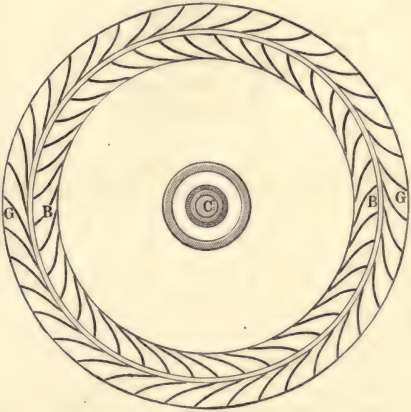
shape and is fastened below to the axle *C D*. The outer diameter is 9·338 ft., the inner 7·987 ft.; the inner depth of wheel $A B = 1·23$ ft., the outer $= 0·999$ ft. This depth, therefore, increases towards the interior, while for the guide-blade apparatus *G G*, the reverse is the case. The number

FIG. 489.



of floats and guide blades is 40, from $\frac{2}{8}$ to $\frac{3}{8}$ inch thick. The shortest distance between two floats is 0·1384 ft., and between two guide blades 0·1467 ft. The wrought-iron axle *C D* passes through a stuffing box in the top of the reservoir, and the end *K* is furnished with a number of rings which play in corresponding circular grooves. By this method of suspension, the enormous weight of the wheel and axle, of 15,200 lbs., is distributed upon a bearing surface of 331 sq. inches, so, that each square inch only sustains 46 lbs. pressure.

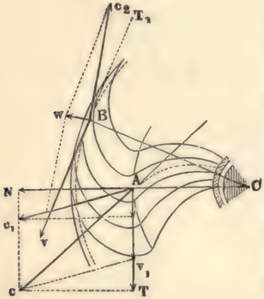
FIG. 490.



Below, the axle passes through a box *H H* in the plate *T T*, by means of which the wheel plate is protected from the pressure of the water. This plate is supported by the rods *N N*. The circular sluice *S S* moves in the space between the wheel and guide-blade apparatus, and sets tight by leather washers upon the plate *T T*. For observation of the level of water above and below the wheel, we have stand pipes, of which one is represented in *U*. The turbine runs, of course, under water. For the determination of the feed, we have a weir at *V* of 14 ft. width. Fig. 490 shows a plan of the wheel and guide blades.

§ **250.—Theory of the Reaction Turbine.**—In order to determine the mechanical relations and the delivery of a Fourneyron turbine, we make use of the following notation:

FIG. 491.



The inner radius $CA$, Fig. 491, or approximately also the outer radius of the reservoir, is $r_1$, the outer radius of the wheel $CB = r$, the inner wheel velocity $= v_1$ and the outer $= v$. The velocity with which the water passes from the guide blades $= c$, the relative velocity with which it enters between the floats $= c_1$, and with which it leaves the floats $= c_2$. The angle $cAT$ which the direction of the water leaving the guide blades makes with the inner wheel circumference $= \alpha$, the angle $c_1AT$ which the water entering the floats makes with this circumference $= \beta$, the angle $c_2BT_1$ which the water leaving the floats makes with the exterior circumference $= \delta$. The sum of the areas of all the exit openings of the guide-blade apparatus $= F$, the sum of the areas of all the entrance openings of the wheel $= F_1$, of all the exit openings $= F_2$. Further, the entire wheel fall from level of water in head race to centre of exit openings of wheel, or when the wheel is under water, to the surface of the under-water, is $h$; the height of the upper-water surface above the centre of the entrance openings of the wheel $= h_1$, and $(h_1 - h) = h_2$ the distance of these last below the exit openings of wheel, or when the wheel is under water, below the surface of the under-water. Finally, the head which measures the pressure of the water at the place where it enters the wheel (without reference to the atmosphere) $= x$.

First, then, for the velocity $c$ with which the water leaves the guide blades, since it is due to the difference of head $h_1 - x$,

$$\frac{c^2}{2g} = h_1 - x,$$

or, more exactly, if the water in the guide-blade apparatus, or upon exit from the same, loses by friction, etc., the head $\zeta \frac{c^2}{2g}$,

$$(1 + \zeta) \frac{c^2}{2g} = h_1 - x.$$

Hence

$$c = \sqrt{\frac{2g(h_1 - x)}{1 + \zeta}},$$

or inversely,

$$x = h_1 - (1 + \zeta) \frac{c^2}{2g}.$$

In order that the water enter the wheel without impact, this velocity $c$ must be decomposed into two others, of which the first must coincide with the inner wheel velocity $v_1$, and the other have the direction of the entering water in the wheel $c_1$. We have, therefore, for this last,

$$c_1^2 = c^2 + v_1^2 - 2 c\, v_1 \cos.\, \alpha.$$

The exit velocity $c_2$ of the water from the wheel is given by the pressure height at entrance $x$, the pressure $h_2$ at exit, and the head corresponding to entrance velocity

$$c_1 = \frac{c_1^2}{2g}$$

and the *centrifugal force* of the water in the wheel, or the head

$$\frac{v^2 - v_1^2}{2g}. \quad \text{(See Art. 235.)}$$

Hence

$$\frac{c_2^2}{2g} = x - h_2 + \frac{c_1^2}{2g} + \frac{v^2 - v_1^2}{2g},$$

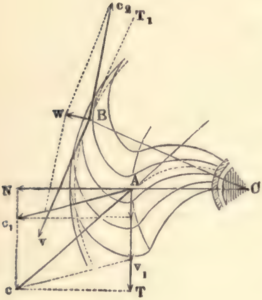
or inserting the above values for $x$ and $c_1$,

$$\frac{c_2^2}{2g} = h_1 - h_2 - (1 + \zeta) \frac{c^2}{2g} + \frac{c^2}{2g} + \frac{v^2}{2g} - \frac{2 c\, v_1 \cos.\, \alpha}{2g},$$

or since $h_1 - h_2 = h$, the total fall of the wheel,

$$c_2^2 = 2gh + v^2 - 2cv_1 \cos.\alpha - \zeta c^2.$$

FIG. 492.



If we assume that the water by friction and its curved motion in the wheel loses the head

$$\frac{\zeta_1 c_2^2}{2g},$$

we have more exactly

$$(1 + \zeta_1) c_2^2 = 2gh + v^2 - 2cv_1 \cos.\alpha - \zeta c^2.$$

Since the feed $Q = Fc = F_1 c_1 = F_2 c_2$, we have also

$$c = \frac{F_2 c_2}{F} \quad \text{and} \quad v_1 = \frac{r_1 v}{r},$$

and therefore finally for the velocity $c_2$ with which the water departs from the wheel,

$$\left[1 + \zeta \left(\frac{F_2}{F}\right)^2 + \zeta_1\right] c_2^2 + 2 \frac{F_2}{F} \frac{r_1}{r} c_2 v \cos.\alpha - v^2 = 2gh.$$

§ **251.—Best Velocity.**—In order to obtain the greatest work from the water, the *absolute velocity* must be as small as possible. Now, this velocity is the diagonal of the parallelogram $Bw$, upon the velocity $c_2$ and $v$. Hence

$$w = \sqrt{c_2^2 + v^2 - 2c_2 v \cos.\delta} = \sqrt{(c_2 - v)^2 + 4c_2 v \left(\sin.\frac{\delta}{2}\right)^2}.$$

We must, therefore, have $v = c_2$ and $\delta$ as small as possible. In order that the water may escape readily, however, we cannot have $\delta = 0$, but must make it small, say from 10° to 20°. If we make $c_2 = v$, we have accordingly the small velocity,

$$w = \sqrt{4 c_2 v \left(sin. \frac{\delta}{2}\right)^2} = 2 v \, sin. \frac{\delta}{2},$$

and the corresponding loss of work,

$$\frac{w^2}{2g} Q \gamma = \frac{4 v^2 \, sin.^2 \frac{\delta}{2}}{2g} Q \gamma.$$

On account of resistance, the relative exit velocity $c_2$ should be indeed somewhat less than $v$ in order to obtain the greatest effect, but since for turbines with guide blades the assumption $v = c_2$ gives, as we shall see hereafter, very nearly the greatest efficiency, and leads to very simple results, we shall in the following accept it. We have, then, from the preceding article,

$$\left[1 + \zeta \left(\frac{F_2}{F}\right)^2 + \zeta_1\right] v^2 + 2 \frac{F_2}{F} \frac{r_1}{r} v^2 \cos. \alpha - v^2 = 2 g h,$$

or

$$\left[2 \frac{F_2}{F} \frac{r_1}{r} \cos. \alpha + \zeta \left(\frac{F_2}{F}\right)^2 + \zeta_1\right] v^2 = 2 g h,$$

and hence for the velocity sought which corresponds nearly to the maximum delivery,

$$v = \sqrt{\frac{2 g h}{2 \frac{F_2}{F} \frac{r_1}{r} \cos. \alpha + \zeta \left(\frac{F_2}{F}\right)^2 + \zeta_1}}.$$

Instead of the ratio $\frac{F_2}{F}$, we may introduce the angle $\beta$ which the direction of $c_1$ makes with the inner circumference. Thus we must have $\overline{AN} = c \, sin. \alpha = c_1 \, sin. \beta$ and

$$A T = c \cos. \alpha = c_1 \cos. \beta + v_1,$$

or

$$\frac{c_1}{c} = \frac{sin. \alpha}{sin. \beta},$$

and

$$c \cos. \alpha - c_1 \cos. \beta = v_1,$$

and

$$\frac{c}{v_1} = \frac{\sin. \beta}{\sin. (\beta - \alpha)}.$$

Moreover

$$F c = F_2 c_2 = F_2 v = \frac{r}{r_1} F_2 v_1,$$

hence

$$\frac{F_2}{F} = \frac{r_1}{r} \frac{c}{v_1} = \frac{r_1}{r} \frac{\sin. \beta}{\sin. (\beta - \alpha)},$$

and the velocity above at the *outer* circumference is

$$v = \sqrt{\frac{2 g h}{2 \left(\frac{r_1}{r}\right)^2 \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{r_1 \sin. \beta}{r \sin. (\beta - \alpha)}\right)^2 + \zeta_1}},$$

while at the *inner* circumference we have

$$v_1 = \frac{r_1}{r} v = \sqrt{\frac{2 g h}{2 \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 + \zeta_1 \left(\frac{r}{r_1}\right)^2}}.$$

Without reference to resistances, we should have

$$v_1 = \sqrt{\frac{g h \sin. (\beta - \alpha)}{\sin. \beta \cos. \alpha}} = \sqrt{g h (1 - \tan. \alpha \cot. \beta)}.$$

§ **252.—Water Pressure.**—By the aid of the formula for $v$, we can now also determine the pressure at the entrance to the wheel. Thus

$$x = h_1 - (1 + \zeta) \frac{c^2}{2 g} = h_1 - (1 + \zeta) \frac{v_1^2}{2 g} \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2$$

$$= h_1 - \frac{(1 + \zeta) h \sin.^2 \beta}{2 \sin. \beta \cos. \alpha \sin.(\beta - \alpha) + \zeta \sin.^2 \beta + \zeta_1 \left(\frac{r}{r_1}\right)^2 [\sin.(\beta - \alpha)]^2}$$

$$= h_1 - \frac{(1+\zeta)h}{1 + \cos. 2\alpha - \cot. \beta \sin. 2\alpha + \zeta + \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{\sin. (\beta - \alpha)}{\sin. \beta}\right)^2}.$$

If for simplicity we neglect the resistances, we have

$$x = h_1 - \frac{h}{1 + \cos. 2\alpha - \cot. \beta \sin. 2\alpha}.$$

If the turbine turns in free air, we have for the turbines of Fourneyron, Cadiat, and Whitelaw, just described, $h_1 = h$, and hence

$$x = \frac{\cos. 2\alpha - \cot. \beta \sin. 2\alpha}{1 + \cos. 2\alpha - \cot. \beta \sin. 2\alpha} h.$$

If, however, the turbine revolves under water, $h_1 = h + h_2$, and hence

$$x = \frac{\cos. 2\alpha - \cot. \beta \sin. 2\alpha}{1 + \cos. 2\alpha - \cot. \beta \sin. 2\alpha} h + h_2.$$

If, in the first case, the pressure is zero, or rather equal to the atmospheric pressure, we have $x = 0$. If, in the second case, it is equal to the pressure of the under-water, we have $x = h_2$. In both cases, however, we have

$$\cos. 2\alpha - \cot. \beta \sin. 2\alpha = 0, \quad \text{or} \quad \tan. \beta = \tan. 2\alpha, \quad \text{or} \quad \beta = 2\alpha.$$

*If, therefore, the entrance angle $\beta$ is twice as great as the angle $\alpha$ of approach, the pressure at the place where the water enters the wheel from the guide blades is equal to the outer air or under-water pressure.*

On the other hand, the inner pressure is greater than the outer when $\beta > 2\alpha$, and less when $\beta < 2\alpha$. This relation is somewhat altered, of course, when we consider the resistances. Thus for equilibrium of outer and inner pressures, we have

$$1 + \cos. 2\alpha - \cot. \beta \sin. 2\alpha + \zeta + \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{\sin. (\beta - \alpha)}{\sin. \beta}\right)^2 = 1 + \zeta,$$

or

$$cot.\,\beta\, sin.\,2\,\alpha = cos.\,2\,\alpha + \zeta_1 \left(\frac{r}{r_1}\right)^2 (cos.\,\alpha - cot.\,\beta\, sin.\,\alpha)^2.$$

If in the last member we put

$$cot.\,\beta = cot.\,2\,\alpha = \frac{cos.\,2\,\alpha}{sin.\,2\,\alpha},$$

we have

$$cot.\,\beta\, sin.\,2\,\alpha = cos.\,2\,\alpha + \zeta_1 \left(\frac{r}{r_1}\right)^2 \left(\frac{sin.\,\alpha}{sin.\,2\,\alpha}\right)^2$$

$$= cos.\,2\,\alpha + \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{4\,cos.^2\,\alpha},$$

and hence

$$tan.\,\beta = \frac{sin.\,2\,\alpha}{cos.\,2\,\alpha + \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{4\,cos.^2\,\alpha}},$$

therefore $\beta$ is somewhat less than $2\,\alpha$.

If we disregard again $\zeta$ and $\zeta_1$, we have, by substituting $\beta \quad 2\,\alpha$ in the value for $v_1$ of the preceding article,

$$v_1 = \sqrt{g\,h\,(1 - tan.\,\alpha\, cot.\,2\,\alpha)} = \sqrt{\frac{g\,h\,(1 + tan.^2\,\alpha)}{2}} = \frac{\sqrt{\tfrac{1}{2}\,g\,h}}{cos.\,\alpha},$$

or since

$$\frac{c}{v_1} = \frac{sin.\,\beta}{sin.\,(\beta - \alpha)} = \frac{sin.\,2\,\alpha}{sin.\,\alpha},\quad c\,sin.\,\alpha = v_1\,sin.\,2\,\alpha,$$

and

$$c = \frac{v_1\,sin.\,2\,\alpha}{sin.\,\alpha} = v_1 \times 2\,cos.\,\alpha.$$

Hence

$$2\,v_1\,cos.\,\alpha = c, \qquad \text{and} \qquad c = \sqrt{2\,g\,h},$$

as of course should be.

If the inner pressure is greater than the outer, we have

$$v_1 > \frac{\sqrt{\tfrac{1}{2}\,g\,h}}{cos.\,\alpha} \qquad \text{and} \qquad c < \sqrt{2\,g\,h},$$

and if the inner pressure is less than the outer,

$$v_1 < \frac{\sqrt{\tfrac{1}{2} g h}}{cos.\ \alpha}, \quad \text{and} \quad c > \sqrt{2 g h}.$$

**§ 253.**—The above relations of the pressure are of great importance in the construction of turbines, because the space between guide-blade ring and wheel is not water-tight, but a small space always remains, through which water may pass out, or air or water pass in. In order that neither may happen, the turbine must be so constructed that the inner pressure where the water enters is just equal to the outer air or water pressure. Hence

$$\beta = 2\,\alpha,$$

or, better,

$$tan.\ \beta = \frac{sin.\ 2\,\alpha}{cos.\ 2\,\alpha + \zeta_1 \left(\frac{r}{r_1}\right)^2 \frac{1}{4\ cos.^2\ \alpha}}.$$

In any case, the delivery will be less, whether water escapes or air enters, for, in one case, a part of the supply water is lost, and, in the second, the entering air impedes the motion of the water. It is therefore necessary, in order to obtain the greatest delivery, to make the clearance as small as possible, and to satisfy as nearly as may be the above equation.

If, however, for a smaller feed, the sluice is lowered so that the area $F$ of the orifice of the guide blades is less, we have a greater velocity $c$, and hence a less pressure $x$. If, now, this was previously equal to the outer air or water pressure, it will now be less, and hence air or water will enter from without, through the clearance. If the turbine revolves in air, this will be sucked in, and for low positions at least of the sluice, it may happen that full discharge is hindered. The wheel no longer discharges with full mouth, but the water runs along the concave side of the floats without entirely filling the space between. The reacticn-wheel then becomes simply a pressure-wheel. Other prejudicial effects due to this, we shall notice further on.

In order, now, to prevent such action when the sluice is lowered, it is preferable so to construct the turbine that, when the sluice is fully raised, there is a slight *excess* of pressure $x$, so that a small amount of water during full action passes out and is lost to the wheel.

**§ 254.—Choice of $\alpha$ and $\beta$.**—It would appear that many very different values might be given to $\alpha$ and $\beta$. The formula

$$v_1 = \sqrt{gh(1 - tan.\,\alpha\; cot.\,\beta)} = \sqrt{gh\left(1 - \frac{tan.\,\alpha}{tan.\,\beta}\right)}$$

gives an imaginary value for $v_1$, when

$$\frac{tan.\,\alpha}{tan.\,\beta} > 1,$$

or when

$$\alpha < 90^\circ \qquad \text{and} \qquad \beta < \alpha,$$

or when

$$\alpha > 90^\circ \qquad \text{and} \qquad \beta > \alpha.$$

These values for $\alpha$ and $\beta$ are therefore excluded. If $\alpha = \beta$, we have $v_1 = 0$, and we see, therefore, that the best velocity of rotation is less, the nearer $\alpha$ and $\beta$ are to each other. The formulæ

$$c = \frac{v_1\, sin.\,\beta}{sin.\,(\beta - \alpha)}$$

and

$$F_2 = \frac{r_1}{r}\,\frac{sin.\,\beta}{sin.\,(\beta - \alpha)}\,F$$

give for $\beta < \alpha$ negative, and therefore also impossible values. It is therefore always necessary that we should have

$$\beta > \alpha \qquad \text{and} \qquad \alpha < 90^\circ.$$

Between these limits we can choose very different values for $\alpha$ and $\beta$, but all do not give equally advantageous constructions. Fourneyron takes

$$\beta = 90^\circ \quad \text{and} \quad \alpha = 30^\circ \text{ to } 33^\circ.$$

Many make $\beta$ smaller, others greater than 90°. Floats constructed with a small value for $\beta$ have a greater curvature than those with an obtuse angle. But greater curvature causes greater resistance, and may even impede full discharge. For these reasons, it is advisable to make the angle $\beta$ rather obtuse than acute, even perhaps 100° to 120°. The angle $\alpha$ would then, for equilibrium of outer and inner pressure, be from 50° to 55°. In order, however, that the guide-blade canals may not be too divergent, and also that for low positions of sluice there may be no inward suction, we may make $\alpha$ from 30° to 40°, and, when the turbine revolves in air, even 25° to 30°. We cannot, however, make $\alpha$ very small, because the area of the aperture diminishes with it, and hence the feed also diminishes, or rather for given feed the wheel becomes too large. Upon the other hand, we have to consider that the losses increase as $v^2$, and hence, under similar circumstances, a wheel has a greater delivery when it revolves slowly. In consequence, we should not have too great a difference between $\alpha$ and $\beta$, and thus avoid an excess of outer over inner pressure.

If $b$ is the height of a column of water equal to the air pressure, the absolute water pressure at entrance is $b + x$. If this pressure height is zero, we have $x = -b$, and the water enters with the maximum velocity

$$c = \sqrt{2g(h_1 - x)} = \sqrt{2g(h_1 + b)}.$$

If $b + x$ is negative, or $x < -b$, we have at entrance a *vacuum*, because the water would flow more rapidly through the wheel canals than through the guide blades; air would then enter from without, and the motion of the water be seriously impeded. If, now, we make $x = -b$ in the formula

$$x = h - \frac{h}{1 + \cos. 2\,\alpha - \cot. \beta \sin. 2\,\alpha},$$

we obtain

$$1 + \cos. 2\,\alpha - \cot. \beta \sin. 2\,\alpha = \frac{h}{h + b},$$

or

$$\tan. \beta = \frac{\sin. 2\,\alpha}{1 + \cos. 2\,\alpha - \dfrac{h}{h+b}} = \frac{(h + b) \sin. 2\,\alpha}{(h + b) \cos. 2\,\alpha + b},$$

and hence the corresponding best velocity of rotation is

$$v_1 = \sqrt{g h (1 - \tan. \alpha \frac{(h + b) \cos. 2\,\alpha + b}{(h + b) \sin. 2\,\alpha}}$$

$$= \frac{h}{\cos. \alpha} \sqrt{\frac{g}{2\,(h + b)}}.$$

§ **255.—Turbine without Guide Blades.**—For turbines without guide blades, we can make $\alpha = 90°$, because here the water flows radially outwards from the reservoir. Here we have the wheels of Combes, Cadiat, and Whitelaw. If in the formula for the best velocity we make $\alpha = 90°$, we have

$$v_1 = \sqrt{\frac{2\,g\,h}{\dfrac{2 \sin. \beta \cos. 90}{\cos. \beta} + \zeta \left(\dfrac{\sin. \beta}{\cos. \beta}\right)^2 + \zeta_1 \left(\dfrac{r}{r_1}\right)^2}}$$

$$= \sqrt{\frac{2\,g\,h}{\zeta \tan.^2 \beta + \zeta_1 \left(\dfrac{r}{r_1}\right)^2}},$$

or without taking account of resistances,

$$v_1 = \sqrt{\frac{2\,g\,h}{0}} = \infty.$$

But the wheel cannot have an infinite velocity, for two reasons: First, this velocity reaches its greatest value when the work at disposal is consumed in friction, or when

$$Q h \gamma = \left(\frac{w^2}{2g} + \zeta \frac{c^2}{2g} + \zeta_1 \frac{c_2^{\,2}}{2g}\right) Q \gamma,$$

*i. e.*, when

$$h = \left[\left(2 \, sin. \frac{\delta}{2}\right)^2 + \zeta \left(\frac{r_1}{r} \, tan. \beta\right)^2 + \zeta_1\right]\frac{v^2}{2g},$$

and therefore

$$v = \sqrt{\frac{2gh}{\left(2 \, sin. \frac{\delta}{2}\right)^2 + \zeta\left(\frac{r_1}{r} \, tan. \beta\right)^2 + \zeta_1}}.$$

Secondly, *full* discharge ceases for $x = -b$, or

$$h - \frac{c^2}{2g} = -b,$$

hence

$$\frac{c^2}{2g} = b + h,$$

or

$$\frac{1}{2g}\left(\frac{r_1}{r} \frac{v \, sin. \beta}{sin. (\beta - 90^\circ)}\right)^2 = b + h,$$

or

$$v = \frac{r}{r_1} \, cot. \beta \sqrt{2g(b+h)},$$

and other relations come into play, because the water does not flow from the reservoir as fast as it would for full-mouth flow through the wheel canals. Moreover, the above formula for $v_1$,

$$v_1 = \sqrt{\frac{2gh}{\zeta \, tan.^2 \beta + \zeta_1 \left(\frac{r}{r_1}\right)^2}},$$

when the resistances are considered, by no means gives $v_1 = \infty$. Even in the best constructions, with smooth and rounded guide-blade apparatus, $\phi$ is not greater than 0·96, and hence $\zeta$ is not less than

$$\zeta = \frac{1}{\phi^2} - 1 = \frac{1}{0{\cdot}96^2} - 1 = 0{\cdot}08,$$

or 8 per cent. For turbines without this apparatus, this resistance indeed disappears, but yet a certain loss is experienced on entrance to the wheel which for the wheels of Combes and Cadiat is about 5, for the Whitelaw wheel 10 or even more per cent, since here the canals are too wide to give to all the liquid threads a certain direction $\beta$. The friction and curvature resistance in the wheel may, as we shall see hereafter, be estimated at 0·05 to 0·15, and we have therefore, taking $\zeta_1 = 0{\cdot}1$, for Combes and Cadiat wheels, or wheel without guide blades, the best velocity,

$$v_1 = \sqrt{\frac{2\,g\,h}{0{\cdot}05\ tan.^2\,\beta + 0{\cdot}1\left(\frac{r}{r^1}\right)^2}},$$

and for the Whitelaw turbine

$$v_1 = \sqrt{\frac{2\,g\,h}{0{\cdot}1\ tan.^2\,\beta + 0{\cdot}1\left(\frac{r}{r_1}\right)^2}}.$$

If we put

$$\beta = 60^\circ \quad \text{and} \quad \frac{r}{r_1} = \tfrac{4}{3},$$

we have in the first case

$$v_1 = \sqrt{\frac{2\,g\,h}{0{\cdot}148 + 0{\cdot}178}} = 1{\cdot}75\sqrt{2\,g\,h},$$

and in the second,

$$v_1 = \sqrt{\frac{2\,g\,h}{0{\cdot}296 + 0{\cdot}178}} = 1{\cdot}45\sqrt{2\,g\,h}.$$

In order, moreover, that in wheels without guide blades, the water may enter without impact, we must have the equation

$$\frac{F_2}{F} = \frac{r_1}{r} \frac{\sin.\beta}{\sin.(\beta - 90^\circ)} - \frac{r_1}{r} \tan.\beta.$$

But since $F_2$ is determined by the position of the sluice, it follows that the maximum delivery is only attained for a certain position of the sluice.

§ **256.—General Theory.**—By placing the absolute velocity $w$ equal to zero, we obtain the maximum delivery for guide-blade turbines only. For turbines without guide blades, as well as for all turbines where the guide-blade angle $\alpha$ is nearly 90°, the influence of the resistances becomes too great to allow us to put $w = 0$ or $v = c_2$.

In order to find the best velocity of rotation for all reaction turbines, we must first find a complete expression for the delivery, and then find the velocity which makes this delivery a maximum.

The work consumed by the resistances, which must be subtracted from the disposable work $Q h \gamma$, is

$$\zeta \frac{c^2}{2g} Q \gamma + \zeta_1 \frac{c_2^2}{2g} Q \gamma,$$

and the work corresponding to the absolute velocity $w$ of the water when it leaves the wheel, is

$$\frac{w^2}{2g} Q \gamma = \left(\frac{c_2^2 + v^2 - 2 c_2 v \cos.\delta}{2g}\right) Q \gamma.$$

The remaining work or wheel delivery is therefore

$$L = \left(h - \zeta \frac{c^2}{2g} - \zeta_1 \frac{c_2^2}{2g} - \frac{c_2^2 + v^2 - 2 c_2 v \cos.\delta}{2g}\right) Q \gamma$$

$$= \left(h - \frac{(1 + \zeta_1) c_2^2 + v^2 - 2 c_2 v \cos.\delta + \zeta_1 c^2}{2g}\right) Q \gamma.$$

But now we have already found (Art. 250)

$$(1 + \zeta_1) c_2^2 = 2 g h + v^2 - 2 c v_1 \cos. \alpha - \zeta c^2,$$

hence we have

$$L = \left(\frac{c v_1 \cos. \alpha + c_2 v \cos. \delta - v^2}{g}\right) Q \gamma.$$

Since further

$$c = \frac{v_1 \sin. \beta}{\sin. (\beta - \alpha)} = \frac{r_1}{r} \frac{v \sin. \beta}{\sin. (\beta - \alpha)}$$

(Art. 251), we have

$$c_2^2 = \frac{2 g h + \left[1 - 2\left(\frac{r_1}{r}\right)^2 \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} - \zeta \left(\frac{r_1}{r}\right)^2 \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2\right] v^2}{1 + \zeta_1}$$

If therefore we denote the expression

$$\left(\frac{r_1}{r}\right)^2 \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} \text{ by } \phi,$$

and the expression

$$1 - 2 \left(\frac{r_1}{r}\right)^2 \cdot \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} - \zeta \left(\frac{r_1}{r}\right)^2 \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 \text{ by } \psi,$$

we have

$$c v_1 \cos. \alpha = \phi v^2,$$

and

$$c_2 v \cos. \delta = v \cos. \delta \sqrt{\frac{2 g h + \psi v^2}{1 + \zeta_1}},$$

or

$$c_2 v \cos. \delta = \frac{\cos. \delta}{\sqrt{1 + \zeta_1}} v \sqrt{2 g h + \psi v^2}.$$

Therefore

$$L = \left(\frac{cos.\,\delta}{\sqrt{1+\zeta_1}}\sqrt{2\,g\,h + \psi\,v^2} - (1-\phi)\,v\right)\frac{v\,Q\,\gamma}{g}$$

$$= \frac{cos.\,\delta\,Q\,\gamma}{g\sqrt{1+\zeta_1}}\left(\sqrt{2\,g\,h + \psi\,v^2} - \frac{v\,(1-\phi)\sqrt{1+\zeta_1}\,v}{cos.\,\delta}\right)v,$$

or

$$L = \frac{cos.\,\delta\,Q\,\gamma}{g\sqrt{1+\zeta_1}}\left(\sqrt{2\,g\,h + \psi\,v^2} - \chi\,v\right)v,$$

where

$$\chi = \frac{(1-\phi)\sqrt{1+\zeta_1}}{cos.\,\delta}.$$

This expression is a maximum when

$$\sqrt{2\,g\,h + \psi\,v^2}\,.\,v - \chi\,v^2$$

is a maximum. Differentiating and putting the first differential coefficient equal to zero, we have

$$\chi\,v = \frac{g\,h + \psi\,v^2}{\sqrt{2\,g\,h + \psi\,v^2}},$$

or

$$v^4 + \frac{2\,g\,h}{\psi}\,v^2 = \frac{g^2\,h^2}{\psi\,(\chi^2 - \psi)}.$$

The solution of this equation gives for the *velocity of rotation*

$$v = \sqrt{\left(\frac{\chi - \sqrt{\chi^4 - \psi}}{\psi\,\sqrt{\chi^2 - \psi}}\right)g\,h},$$

where

$$\psi = 1 - 2\left(\frac{r_1}{r}\right)^2\frac{sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta-\alpha)} - \zeta\left(\frac{r_1}{r}\right)^2\left(\frac{sin.\,\beta}{sin.\,(\beta-\alpha)}\right)^2,$$

and

$$\chi = \left[1 - \left(\frac{r_1}{r}\right)^2\frac{sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta-\alpha)}\right]\frac{\sqrt{1+\zeta_1}}{cos.\,\delta}.$$

If we take $\zeta$ and $\zeta_1$ as also $\delta = 0$, *i. e.*, if we neglect the resistances and other losses, we have

$$\psi = 1 - 2\left(\frac{r_1}{r}\right)^2 \frac{\sin.\beta \cos.\alpha}{\sin.(\beta - \alpha)}$$

and

$$\chi = 1 - \left(\frac{r_1}{r}\right)^2 \frac{\sin.\beta \cos.\alpha}{\sin.(\beta - \alpha)}.$$

Therefore,

$$\chi^2 - \psi = \left(\frac{r_1}{r}\right)^4 \left(\frac{\sin.\beta \cos.\alpha}{\sin.(\beta - \alpha)}\right)^2,$$

$$\sqrt{\chi^2 - \psi} = \left(\frac{r_1}{r}\right)^2 \frac{\sin.\beta \cos.\alpha}{\sin.(\beta - \alpha)}$$

$$\chi - \sqrt{\chi^2 - \psi} = 1 - 2\left(\frac{r_1}{r}\right)^2 \frac{\sin.\beta \cos.\alpha}{\sin.(\beta - \alpha)} = \psi,$$

and hence

$$v = \sqrt{\frac{1}{\sqrt{\chi^2 - \psi}}\, g\, h} = \frac{r}{r_1}\sqrt{\frac{g\, h \sin.(\beta - \alpha)}{\sin.\beta \cos.\alpha}},$$

just as has already been found in Art. 251.

If in the expression above for the delivery, viz.,

$$L = \frac{\cos.\delta\, Q\, \gamma}{g\sqrt{1 + \zeta_1}} \left(\sqrt{2\, g\, h + \psi\, v^2} - \chi\, v\right) v,$$

we insert the first value for $v$ above, viz.,

$$v = \sqrt{\left(\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi\, \sqrt{\chi^2 - \psi}}\right) g\, h},$$

we have for the maximum delivery,

$$L = \left(\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi}\right) \frac{\cos.\delta}{\sqrt{1 + \zeta_1}}\, Q\, h\, \gamma.$$

Since, from the above, when we neglect the resistances, we have

$$\chi - \sqrt{\chi^2 - \psi} = \psi \quad \text{and} \quad \sqrt{1 + \zeta_1} = 1 \quad \text{and} \quad cos.\ \delta = 1,$$

we have, as should be the case,

$$L = Q\, h\, \gamma,$$

or the maximum delivery equal to the whole disposable work.

If, now, by the aid of the formulæ

$$v = \sqrt{\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi \sqrt{\chi^2 - \psi}}\, g\, h} \quad \text{and} \quad v_1 = \frac{r_1}{r} v,$$

we have determined the velocities of rotation $v$ and $v_1$, we can then determine the velocities

$$c = \frac{v_1\, sin.\ \beta}{sin.\ (\beta - \alpha)},\ c_1 = \frac{c\ sin.\ \alpha}{sin.\ \beta}$$

and

$$c_2 = \sqrt{\frac{2\, g\, h + \psi\, v^2}{1 + \zeta_1}} = \sqrt{\frac{\chi + \sqrt{\chi^2 - \psi}}{(1 + \zeta_1)\, \sqrt{\chi^2 - \psi}}\, g\, h},$$

and, finally, can find the cross-sections

$$F = \frac{Q}{c},\ F_1 = \frac{Q}{c_1} \text{ and } F_2 = \frac{Q}{c_2}.$$

If we have to do with a turbine without guide blades, we may use the same formulæ, only we have here

$$cos.\ \alpha = cos.\ 90° = 0,$$

and hence

$$\psi = 1 - \zeta \left(\frac{r_1}{r}\right)^2 tan.^2\ \beta$$

and

$$\chi = \frac{\sqrt{1 + \zeta_1}}{cos.\ \delta}.$$

§ **257.—Influence of Position of Sluice.**—Turbines stand in one respect essentially below the over and middle shot water-wheel. If for these latter we have a smaller feed or wish a smaller delivery, and therefore lower the sluice, then, as we know, by reason of the smaller coefficient of fill $\varepsilon$, the efficiency is rather greater than smaller. For a turbine, however, the opposite is the case. Here the efficiency is less for lower position of sluice, because the water then enters the wheel with impact. This is the more unfavorable as the greatest economy is desirable just at the very time when the feed is least, rather than when we have an excess of water at disposal. The loss of work by low position of sluice may also be very considerable.

Thus, if we decompose the velocities $c$ and $c_1$ into their radial and tangential components,

$$c\, sin.\, \alpha, \quad c\, cos.\, \alpha, \quad \text{and} \quad c_1\, sin.\, \beta, \quad c_1\, cos.\, \beta,$$

and subtract each two, we have the relative velocities

$$c\, sin.\, \alpha - c_1\, sin.\, \beta \quad \text{and} \quad c\, cos.\, \alpha - c_1\, cos.\, \beta.$$

Since, however, the water has also the velocity of the wheel, this last relative velocity is in reality

$$= c\, cos.\, \alpha - c_1\, cos.\, \beta - v_1.$$

The height due to these velocities is

$$y = \frac{1}{2g}\left[(c\, sin.\, \alpha - c_1\, sin.\, \beta)^2 + (c\, cos.\, \alpha - c_1\, cos.\, \beta - v_1)^2\right],$$

which head is lost if these velocities are suddenly destroyed (Vol. I., Art. 436). The corresponding loss of work is then

$$Y = yQ\gamma = \left[(c\, sin.\, \alpha - c_1\, sin.\, \beta)^2 + (c\, cos.\, \alpha - c_1\, cos.\beta - v_1)^2\right] \frac{Q\,\gamma}{2g}.$$

If in this formula we make

$$c_2 = v \qquad \text{and} \qquad v_1 = \frac{r_1}{r}\, v,$$

also

$$c = \frac{F_2}{F} v \qquad \text{and} \qquad c_1 = \frac{F_2}{F_1} v,$$

we have

$$Y = \left[ \left( \frac{F_2 \sin. \alpha}{F} - \frac{F_2 \sin. \beta}{F_1} \right)^2 + \left( \frac{F_2 \cos. \alpha}{F} - \frac{F_2 \cos. \beta}{F_1} - \frac{r_1}{r} \right)^2 \right] \frac{v^2}{2g} Q \gamma.$$

From this expression, we can determine what work is lost to the turbine when the equations

$$F_1 \sin. \alpha = F \sin. \beta \qquad \text{and} \qquad F_1 \cos. \alpha = F \cos. \beta + \frac{F F_1}{F_2} \frac{r_1}{r}$$

no longer hold good.

But if these equations are satisfied for fully opened sluice, they will no longer hold good for any lower position of sluice when $F$ has a less value $F_x$. The loss of work in such case if

$$c_2 = v = \sqrt{\frac{g h \sin. (\beta - \alpha)}{\sin. \beta \cos. \alpha}},$$

is

$$Y = \left[ \left( \frac{F_2 \sin. \alpha}{F_x} - \frac{F_2 \sin. \beta}{F_1} \right)^2 + \left( \frac{F_2 \cos. \alpha}{F_x} - \frac{F_2 \cos. \beta}{F_1} - \frac{r_1}{r} \right)^2 \right] \frac{v^2}{2g} Q \gamma.$$

If in this we put

$$F \sin. \beta = F_1 \sin. \alpha \qquad \text{and} \qquad F \cos. \beta + \frac{F F_1}{F_2} \frac{r_1}{r} = F_1 \cos. \alpha,$$

we have

$$Y = \left[ \left( \frac{1}{F_x} - \frac{1}{F} \right)^2 (F_2 \sin. \alpha)^2 + \left( \frac{1}{F_x} - \frac{1}{F} \right)^2 (F_2 \cos. \alpha)^2 \right] \frac{v^2}{2g} Q \gamma$$

$$= \left( \frac{F_2}{F_x} - \frac{F_2}{F} \right)^2 \frac{v^2}{2g} Q \gamma.$$

If, for example, we put

$$\frac{v_1^2}{2g} = \tfrac{1}{2}h,$$

which for a Fourneyron turbine is permissible, we have

$$Y = \left(\frac{F_2}{F_z} - \frac{F_2}{F}\right)^2 \left(\frac{r}{r_1}\right)^2 \tfrac{1}{2} Q h \gamma,$$

or for half-open sluice, since $F_z = \frac{1}{2} F$,

$$Y = \tfrac{1}{2} \left(\frac{F_2\, r}{F\, r_1}\right)^2 Q h \gamma.$$

We see at once that we may reduce this loss of work by making the ratios $\frac{F_2}{F}$ and $\frac{r}{r_1}$ small, or generally making the exit orifices of the wheel and the outer radius small, and the orifices of the guide blades and radius of the reservoir, large. Since

$$\frac{F_2}{F} = \frac{r_1 \sin. \beta}{r \sin. (\beta - \alpha)},$$

we have

$$Y = \tfrac{1}{2} \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 Q h \gamma,$$

and, therefore, for

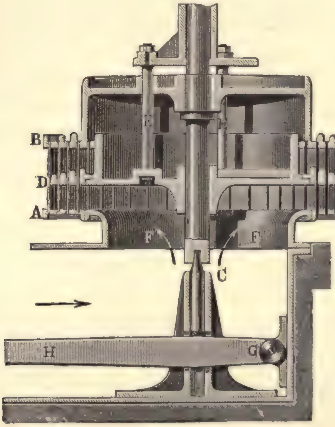
$$\beta = 90^\circ \quad \text{and} \quad \alpha = 20^\circ,\ Y = 0{\cdot}57\, Q h \gamma.$$

In this case, therefore, 57 per cent of the disposable work is lost.

In general, for low positions of sluice, when $F_z < \frac{1}{2} F$, full discharge ceases, the water no longer entirely fills the canals of the wheel, nor do these canals discharge with "full mouth." The turbine then becomes a *pressure turbine*.

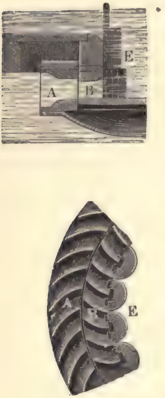
§ **258.—Sluice Apparatus.**—In order to avoid or at least reduce the loss of work consequent upon a lowering of the sluice, we have several arrangements. Fourneyron, for this purpose, divides the wheel by circular horizontal partitions into *stories* (Fig. 486, Art. 248), so that one or more of these stories may be shut off while the water flows through those remaining. These wheels naturally do not completely answer the purpose, intermediate positions of sluice are still prejudicial. Combes (Fig. 493) has between the upper and lower crowns *A A* and *B B* a plate *D D* through which both guide blades and floats pass; the portion or rim through which the floats pass resting upon the guide-blade portion by an elbow so that it can revolve with the floats. This plate is raised by rods *E*, *E* . . . sliding upon the floats and guide blades, and thus forms the sluice. Thus, for any position of sluice, we have a wheel of corresponding depth in which the water produces its proper action. This wheel fulfils its purpose completely, but is rather difficult and expensive of execution.

FIG. 493.



In the construction of Laurent and Deckherr (see Armengaud, Publ. Ind., Vol. 6, also "Der Ingenieur," Bd. II.), both the upper crown and the wheel plate are movable, each being pierced for the floats and guide blades.

FIG. 494.



Callon and Gentilhomme have an apparatus, Fig. 494, by means of which the guide-blade apparatus may be partially shut off and two or more canals partially closed at a time. The discharge is thus regulated and the water still enters without shock. This wheel also has a certain degree of incompleteness, as the water does not flow in unbroken streams through the canals. By the alternate emptying and filling of the wheel canals, the velocities $c$, $c_1$ and $c_2$ are subject to incessant variations, when $x$ is not $= 0$ or $\beta$ not $= 2\,\alpha$. Thus, for example, for an unfilled wheel canal the velocity is

$$c = \sqrt{2\,g\,h},$$

while for a full one it is

$$c = \sqrt{2\,g\,(h - x)}.$$

Thus the velocity $c$ oscillates for each full and only partially full canal between the limits,

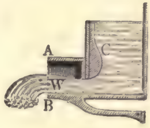
$$c = \sqrt{2\,g\,(h - x)} \quad \text{and} \quad c = \sqrt{2\,g\,h}.$$

If, now, the maximum delivery can only be obtained for a certain value of $v$ and $c_2 = \frac{F\,c}{F_2}$, we see that for a varying value of $c_2 = \frac{F\,c}{F_2}$ it cannot be reached.

In the turbine of Gentilhomme, the same result is obtained by circular sectors moved by gearing so as to close a portion of the guide-blade apparatus. Such a construction is still more imperfect than that of Callon.

**§ 259.—Pressure Turbines.**—We can now make a comparison between the reaction turbines thus far considered and *impact* and *pressure* turbines, which the first always become when the sluice $C$ (see Fig. 495) is lowered more than half of the wheel depth. Since the water $W$ only fills a part of the wheel canal, the remaining portion is for revolution in the air, filled with air, and the pressure at entrance is equal to the atmospheric pressure. Hence the velocity $c = \sqrt{2gh}$ and is independent of the velocity of the wheel. We have then for the exit velocity,

FIG. 495.



$$c_2^2 = 2gh + v^2 - 2cv_1 \cos. \alpha,$$

and for the maximum delivery $c_2 = v$. For these turbines, then, we have $2cv_1 \cos. \alpha = 2gh$, or substituting

$$c = \sqrt{2gh},$$

$$v_1 = \frac{\sqrt{2gh}}{2 \cos. \alpha}.$$

For the reaction turbine, we have already found

$$v_1 = \sqrt{gh(1 - \tan. \alpha \cot. \beta)}.$$

We see, therefore, that the conditions for maximum delivery for both coincide when $\frac{1}{2\cos.^2 \alpha} = 1 - \tan. \alpha \cot. \beta$ or $\tan. \beta = \tan. 2\alpha$ or $\beta = 2\alpha$; a relation which we have already found for $x = 0$.

There is, then, this essential difference between the two classes of turbines, that the velocity for maximum delivery of the first class is independent of $\beta$, while for the second it depends upon $\beta$, and only for $\beta = 2\alpha$ is this velocity the same for both. While, therefore, for reaction turbines, the velocity $v_1$ can by variation of $\beta$ vary between considerable limits, for pressure turbines no such variation is permissible.

In regard to the delivery of both wheels, we may remark the following. If in a reaction turbine we lower the sluice gradually, the efficiency diminishes. At the moment when the water no longer fills the wheel canals, the reaction ceases and the wheel becomes a pressure turbine. At this moment, the efficiency suddenly increases, because the loss of work due to the sudden change of velocity ceases. For still lower positions of sluice, the efficiency gradually diminishes. This would seem to indicate an advantage of the pressure over the reaction turbine, but this advantage must not be overrated, and only holds when the wheel is fed with very variable flow and revolves above water. Since the water has in the wheel more space than its velocity requires, it takes irregular side motions, and not only does not depart with the above value of $c_2$, but also loses a part of its inherent work, which is consumed by special resistances due to the irregular motion and agitation of the water. Many observations give the most certain proof of this, and it may be observed upon any turbine, when we let it revolve with the best velocity, first as reaction and then as pressure turbine. Always the turbine with full discharge and sluice fully opened gives a greater efficiency than for lower position of sluice and imperfect discharge.

Turbines which run under water always discharge with full canal, and are therefore always reaction wheels. From them we may also naturally expect a greater efficiency for fully opened sluice than from a pressure turbine running in air. On the other hand, we may consider it certain that for low positions of sluice, where the sluice opening is $\frac{2}{3}$ or less of the wheel depth, the efficiency of the reaction is less than for the pressure wheel. We see from this the great advantage of wheels divided into horizontal partitions or stories, or of wheels such as that of Combes described in the preceding article.

REMARK.—The older Fourneyron turbines were merely pressure wheels. Since, however, the repeated proofs of the greater efficiency of the reaction wheel, they are now almost exclusively constructed as reaction wheels.

§ **260.—Delivery of the Reaction Turbine.**—We can now determine the delivery of a *reaction turbine* of *outward flow.* The disposable work for the feed $Q$ and fall $h$ is

$$L = Q\,h\,\gamma.$$

From this we have, however, to subtract the losses from friction, etc., which the water experiences in passing through the wheel and guide-blade canals. Since the water departs from the guide blades with the velocity $c$, we can put the loss of head here

$$h_1 = \zeta\,\frac{c^2}{2g},$$

and since it departs from the wheel with the velocity $c_2$, we have the loss of head

$$h_2 = \zeta_1\,\frac{c_2^{\,2}}{2g}.$$

According to the author's experiments, for well-constructed canals the coefficients of resistance $\zeta = \zeta_1 = 0 \cdot 05$ to $0 \cdot 10$. (See "Versuche über den Widerstand, welchen das Wasser beim Durchgange durch die Turbinencanäle erleidet," in "Poly. Centralblatt," 1850, Lieferung III.) We have also the loss of head $\frac{w^2}{2g}$ due to the velocity of the departing water, so that the *effective* delivery of the turbine is

$$L_1 = [\,h - (h_1 + h_2 + h_3)\,]\,Q\,\gamma = \left(h - \frac{\zeta\,c^2 + \zeta_1\,c_2^{\,2} + w^2}{2g}\right) Q\,\gamma.$$

For the best effect, we have $c_2 = v$, also $w = 2\,v\,sin.\,\frac{\delta}{2}$, and since $c\,r_1\,sin.\,\alpha = c_2\,r\,sin.\,\delta$,

$$c = \frac{r \sin. \delta}{r_1 \sin. \alpha} c_2 = \frac{r \sin. \delta}{r_1 \sin. \alpha} v,$$

hence when $\zeta = \zeta_1$,

$$L_1 = \left[h - \left(\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 4 \left(\sin. \frac{\delta}{2}\right)^2\right) \frac{v^2}{2 g}\right] Q \gamma,$$

or

$$L_1 = \left[1 - \left(\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 4 \left(\sin. \frac{\delta}{2}\right)^2\right) \frac{v^2}{2 g h}\right] Q h \gamma.$$

The *efficiency* is, therefore,

$$\eta = \frac{L_1}{L} = \frac{L_1}{Q h \gamma} = 1 - \left(\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 4 \left(\sin. \frac{\delta}{2}\right)^2\right) \frac{v^2}{2 g h}.$$

But according to Art. 89, we have

$$\frac{v^2}{2 g h} = \frac{1}{\zeta \left[1 + \left(\frac{r_1 \sin. \beta}{r \sin. (\beta - \alpha)}\right)^2\right] + 2 \left(\frac{r_1}{r}\right)^2 \frac{\sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)}},$$

or since

$$c = \frac{v_1 \sin. \beta}{\sin. (\beta - \alpha)} = \frac{r_1 v \sin. \beta}{r \sin. (\beta - \alpha)} = \frac{r \sin. \delta}{r_1 \sin. \alpha} v,$$

or

$$\sin. \delta = \left(\frac{r_1}{r}\right)^2 \frac{\sin. \alpha \sin. \beta}{\sin. (\beta - \alpha)},$$

$$\frac{v^2}{2 g h} = \frac{1}{\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 2 \cot. \alpha \sin. \delta}.$$

By insertion of this value of $\frac{v^2}{2 g h}$, we have for the efficiency of the turbine,

$$\eta = 1 - \frac{\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 4 \left(\sin. \frac{\delta}{2}\right)^2}{\zeta \left[1 + \left(\frac{r \sin. \delta}{r_1 \sin. \alpha}\right)^2\right] + 2 \cot. \alpha \sin. \delta}.$$

From the delivery thus found, we have to subtract the work due to axle friction. If $G$ is the weight of the wheel, $a$ the radius of the axle and $\phi$ the coefficient of friction, we have for this loss of work,

$$L_2 = \frac{2}{3}\,\phi\,G\,\frac{a}{r}\,v \text{ (Weisbach, Vol. I., Art. 188).}$$

The above formulæ hold good not only for outward but also for inward flow turbines, only we have to reverse $v$ and $v_1$, and $r$ and $r_1$—*i. e.*, $r$ is the inner and $r_1$ the outer radius, and $v$ the inner and $v_1$ the outer velocity of wheel.

Moreover, for turbines running under water, $h$ is taken from water surface to water surface; for turbines running in air from head-water surface to centre of exit openings of wheel. In the last case, a part of the total fall, from centre of openings to under-water, is lost; on the other hand, for the turbine under water, we have a loss due to friction of the water on the wheel.

REMARK.—For high-pressure turbines, we have also a loss of work due to friction of water in the conduit pipe.

§ **261.**—Since, on account of the resistance to motion of the water in the wheel and guide-blade canals, the best speed is not exactly $v = c_2$, much more is this the case when the water enters with impact. If we let the wheel revolve *not* with the best velocity, but assume that the sluice is fully open, we have

$$F\,c = F_1\,c_1 \qquad \text{or} \qquad c \sin.\alpha = c_1 \sin.\beta,$$

and therefore for the velocity of exit $c_2$ we have, instead of

$$\left[1 + \zeta\left(\frac{F_2}{F}\right)^2 + \zeta_1\right] c_2^2 + \frac{2\,F_2}{F}\,\frac{r_1}{r}\,c_2\,v\cos.\alpha - v^2 = 2\,g\,h,$$

as found from Art. 250, the expression

$$\left[1 + \zeta\left(\frac{F_2}{F}\right)^2 + \zeta_1\right] c_2^2 + \frac{2\,F_2}{F}\,\frac{r_1}{r}\,c_2\,v\cos.\alpha - v^2 = 2\,g\,(h - y),$$

or, from Art. 257,

$$\left[1+\zeta\left(\frac{F_2}{F}\right)^2+\zeta_1\right]c_2^2+\left[\left(\frac{F_2 \cos.\alpha}{F}-\frac{F_2 \cos.\beta}{F_1}\right)c_2-\frac{r_1}{r}v\right]^2$$

$$+\frac{2 F_2}{F}\frac{r_1}{r}c_2 v \cos.\alpha - v^2 = 2 g h.$$

By the aid of this expression, we can find $c_2$ in terms of $v$, and then substituting in the expression for the delivery,

$$L_1=\left(h-y-\frac{\zeta c^2+\zeta_1 c_2^2+w^2}{2g}\right)Q\gamma$$

$$=\left[h-\frac{1}{2g}\left(\left[\zeta\left(\frac{F_2}{F}\right)^2+\zeta_1\right]c_2^2\right.\right.$$

$$+\left[\left(\frac{F_2 \cos.\alpha}{F}-\frac{F_2 \cos.\beta}{F_1}\right)c_2-\frac{r_1}{r}v\right]^2$$

$$\left.\left.+(c_2^2-2c_2 v \cos.\delta+v^2)\right)\right]Q\gamma,$$

we have the delivery for any velocity $v$ of rotation.

If the turbine revolves without performing work, $L_1=0$, and

$$\left[\zeta\left(\frac{F_2}{F}\right)^2+\zeta_1\right]c_2^2+\left[\left(\frac{F_2 \cos.\alpha}{F}-\frac{F_2 \cos.\beta}{F_1}\right)c_2-\frac{r_1}{r}v\right]^2$$

$$+c_2^2-2c_2 v \cos.\delta+v^2=2gh.$$

If we subtract this from the above equation for $c_2$, we have for the maximum velocity of rotation $v_0$, the simple expression,

$$2v_0^2=2\frac{F_2}{F}\frac{r_1}{r}c_2 v_0 \cos.\alpha+2c_2 v_0 \cos.\delta,$$

or

$$v_0=\left(\frac{F_2}{F}\frac{r_1}{r}\cos.\alpha+\cos.\delta\right)c_2,$$

and also

$$c_2 = \frac{v_0}{\frac{F_2}{F}\frac{r_1}{r}\, cos.\ \alpha + cos.\ \delta}.$$

If we substitute this value of $c_2$ in the equation,

$$\left[1 + \zeta\left(\frac{F_2}{F}\right)^2 + \zeta_1\right] c_2^2 + \left[\left(\frac{F_2\, cos.\ \alpha}{F} - \frac{F_2\, cos.\ \beta}{F_1}\right)c_2 - \frac{r_1}{r}\, v_0\right]^2$$

$$+ \frac{2\, F_2}{F}\frac{r_1}{r}\, c_2\, v_0\, cos.\ \alpha - v_0^2 = 2\, g\, h,$$

we have a formula for the determination of the velocity $v_0$ with which the wheel revolves when unloaded, and may then compare this with the velocity $c_2 = v$, for which the delivery is a maximum when the water enters without impact.

REMARK.—For ordinary guide-blade turbines, $\delta$ is nearly $= \alpha$ and small, hence $cos.\ \alpha = cos.\ \delta$ is nearly $= 1$, and $\frac{F_2}{F} = \frac{r}{r_1}$, and hence

$$c_2 = \frac{v_0}{\frac{F_2}{F}\frac{r_1}{r}\, cos.\ \alpha + cos.\ \delta}$$

is nearly $= \frac{v_0}{2}$. If we take now $\frac{r_1}{r} = \frac{3}{4}$ and $\zeta_1 = \zeta = 0 \cdot 1$, we have

$$\left[1 + \zeta\left(\frac{F_2}{F}\right)^2 + \zeta_1\right] c_2^2 = (1 + 0 \cdot 1 \times \tfrac{16}{9} + 0 \cdot 1)\,\frac{v_0^2}{4} = 0 \cdot 32\, v_0^2.$$

Further, taking $F_1 = F$ and $cos.\ \beta = cos.\ \alpha$,

$$\left[\left(\frac{F_2}{F}\, cos.\ \alpha - \frac{F_2}{F_1}\, cos.\ \beta\right) c_2 - \frac{r_1}{r}\, v_0\right]^2 = \left(\frac{r\, c_2}{2\, r_1} - \frac{r_1\, v_0}{r}\right)^2$$

$$= (\tfrac{1}{3} - \tfrac{3}{4})^2\, v_0^2 = 0 \cdot 17\, v_0^2,$$

and

$$2\,\frac{F_2}{F}\frac{r_1}{r}\, c_2\, v_0\, cos.\ \alpha = v_0^2,$$

so that we have now

$$(0 \cdot 32 + 0 \cdot 17 + 1 - 1)\, v_0^2 = 2\, g\, h, \qquad \text{or} \qquad 0 \cdot 49\,\frac{v_0^2}{2\, g} = h.$$

For the velocity $v = c_2$ of the wheel, for which we obtain nearly the maximum delivery, we have approximately

$$\left(1 + \zeta \frac{F_2}{F} + \zeta_1\right) c_2^2 = 1 \cdot 28\, v^2,$$

also

$$\left[\left(\frac{F_2}{F} \cos.\, \alpha - \frac{F_2}{F_1} \cos.\, \beta\right) c_2 - \frac{r_1}{r} v\right]^2 = (\tfrac{2}{3} - \tfrac{3}{4})^2\, v^2,$$

approximately $= 0 \cdot 01$, and

$$2 \frac{F_2}{F} \frac{r_1}{r} c_2\, v \cos.\, \alpha = 2\, v^2$$

hence

$$(1 \cdot 29 + 2 - 1)\, v^2 = 2\, g\, h, \qquad \text{or} \qquad 2 \cdot 29 \frac{v^2}{2\, g} = h.$$

Hence we have

$$\frac{v_0^2}{v^2} = \frac{2 \cdot 29}{0 \cdot 49},$$

or nearly $= 5$, and

$$\frac{v_0}{v} = \sqrt{5} = 2 \cdot 22.$$

By reason of axle friction, this ratio is still smaller. In fact, experiments give generally the ratio $\frac{v_0}{v} = 2$; *i. e.*, the unloaded turbine runs twice as fast as when it performs its maximum work.

§ **262.—Arrangement of Guide-Blade Turbines.**—We have now all the most essential rules for the arrangement, calculation, and construction of turbines of *outward flow*. In any case, we can consider the feed $Q$ and the fall $h$ as given; or if, instead of $Q$, the delivery $W$ were given, we could find $Q$ from $L$, and the efficiency $\eta$ (about $0 \cdot 75$) by the formula

$$Q = \frac{L}{\eta\, h\, \gamma}.$$

The remaining quantities, $r$, $r_1$, $\alpha$, $\beta$, $\delta$, $v$, $n$, $c$, etc., are some arbitrary, some empirical, some theoretically determined. First we take the angle $\alpha$ arbitrarily. For wheels without guide blades, it is of course 90°; with guide blades, which we now consider, we have

$$\alpha = 20^\circ \text{ to } 30^\circ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the first for high, the last for low falls, in order, in the one case, not to have too wide and in the other too narrow orifices, or too small and too large wheels.

The entrance angle $\beta$ is to a certain degree dependent upon $\alpha$. In order that the water pressure upon entrance may be equal to the outer pressure, we must have $\beta = 2\,\alpha$, but because this pressure diminishes when the sluice descends, in order to prevent a negative pressure we must have $\beta$ greater than $2\,\alpha$. Probably the best ratio is

$$\beta = 2\,\alpha + 20^\circ \quad \text{to} \quad 2\,\alpha + 30^\circ. \quad . \quad . \quad . \quad . \quad (2).$$

The ratio $\nu = \frac{r}{r_1}$ of the radii is to be taken between the limits 1·25 to 1·5 . . . . . . . . . . . . . . (3).

For apparent reasons, the smaller ratio is to be taken for a large value of $\beta$ or a large wheel, the smaller for a smaller value of $\beta$ or a smaller wheel.

The *exit angle* $\delta$ is given by the formula,

$$\sin.\ \delta = \left(\frac{r_1}{r}\right)^2 \frac{\sin.\ \alpha\ \sin.\ \beta}{\sin.\ (\beta - \alpha)} = \frac{\sin.\ \alpha\ \sin.\ \beta}{\nu^2\ \sin.\ (\beta - \alpha)} \quad . \quad (4).$$

In order that the departing water carry away as little work as possible, this angle should not be more than 20°, and hence $\alpha$, $\beta$, and $\nu = \frac{r}{r_1}$ must be so taken that $\delta$ shall be less than 20°.

In many wheels, as in those of Combes and Callon, an attempt is made to diminish $\delta$ by making the exterior depth of wheel greater than interior. Since, however, the full discharge of water is liable to be thus impeded, this construction requires to be applied with caution.

In order, further, to determine the radius of the wheel and of the discharge reservoir, we shall make the condition in correspondence with the best approved practice, that the velocity of water in the reservoir shall not exceed 3 ft. We can therefore put $Q = 3\,\pi\, r_1^2$, or inversely for the *inner radius* of wheel,

$$r_1 = \sqrt{\frac{Q}{3\pi}} = 0 \cdot 326 \sqrt{Q} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where $r_1$ is in feet and $Q$ in cubic feet.

We have then for the *outer radius* of wheel,

$$r = \nu\, r_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

The *inner velocity* is given by the formula

$$v_1 = \sqrt{\frac{2\,g\,h}{\frac{2\ sin.\ \beta\ cos.\ \alpha}{sin.\ (\beta - \alpha)} + \zeta\left(\frac{sin.\ \beta}{sin.\ (\beta - \alpha)}\right)^2 + \zeta_1\left(\frac{r}{r_1}\right)^2}} \quad (7).$$

We have then the velocity $c$ with which the water issues from the guide blades,

$$c = \frac{v_1\ sin.\ \beta}{sin.\ (\beta - \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

and for the cross-section $F$,

$$F = \frac{Q}{c} = \frac{Q\ sin.\ (\beta - \alpha)}{v_1\ sin.\ \beta} \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Further, the *entrance velocity*,

$$c_1 = \frac{c\ sin.\ \alpha}{sin.\ \beta} = \frac{v_1\ sin.\ \alpha}{sin.\ (\beta - \alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

and the cross-section,

$$F_1 = \frac{Q}{c_1} = \frac{Q\ sin.\ (\beta - \alpha)}{v_1\ sin.\ \alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11).$$

Finally, the outer wheel, as also the exit velocity,

$$v = c_2 = \frac{r}{r_1} v_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

and the exit orifice area,

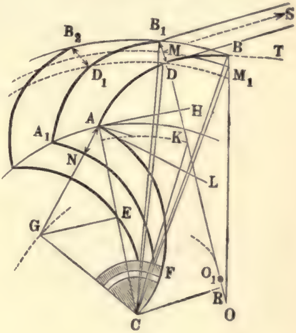
$$F_2 = \frac{Q}{c_2} = \frac{r_1}{r}\frac{Q}{v_1} = \frac{r_1}{r}\frac{F\,c}{v_1} \quad . \quad . \quad . \quad . \quad (13).$$

Moreover, the number of revolutions of the wheel per minute is

$$u = \frac{30\,v}{\pi\,r} = 9\cdot 55\,\frac{v}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

§ **263.**—It still remains to deduce rules for the calculation of the number of wheel floats and the dimensions of the wheel orifices. The exit orifices of the wheel, which together possess the area $F_2 = \frac{Q}{c_2}$, do not form the outer circumference of the wheel, but they are given by the cross-sections $B_1\,D$, $B_2\,D_1$, etc., Fig. 496. Also $r$ is not the radius $C\,B_1$ of the outer circumference, but the distance $C\,M$ of the centre of the orifice $B_1\,D$ from the axis $C$, and $v$ is not the velocity of $B$, but of $M$. If, now, $\delta$ is the angle $S\,M\,T$ which the axis of the departing stream through $B_1\,D$ makes with the tangent $M\,T$ or normal to $C\,M$, and if $n$ is the number of floats, $s$ their thickness, $d$ the width $B_1\,D$ of the orifice, $e$ the depth of wheel or height of float, and $\lambda$ the ratio $\frac{e}{d}$, then we have for the exit area

FIG. 496.



$$F_2 = n\,d\,e = n\,\lambda\,d^2 = \frac{n\,e^2}{\lambda} \qquad \text{or} \qquad n = \frac{\lambda\,F_2}{e^2}.$$

Since each float occupies the space $n\,s\,e$, we have also

$$F_2 = (2\,\pi\,r\,sin.\,\delta - n\,s)\,e = \left(2\,\pi\,r\,sin.\,\delta - \frac{\lambda\,F_2\,s}{e^2}\right)\,e,$$

hence

$$e = \frac{F_2}{2\,\pi\,r\,sin.\,\delta - \frac{\lambda\,F_2\,s}{e^2}},$$

or approximately,

$$e = \frac{F_2}{2\,\pi\,r\,sin.\,\delta}\left(1 + \frac{\lambda\,F_2\,s}{2\,\pi\,r\,e^2\,sin.\,\delta}\right)$$

$$= \frac{F_2}{2\,\pi\,r\,sin.\,\delta}\left(1 + \frac{2\,\pi\,r\,sin.\,\delta\,\lambda\,s}{F_2}\right).$$

The ratio

$$\lambda = \frac{e}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is taken from 2 to 5; the first value for long and slightly curved, and the second value for short and more sharply curved floats, in order that we may have full discharge. We have then for wheel depth,

$$e = \frac{F_2}{2\,\pi\,r\,sin.\,\delta}\left(1 + 2\,\pi\,r\,sin.\,\delta\,\frac{\lambda\,s}{F_2}\right) \quad . \quad . \quad . \quad (2),$$

for the width of orifices

$$d = \frac{e}{\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

and the number of floats

$$n = \frac{\lambda\,F_2}{e^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

As to the number $n_1$ of the guide blades, we can determine it under the following assumption:

We have above

$$\frac{F}{F_2} = \frac{2\,\pi\,r_1\,sin.\,\alpha}{2\,\pi\,r\,sin.\,\delta}.$$

But if $s_1$ is the thickness of the guide blades,

$$\frac{F}{F_2} = \frac{2\,\pi\,r_1\,sin.\,\alpha - n_1\,s_1}{2\,\pi\,r\,sin.\,\delta - n\,s}.$$

If, therefore, both equations hold good, we have

$$\frac{n_1\,s_1}{n\,s} = \frac{r_1\,sin.\,\alpha}{r\,sin.\,\delta},$$

or since $s_1 = s$ generally, the ratio of the number of guide blades to the number of wheel floats is

$$\frac{n_1}{n} = \frac{sin.\ \alpha}{v\ sin.\ \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

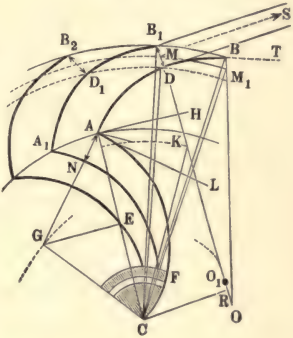
§ **264.—Float Construction.**—The floats are, as a rule, of circular curvature; for the guide blades, one arc is sufficient, but for the floats two arcs tangent to each other are necessary. We have then to find the radii of these arcs and to determine how they are to be joined.

Describe with the radius $CM = r$, Fig. 497, a circle; draw the tangent $MT$ and lay off the exit angle $SMT = \delta$, which may be determined as in the preceding articles. By the aid of the angle $\phi = \frac{360°}{n}$, we can then determine

$$\tfrac{1}{2}\, d_1 = r\ sin.\ \delta\ tan.\ \frac{\phi}{2},$$

and lay this off upon both sides of $M$ in $MB_1 = MD$ at right angles to $MS$. Further, draw the radius $CB_1$, lay off the angle $B_1CB = \phi$, and describe the circles $B_1B$ and $D_1D$ with $C$ as centre. The first of these circles gives the outer circumference, and the points $B$ and $B_1$ are the outer float ends. If, now, we draw $BO$ so that the angle $BOD = BCB_1 = \phi$, we have the centre $O$ and $BO = DO =$ the radius of the outer float end $BD$. If, now, we make $B_1O_1 = DO$, we have the centre $O_1$ for the next float end $B_1D_1$. This construction is proved as follows:

FIG. 497.



The chord joining the points $M$ and $M_1$ is

$$MM_1 = 2\ CM\ sin.\ \frac{\phi}{2} = 2\ r\ sin.\ \frac{\phi}{2}.$$

Further, the angle $M\,O\,M_1 = \phi$, and the angle

$$O\,M\,M_1 = 90^\circ - S\,M\,M_1 = 90^\circ - (S\,M\,T + T\,M\,M_1)$$

$$= 90^\circ - \left(\frac{\phi}{2} + \delta\right).$$

Finally, the angle

$$M\,M_1\,O = 180^\circ - (M\,O\,M_1 + O\,M\,M_1) = 90^\circ - \left(\frac{\phi}{2} - \delta\right).$$

Hence, since

$$\frac{O\,M_1}{M\,M_1} = \frac{\sin.\, O\,M\,M_1}{\sin.\, M\,O\,M_1} \quad \text{and} \quad \frac{O\,M}{M\,M_1} = \frac{\sin.\, O\,M_1\,M}{\sin.\, M\,O\,M_1},$$

we have

$$O\,M_1 = \frac{2\,r\,\sin.\frac{\phi}{2}\,\sin.\left[90^\circ - \left(\frac{\phi}{2} + \delta\right)\right]}{\sin.\,\phi} = \frac{r\,\cos.\,\left(\frac{\phi}{2} + \delta\right)}{\cos.\,\frac{\phi}{2}}$$

$$= r\,\cos.\,\delta - r\,\sin.\,\delta\,\tan.\,\frac{\phi}{2}$$

and

$$O\,M = \frac{2\,r\,\sin.\frac{\phi}{2}\,\sin.\left[90^\circ - \left(\frac{\phi}{2} - \delta\right)\right]}{\sin.\,\phi} = \frac{r\,\cos.\,\left(\frac{\phi}{2} - \delta\right)}{\cos.\,\frac{\phi}{2}}$$

$$= r\,\cos.\,\delta + r\,\sin.\,\delta\,\tan.\,\frac{\phi}{2}.$$

Since, now, however,

$$M\,D = M\,B_1 = M_1\,B = \frac{d_1}{2} = r\,\sin.\,\delta\,\tan.\frac{\phi}{2},$$

we have

$$O\,B = O\,M_1 + M_1\,B = r\,\cos.\,\delta$$

and

$$O\,D = O\,M - M\,D = r\,\cos.\,\delta.$$

Therefore the radius of the outer float end $BD$ is

$$r_2 = OB = OD = r \cos. \delta,$$

and it may be readily found by drawing from $C$ a parallel $CR$ to $MS$, and from $M$ a perpendicular $MR$ to $MS$. The length $MR$ is then the length $r \cos. \delta$ of the radius sought—viz., $OB = OD = O_1 B_1$.

By this construction, the float end $B_1$ is parallel to the element at $D$ of the next float, and the stream issues without contraction. If this parallelism does not obtain, there is always a disadvantage. If the tangents at $B_1$ and $D$ diverge outwards, there is danger of not obtaining full discharge; if they converge, there is partial contraction and the stream strikes against the outer surface of $BD$.

The remaining portion $DA$ of the float has also a circular curvature. The radius $KD = KA = r_3$ of this arc is found as follows:

In the triangle $CMK$, we have

$$CM = r, \quad MK = r_3 + \frac{d_1}{2}, \quad \text{and} \quad CMK = SMT = \delta,$$

hence

$$CK^2 = r^2 + \left(r_3 + \frac{d_1}{2}\right)^2 - 2\,r \left(r_3 + \frac{d_1}{2}\right) \cos. \delta.$$

In the triangle $CAK$, on the other hand, we have

$$CA = r_1, \qquad AK = r_3, \qquad \text{and} \qquad CAK = 180° - \beta,$$

hence

$$CK^2 = r_1^2 + r_3^2 + 2\,r_1\,r_3 \cos. \beta.$$

Equating both expressions, we have

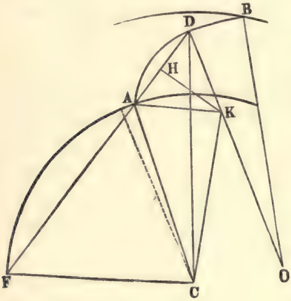
$$r^2 + r_3\,d_1 + \frac{d_1^2}{4} - 2\,r\,r_3 \cos. \delta - r\,d_1 \cos. \delta = r_1^2 + 2\,r_1\,r_3 \cos. \beta,$$

and hence we have for the radius desired,

$$r_3 = \frac{r^2 - r_1^2 - r\,d_1 \cos. \delta + \dfrac{d_1^2}{4}}{2\,(r \cos. \delta + r_1 \cos. \beta) - d_1}.$$

We can find this radius by construction in the following manner: Lay off at $C$ from $CD$, Fig. 498, the given angle $DCF = \delta + 180 - \beta$, make $CF = CA = r_1$ and draw $DF$. The intersection of $DF$ with the inner circumference at $A$ is the foot of the required arc, whose centre $K$ will be given by drawing at the middle $H$ of $AD$ a perpendicular $HK$ and producing it to intersection with $DO$. The construction may be proved as follows:

FIG. 498.



Since $CF = CA = r_1$ and $KA = KD = r_3$, also the angles $CAF$ and $CFA$ are equal, and also the angles $DAK$ and $ADK$ are equal, we have

$$CAK = 180^\circ - FAC - KAD = 180^\circ - CFA - ADK$$

$$= 180^\circ - CFA - CDF - CDK.$$

But

$$180^\circ - CFA - CDF = DCF = \delta + 180 - \beta \text{ and } CDK = \delta,$$

hence

$$CAK = \delta + 180 - \beta - CDK = 180 - \beta.$$

Since this angle is enclosed by the radii $CA$ and $KA$ of the arcs $AF$ and $DA$, the angle under which these arcs themselves meet at $A$ is $= 180 - CAK = \beta$, as should be.

Finally, as to the radius of the guide blades, we have in the preceding figure to draw $AL$, making the given angle $\alpha$ with the tangent $AH$, then draw a perpendicular to $AL$, and where it intersects a perpendicular through the middle point $E$ of $CA$, we have the centre $G$ for the guide blade $AF$, which we can now draw and continue entirely round to axle, or only as far as the pipe which surrounds the axle. The radius $GA = GC = r_4$ of the guide blade is

$$r_4 = \frac{r_1}{2\ cos.\ \alpha}.$$

The centres for the other floats and guide blades are found in the circles described with the radii $CO$, $CK$ and $CG$.

EXAMPLE.—Required the calculation and arrangement of the parts for a Fourneyron turbine, for a fall of 5 ft. and a feed $Q = 30$ cubic ft.

If we take

(1) $\alpha = 30°$,

(2) $\beta = 100°$,

and

(3) $\nu = \frac{r}{r_1} = 1 \cdot 35$,

we have

$$sin.\ \delta = \frac{sin.\ \alpha\ sin.\ \beta}{\nu^2\ sin.\ (\beta - \alpha)} = \frac{sin.\ 30°\ sin.\ 80°}{1 \cdot 35^2\ sin.\ 70°} = 0 \cdot 28752,$$

and therefore

(4) $\delta = 16°\ 42'$.

Further, we have for the inner radius,

(5) $r_1 = 0 \cdot 326 \sqrt{Q} = 0 \cdot 326 \sqrt{30} = 1 \cdot 785$ ft.,

or we may put $r_1 = 1 \cdot 80$ ft., and therefore the outer radius is

(6) $r = \nu r_1 = 1 \cdot 35 \times 1 \cdot 8 = 2 \cdot 43$ ft.,

for which we will take $2 \cdot 45$ ft., so that the breadth of crown is

$$r - r_1 = 2 \cdot 45 - 1 \cdot 80 = 0 \cdot 65 \text{ ft.}$$

Without reference to resistances, we should have for the velocity of the inner circumference,

$$v_1 = \sqrt{g h (1 - tan.\ \alpha\ cot.\ \beta)} = \sqrt{5 \times 32 \cdot 2 (1 + tan.\ 30°\ cot.\ 80°)}$$

$$= \sqrt{156 \cdot 25 \times 1 \cdot 10182} = 13 \cdot 105 \text{ ft.}$$

With reference to hydraulic resistances, however, we have, when $\zeta = \zeta_1 = 0{\cdot}075$,

$$(7)\quad v_1 = \sqrt{\left(\frac{2gh}{\dfrac{2\,sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta-\alpha)} + \zeta\left[\left(\dfrac{sin.\,\beta}{sin.\,(\beta-\alpha)}\right)^2 + \nu^2\right]}\right)}$$

$$= \sqrt{\left(\frac{64{\cdot}4 \times 5}{\dfrac{2\,sin.\,80^\circ\,cos.\,30^\circ}{sin.\,70^\circ} + 0{\cdot}075\left[\left(\dfrac{sin.\,80^\circ}{sin.\,70^\circ}\right)^2 + 1{\cdot}35^2\right]}\right)}$$

$$= \sqrt{\frac{322}{1{\cdot}8152 + 0{\cdot}075 \times 2{\cdot}9208}} = \sqrt{\frac{322}{2{\cdot}03426}} = 12{\cdot}394 \text{ ft.}$$

We have now the outer velocity

$$(8)\quad v = \nu v_1 = 1{\cdot}35 \times 12{\cdot}394 = 16{\cdot}732 \text{ ft.},$$

the velocity of water upon exit from guide blades,

$$(9)\quad c = \frac{v_1\,sin.\,\beta}{sin.\,(\beta-\alpha)} = \frac{12{\cdot}394\,sin.\,80^\circ}{sin.\,70^\circ} = 12{\cdot}989 \text{ ft.},$$

further, the relative velocity of the entering water,

$$(10)\quad c_1 = \frac{c\,sin.\,\alpha}{sin.\,\beta} = \frac{v_1\,sin.\,\alpha}{sin.\,(\beta-\alpha)} = 6{\cdot}595 \text{ ft.},$$

and the relative velocity of exit $c_2 = v = 16{\cdot}732$ ft. Finally, the absolute exit velocity is

$$(11)\quad w = 2\,v\,sin\,\frac{\delta}{2} = 2 \times 16{\cdot}732\,sin.\,8^\circ\,21' = 4{\cdot}860 \text{ ft.}$$

The number of revolutions of the wheel per minute is

$$(12)\quad u = 9{\cdot}55\,\frac{v}{r} = 9{\cdot}55\,\frac{16{\cdot}732}{2{\cdot}45} = 65{\cdot}22.$$

We have now for the area of cross-section of the orifices,

$$(13)\quad F = \frac{Q}{c} = \frac{30}{12{\cdot}989} = 2{\cdot}3096 \text{ sq. ft.},$$

and

$$(14)\quad F_2 = \frac{Q}{c_2} = \frac{Q}{v} = \frac{30}{16{\cdot}732} = 1{\cdot}7930 \text{ sq. ft.}$$

If, further, we take the ratio $\lambda = \frac{e}{d}$ of the dimensions of the exit orifices $= 4$, and the thickness of a float $s = 0 \cdot 02$ ft., we have

$$(15) \quad e = \frac{F_2}{2\pi r \sin. \delta}\left(1 + 2\pi r \sin. \delta \frac{\lambda s}{F_2}\right)$$

$$= \frac{1 \cdot 793}{2\pi \times 2 \cdot 45 \sin. 16^\circ 42'}\left(1 + \frac{2\pi \times 2 \cdot 45 \sin. 16^\circ 42' \times 4 \times 0 \cdot 02}{1 \cdot 793}\right)$$

$$= \frac{1 \cdot 793}{4 \cdot 424}\left(1 + \frac{4 \cdot 424 \times 0 \cdot 08}{1 \cdot 793}\right) = 0 \cdot 4053(1 + 0 \cdot 1974)$$

$$= 0 \cdot 485 \text{ ft.} = 5 \cdot 82 \text{ inches};$$

also

$$(16) \quad d = \frac{e}{\lambda} = \frac{0 \cdot 485}{4} = 0 \cdot 12125 \text{ ft.} = 1 \cdot 45 \text{ inches},$$

and

$$(17) \quad n = \frac{\lambda F_2}{e^2} = \frac{1 \cdot 793 \times 4}{0 \cdot 485^2} = 30,$$

for which we may take 32 for the sake of more convenient division. Finally, the number of guide blades, when we take them equally thick, is

$$(18) \quad n_1 = \frac{n s \sin. \alpha}{v s_1 \sin. \delta} = \frac{32 \sin. 30^\circ}{1 \cdot 35 \sin. 16^\circ 42'} = 40.$$

In general, however, we never make the number of guide blades greater than the number of floats. The angle of division for 32 floats is

$$(19) \quad \phi = \frac{360}{32} = 11\tfrac{1}{4}^\circ,$$

and hence the half theoretical width of orifice, without reference to thickness of float $s$, is

$$(20) \quad \frac{d_1}{2} = r \sin. \delta \tan. \frac{\phi}{2} = 2 \cdot 45 \times 0 \cdot 28752 \tan. 5^\circ 37\tfrac{1}{2}' = 0 \cdot 06938 \text{ ft.},$$

or 0·8325 inches. The entire width is therefore

$$(21) \quad d_1 = 0 \cdot 13876 \text{ ft.} = 1 \cdot 6651 \text{ inches.}$$

The radius of curvature for the outer float end is

$$(22) \quad r_2 = r \cos. \delta = 2 \cdot 45 \cos. 16^\circ 42' = 2 \cdot 347 \text{ ft.},$$

and for the inner portion

$$(23)\quad r_3 = \frac{r^2 - r_1^2 - r\,d_1\,cos.\,\delta + \frac{1}{4}\,d_1^2}{2\,(r\,cos.\,\delta + r_1\,cos.\,\beta) - d_1}$$

$$= \frac{2\cdot45^2 - 1\cdot80^2 - 2\cdot45 \times 0\cdot13876\;cos.\;16^\circ\,42' + \frac{1}{4} \times 0\cdot13876^2}{2\,(2\cdot45\;cos.\;16^\circ\,42' + 1\cdot80\,cos.\;100^\circ) - 0\cdot13876}$$

$$= \frac{2\cdot7673 - 0\cdot3256}{2 \times 2\cdot0341 - 0\cdot13876} = \frac{2\cdot4417}{3\cdot9294} = 0\cdot6214 \text{ ft.}$$

For the centre angle of this arc, we have

$$\phi_1 = 180^\circ - \beta - \delta + \sigma - \tau,$$

where $\sigma = A\,C\,K$ and $\tau = M\,C\,K$, or

$$tan.\;\sigma = \frac{r_3\;sin.\;\beta}{r_1 + r_3\;cos.\;\beta}$$

and

$$tan.\;\tau = \frac{\left(r_3 + \frac{d_1}{2}\right)\;sin.\;\delta}{r - \left(r_3 + \frac{d_1}{2}\right)\;cos.\;\delta}.$$

We have, then, in the present case,

$$tan.\;\sigma = \frac{0\cdot6214\;sin.\;80^\circ}{1\cdot80 - 0\cdot6214\;cos.\;80^\circ} \qquad \text{or} \qquad \sigma = 19^\circ\,53',$$

and

$$tan.\;\tau = \frac{0\cdot6908\;sin.\;16^\circ\,42'}{2\cdot45 - 0\cdot6908\;cos.\;16^\circ\,42'} \qquad \text{or} \qquad \tau = 6^\circ\,20',$$

hence the centre angle of the inner arc of the float is

$$(24)\quad \phi_1 = 180^\circ - 100^\circ - 16^\circ\,42' + 19^\circ\,53' - 6^\circ\,20' = 76^\circ\,51'.$$

Finally, the radius for the guide blades is

$$(25)\quad r_4 = \frac{r_1}{2\;cos.\;\alpha} = \frac{1\cdot8}{2\;cos.\;30^\circ} = 1\cdot0392 \text{ ft.}$$

The work inherent in the water is

$$L = Q\,h\,\gamma = 30 \times 5 \times 62\cdot5 = 9262\cdot5 \text{ ft. lbs.},$$

and the work of the turbine is

$$L_1 = \left(1 - \frac{\zeta(c^2 + v^2) + w^2}{2gh}\right) Q h \gamma$$

$$= \left(1 - 0\cdot016 \frac{0\cdot075\,(12\cdot989^2 + 16\cdot732^2) + 4\cdot860^2}{5}\right) 9262\cdot5$$

$$= [1 - 0\cdot0032\,(0\cdot075 \times 448 + 23\cdot62)]\, 9262\cdot5$$

$$= (1 - 0\cdot1830)\, 9262\cdot5 = 0\cdot817 \times 9262\cdot5 = 7567\cdot5 \text{ ft. lbs.}$$

If this turbine revolves in air, since the depth $e = 0\cdot485$ ft., we have at least $\frac{1}{2}$ a foot to be taken from the fall, or a loss of work of

$$30 \times 0\cdot5 \times 62\cdot5 = 926\cdot25 \text{ ft. lbs.}$$

In order to estimate the loss of water by leakage, we must know the pressure height $x$ behind the sluice. This is

$$x = h - (1 + \zeta)\frac{c^2}{2g} = 5 - 1\cdot075 \times 0\cdot016 \times 12\cdot989^2 = 5 - 2\cdot9019 = 2\cdot0981 \text{ ft.},$$

and hence the corresponding velocity is

$$w_1 = \sqrt{2gx} = 7\cdot906\,\sqrt{2\cdot0981} = 11\cdot45 \text{ ft.}$$

If, now, the circular crack between wheel and sluice is 0·0099 ft. wide, its cross-section is

$$2\pi r \times 0\cdot0099 = 0\cdot0393 \text{ sq. ft.},$$

and if the coefficient of efflux $\mu = 0\cdot7$, the water lost by leakage is

$$Q_1 = 0\cdot7 \times 0\cdot0393 \times w_1 = 0\cdot315 \text{ cubic ft.},$$

and the corresponding loss of work is

$$Q_1 h \gamma = 0\cdot315 \times 5 \times 62\cdot5 = 97\cdot25 \text{ ft. lbs.}$$

Finally, work is lost by axle friction. If the wheel, etc., weighs 3000 lbs., the radius of axle $= \frac{1}{8}$ ft., and $\phi = 0\cdot075$, then

$$\phi G \frac{a}{r} v = 0\cdot075 \times 3000 \frac{16\cdot732}{8 \times 2\cdot45} = 192 \text{ ft. lbs.}$$

Taking into account, then, these last three losses, we have

$$926\cdot25 + 97\cdot25 + 192 = 1215\cdot5 \text{ ft. lbs.}$$

of lost work, and hence the effective work is

$$L_1 = 7567 \cdot 5 - 1215 \cdot 5 = 6352 \text{ ft. lbs.},$$

and the efficiency is hence

$$\eta = \frac{6352}{9262 \cdot 5} = 0 \cdot 686.$$

§ **265.—Turbine without Guide Blades.**—The dimensions of the turbine without guide blades are calculated only in part, like the guide-blade turbine. The water now takes the shortest path, viz., radial from the reservoir, and hence $\alpha = 90°$. The angle $\beta$ is greater, from 140° to 160°, in order to give a small negative pressure ($x$) at entrance and prevent sucking in of air. The radius ratio

$$v = \frac{r}{r_1}$$

we take only 1·15 to 1·30, because otherwise, by reason of the large value of $\beta$, the wheel canals would be too long. In order to diminish as much as possible the loss of work on entrance of the water to the wheel, we allow it to enter with a velocity of only 2 ft., and have therefore the inner radius

$$(1) \quad r_1 = \sqrt{\frac{Q}{2\pi}} = 0 \cdot 4 \sqrt{Q} \text{ ft.},$$

and the outer radius

$$(2) \quad r = v\, r_1 = 0 \cdot 4\, v \sqrt{Q}.$$

If, also, we put

$$1 - \zeta \left(\frac{r_1}{r}\right)^2 tan.^2 \beta = 1 - \zeta \frac{tan.^2 \beta}{v^2} = \psi,$$

and

$$\frac{\sqrt{1 + \zeta_1}}{cos.\, \delta} = \chi,$$

where generally $\zeta = \zeta_1 = 0{\cdot}075$ and $\delta$, 10° to 20°, we have for the best velocity of rotation

$$(3) \quad v = \sqrt{\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi \sqrt{\chi^2 - \psi}} g h},$$

and

$$(4) \quad v_1 = \frac{r_1}{r} v = \frac{v}{\nu},$$

and accordingly

$$(5) \quad c = - v_1 \, tan. \, \beta$$

and

$$(6) \quad c_2 = \sqrt{\frac{2 g h + \psi v^2}{1 + \zeta_1}}.$$

The number of revolutions is

$$(7) \quad u = \frac{30 v}{\pi r} = 9{\cdot}55 \frac{v}{r}.$$

Then follow the areas of the orifices,

$$(8) \quad F = \frac{Q}{c},$$

$$(9) \quad F_2 = \frac{Q}{c_2},$$

and the wheel height

$$(10) \quad e = \frac{F}{2 \pi r_1}.$$

If, further,

$$\lambda = \frac{e}{d},$$

we have $n d e = F_2$, $n e = \lambda F_2$, and

$$(11) \quad n = \frac{\lambda F_2}{e^2},$$

and, lastly, since

$$(2\,\pi\, r\, sin.\,\delta - n\, s)\, e = F_2,$$

when $s$ is the thickness of float,

$$(12)\quad sin.\,\delta = \frac{F_2 + n\,s\,e}{2\,\pi\, r\, e} = \frac{(e + \lambda\, s)\, F_2}{2\,\pi\, r\, e^2}.$$

If $\delta$ comes out too great, much over 15°, we must take either $\beta$ or $\nu$ greater.

EXAMPLE.—Required for a fall of 5 ft. and feed $Q = 30$ cubic ft. to calculate and arrange the dimensions for a Cadiat turbine. (Compare with preceding example.)

If we take $\beta = 150°$ and $\nu = 1\cdot2$, we have

$$(1)\quad r_1 = 0\cdot4\sqrt{Q} = 0\cdot4\sqrt{30} = 2\cdot19,$$

or, safer, 2·25 ft., and

$$(2)\quad r = \nu\, r_1 = 1\cdot2 \times 2\cdot25 = 2\cdot70 \text{ ft.}$$

Putting $\zeta = \zeta_1 = 0\cdot075$ and $\delta$ for the present $= 15°$, we have

$$\psi = 1 - \zeta\left(\frac{r_1}{r}\right)^2 tan.^2\,\beta = 1 - 0\cdot075\,\frac{tan.^2\,30}{1\cdot44} = 0\cdot9826,$$

and

$$\chi = \frac{\sqrt{1 + \zeta_1}}{cos.\ \delta} = \frac{\sqrt{1\cdot075}}{cos.\ 15°} = 1\cdot0734,$$

hence

$$(3)\quad v = \sqrt{\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi\sqrt{\chi^2 - \psi}}\, g\, h} = \sqrt{\frac{1\cdot0734 - 0\cdot4118}{0\cdot9826 \times 0\cdot4118}} \times 32\cdot2 \times 5$$

$$= \sqrt{\frac{0\cdot6616 \times 156\cdot25}{0\cdot9826 \times 0\cdot4118}} = 15\cdot985 \text{ ft.},$$

and

$$(4)\quad v_1 = \frac{v}{\nu} = \frac{15\cdot985}{1\cdot2} = 13\cdot321 \text{ ft.}$$

Also

$$(5)\quad c = -\, v_1\, tan.\,\beta = 13\cdot321\ tan.\ 30 = 7\cdot692 \text{ ft.},$$

and

$$(6)\quad c_2 = \sqrt{\frac{2\,g\,h + \psi\, v^2}{1 + \zeta_1}} = \sqrt{\frac{312\cdot5 + 251\cdot1}{1\cdot075}} = 22\cdot897 \text{ ft.}$$

The number of revolutions is

$$(7) \quad u = 9\cdot55\,\frac{v}{r} = 9\cdot55\,\frac{15\cdot985}{2\cdot70} = 56\cdot54.$$

We have now the area of orifices,

$$(8) \quad F = \frac{Q}{c} = \frac{30}{7\cdot692} = 3\cdot900,$$

$$(9) \quad F_2 = \frac{Q}{c_2} = \frac{30}{22\cdot897} = 1\cdot3102 \text{ sq. ft.},$$

and for the necessary depth of wheel,

$$(10) \quad e = \frac{F}{2\,\pi\,r_1} = \frac{3\cdot900}{2\,\pi \times 2\cdot25} = 0\cdot2759 \text{ ft.}$$

If we take $\lambda = \frac{e}{d} = 2$, we have for the number of floats,

$$(11) \quad n = \frac{\lambda\,F_2}{e^2} = \frac{2 \times 1\cdot3102}{0\cdot2759^2} = 34,$$

or, for convenience of division, 32, and if the thickness of float is 0·015 ft., we have

$$\sin.\,\delta = \frac{F_2 + n\,s\,e}{2\,\pi\,r\,e} = \frac{1\cdot3102 + 32 \times 0\cdot015 \times 0\cdot2759}{2\,\pi \times 2\cdot7 \times 0\cdot2759}$$

$$= \frac{1\cdot3102 + 0\cdot132}{5\cdot4 \times 0\cdot2759\,\pi} = \frac{1\cdot442}{4\cdot681} = 0\cdot3081,$$

and hence the exit angle $\delta = 17^\circ\,56'$.

The efficiency of this wheel is, without reference to loss of water, axle friction, etc.,

$$\eta = \left(v\sqrt{2\,g\,h + \psi\,v^2} - \phi\,v^2\right)\frac{Q\,\gamma}{\chi\,g\,Q\,h\,\gamma} = \frac{\chi - \sqrt{\chi^2 - \psi}}{\chi\,\psi} = \frac{0\cdot6616}{0\cdot9826 \times 1\cdot0734} = 0\cdot627.$$

(Compare preceding example.)

§ **266.—Whitelaw or Scottish Turbine.**—The Whitelaw or Scottish turbine must be treated somewhat differently from the Cadiat turbine, inasmuch as the water, by reason of the great breadth of the canals, enters the wheel with impact, and a greater choice is allowed in the form and size of the canals. We can thus make the angle $\delta$ much smaller than for wheels of many canals. By reason of the small number of canals, the Whitelaw turbine is especially suited to high falls and small feed. The width of the conduit pipe or the exit reservoir is, when we allow at most a velocity of entrance of 6 ft., given by

$$r_1 = \frac{Q}{\sqrt{6\pi}} = 0 \cdot 23 \sqrt{Q.}$$

We make the outer radius $r$ two, three, or four times $r_1$, according as the number of canals is four, three, or two. The velocities $v$, $v_1$, and $c$, and hence the areas $F_1$ and $F_2$, are determined as for turbines without guide blades, just as in the preceding article. For the depth of wheel, we have

$$e = \frac{F}{2\pi r_1},$$

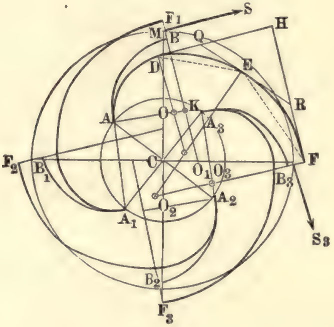
and for the outer width of the wheel canals,

$$d = \frac{F_2}{n e}.$$

In determining the velocity $v$, the coefficient of resistance at entrance $\zeta$ must be taken greater than 0·075, since a slight shock due to the very different directions of the water entering the wheel cannot be avoided. We may, perhaps, without perceptible error, put $\zeta = 0 \cdot 10$. Since, also, the canals are very long, $\zeta_1$ would be greater than for wheel turbines, were this not compensated by the greater width of the canals. We must, however, take $\zeta_1$ at least $= 0 \cdot 075$.

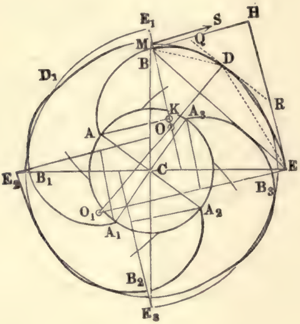
The axis of the canal $ADEFK$, Fig. 499, is in general an Archimedes spiral, though it may also be formed by two or three circle arcs $AD$, $DE$, $EF$. For this purpose, we divide the circumference into as many equal parts as there are exit tubes—for instance, in the figure into four, and at each point draw a line as $MS$, which makes the angle $\delta$ with the tangent, or $SMC = 90° + \delta°$. Then upon each side of $M$, at right angles to $M S$, we lay off the half width of orifice $\frac{1}{2} d_1 = M B = M F_1$, and then describe, according to the rule given in Art. 101, from the centre $K$ in the prolongation of $F_1 B$ an arc $A B_1$ which intersects the inner circumference at $A$ under the angle $\beta$, and is parallel to $M S$ at $B$. In this way, the inner curves of the exit pipes are formed. The outer curves are composed of three arcs $A D$, $D E$, and $E F$, tangent at $D$ and $E$. The arc $A D$ has the least radius $O A = O D$, and, like $A B$, cuts the inner circumference under the angle $\beta$. The arc $E F$ has the greatest radius $O_2 E = O_2 F$, and is parallel at $F$ to the issuing stream. So also is $A_3 B_3$. By construction of the arc $A_3 B_3 = A B$, the points $A A_3 B_3$ and $F$ of the exit pipe are determined, and it is easy, by repetition of the construction, to draw the other pipes.

FIG. 499.



For a very small number of exit pipes, it is not necessary to have separate pipes. They may follow one after the other, as shown in Fig. 500, without any space between. In this case, the arc $A B$ forms the separating partition between two pipes, and the outer arc $B D E$ is tangent to $A B$ at $B$. The centres $O$ and $O_1$ of the arcs $B D$ and $D E$ can be easily determined as follows: Join the end points $B$ and $E$ by a line and draw $B H$ and $E H$, making, with the radii $C B$ and $C E$, the angles $C B H = 90° + \delta$ and $C E H = 90° - \delta$. These lines form with $B E$ a triangle $B E H$. Now, bisect the angle $E B H$ and $B E H$ by the lines $B D$ and $E D$, draw through $D$, $Q R$ parallel to $B E$ and $D O O_1$, at right angles to $B E$, as also $B O$ and $E O_1$ at right angles to $B H$ and $E H$. The intersections $O$ and $O_1$ between each two perpendiculars are the centres of the arcs $B D$ and $D E$.

FIG. 500.



The correctness of the construction is easily seen. By the bisection of the angles $E B H$ and $B E H$, and by drawing the parallel $Q R$, the angles $O B D$ and $O D B$, and hence the lines $O B$ and $O D$, are equal. In like manner, $O_1 D E$ and $O_1 E D$, and therefore $O_1 D$ and $O_1 E$, are equal.

EXAMPLE.—Required to calculate the dimensions and delivery of a Whitelaw turbine, for a fall of 150 ft. and a feed of $1\frac{1}{2}$ cubic ft. per second.

The inner radius is

$$r_1 = 0{\cdot}23\sqrt{Q} = 0{\cdot}23\sqrt{1{\cdot}5} = 0{\cdot}282 \text{ ft.}$$

We take it, then, at 0·3 ft., and the width of the conduit pipe = 0·75 ft., and supposing that we have but two exit pipes or canals, and hence the outer radius $r = 4 r_1 = 1{\cdot}2$ ft. If, now, we assume $\beta = 150°$ and $\delta = 10°$, and put $\zeta = \zeta_1 = 0{\cdot}10$, we have

$$\psi = 1 - 0{\cdot}1 \left(\frac{r_1}{r}\right)^2 tan.^2\, \beta = 1 - 0{\cdot}1 \times \tfrac{1}{16}\, tan.^2\, 30 = 1 - 0{\cdot}0021 = 0{\cdot}9979,$$

and

$$\chi = \frac{\sqrt{1 + \zeta_1}}{cos.\, \delta} = \frac{\sqrt{1{\cdot}1}}{cos.\, 10^\circ} = 1{\cdot}0650.$$

Of the fall $h = 150$ ft., the friction of the water in the conduit pipe, which is 0·75 ft. wide and perhaps 200 ft. long (Weisbach, Vol. I., Arts. 427–429), consumes a head of

$$h_2 = 0{\cdot}0213 \times 0{\cdot}016 \left(\frac{4}{\pi}\right)^2 \frac{l\, Q^2}{d^5} = 0{\cdot}0003408 \left(\frac{4}{\pi}\right)^2 \frac{200 \times 1{\cdot}5^2}{0{\cdot}75^5}$$

$$= 0{\cdot}0003408 \times 1{\cdot}621\, \frac{200 \times 256}{27} = 0{\cdot}03408 \times 1{\cdot}621\, \frac{512}{27} = 1{\cdot}05 \text{ ft.}$$

The effective head, therefore, is

$$h_1 = h - h_2 = 150 - 1{\cdot}05 = 148{\cdot}95 \text{ ft.}$$

For the best velocity, we have

$$\frac{v^2}{2gh} = \frac{\chi - \sqrt{\chi^2 - \psi}}{2\psi \sqrt{\chi^2 - \psi}} = \frac{1{\cdot}065 - \sqrt{1{\cdot}1342 - 0{\cdot}9979}}{1{\cdot}9958 \sqrt{1{\cdot}1342 - 0{\cdot}9979}}$$

$$= \frac{1{\cdot}065 - \sqrt{0{\cdot}1363}}{1{\cdot}9958 \times 0{\cdot}3692} = \frac{0{\cdot}6958}{0{\cdot}7369} = 0{\cdot}9443,$$

and hence

$$v = \sqrt{0{\cdot}9443}\, \sqrt{2gh} = 0{\cdot}9718 \times 7{\cdot}906 \sqrt{150} = 94{\cdot}10 \text{ ft.}$$

The other velocities are

$$v_1 = \frac{r_1}{r} v = \frac{v}{4} = 23{\cdot}525 \text{ ft.}$$

$$c = - v_1\, tan.\, \beta = 23{\cdot}525\, tan.\, 30^\circ = 13{\cdot}58 \text{ ft.}$$

$$c_2 = \sqrt{\frac{2gh + \psi v^2}{1 + \zeta_1}} = \sqrt{\frac{9309 + 8836}{1{\cdot}1}} = \sqrt{\frac{18145}{1{\cdot}1}} = 128{\cdot}43 \text{ ft.}$$

Therefore, the areas of apertures are,

$$F = \frac{Q}{c} = \frac{1{\cdot}5}{13{\cdot}58} = 0{\cdot}11044 \text{ sq. ft.}$$

and

$$F_2 = \frac{Q}{c_2} = \frac{1\cdot 5}{128\cdot 43} = 0\cdot 01168 \text{ sq. ft.}$$

The depth of wheel is

$$e = \frac{F}{2\pi r_1} = \frac{0\cdot 11044}{0\cdot 6\,\pi} = 0\cdot 05859 \text{ ft.} = 0\cdot 703 \text{ inches,}$$

and the width of exit aperture, since there are two of them.

$$d = \frac{F_2}{n\,e} = \frac{0\cdot 01168}{2 \times 0\cdot 05859} = 0\cdot 09967 \text{ ft.} = 1\cdot 196 \text{ inches.}$$

The ratio $\frac{e}{d}$ is $\frac{0\cdot 05859}{0\cdot 09967} = 0\cdot 5879$ only. In order to make it greater, we must take three or more canals or exit pipes.

The efficiency of this wheel, without reference to friction on axle and in the conduit pipe, is

$$\eta = \frac{\chi - \sqrt{\chi^2 - \psi}}{\psi\,\chi} = \frac{0\cdot 6958}{0\cdot 9979 \times 1\cdot 0650} = 0\cdot 6547.$$

§ **267.—Reaction-Wheel with Radial Arms.**—In reaction-wheels where the axes of the arms diverge radially from the reservoir, the water suffers impact when it enters the wheel and a corresponding loss of work, and if these arms are not curved at all, so that the water issues through side openings, there is also impact of the water against the ends of the arms or tubes, which also occasions a loss of work. Since, in general, curved arms are now used, we shall discuss here only the loss due to impact at entrance. The velocity of exit is determined in this case by the formula

$$(1 + \zeta_1)\,c_2^2 = 2gx + c^2 + v^2 - v_1^2,$$

or, since

$$2gx + c^2 = 2gh - \zeta c^2,$$

by

$$(1 + \zeta_1)\,c_2^2 = 2gh + v^2\left[1 - \left(\frac{r_1}{r}\right)^2\right] - \zeta c^2;$$

we have, therefore,

$$c_2 = \sqrt{\frac{2gh - \zeta c^2 + \left[1 - \left(\frac{r_1}{r}\right)^2\right] v^2}{1 + \zeta_1}}$$

The velocity height corresponding to the loss of work of the wheel is, since the water upon entrance must suddenly assume the tangential velocity $v_1$,

$$y = (c_2^2 + v^2 - 2\, c_2\, v \cos.\delta + v_1^2 + \zeta_1 c_2^2 + \zeta c^2)\frac{1}{2g}$$

$$= \left((1 + \zeta_1) c_2^2 + \zeta c^2 + v^2 \left[1 + \left(\frac{r_1}{r}\right)^2\right] - 2\, v\, c_2 \cos.\delta\right)\frac{1}{2g}$$

$$= \left(gh + v^2 - v \cos.\delta \sqrt{\frac{2gh - \zeta c^2 + \left[1 - \left(\frac{r_1}{r}\right)^2\right] v^2}{1 + \zeta_1}}\right)\frac{1}{g},$$

and hence the effective delivery of the wheel is

$$L = \left(v \cos.\delta \sqrt{\frac{2gh - \zeta c^2 + \left[1 - \left(\frac{r_1}{r}\right)^2\right] v^2}{1 + \zeta_1}} - v^2\right)\frac{Q\gamma}{g}$$

$$= (v \sqrt{2gh - \zeta c^2 + \psi v^2} - \chi v^2)\frac{Q\gamma}{\chi g},$$

if we denote $1 - \left(\frac{r_1}{r}\right)^3$ by $\chi$, and $\frac{\sqrt{1 + \zeta_1}}{\cos.\delta}$ by $\chi$.

In general, $\zeta$ is so small that we can put

$$L = (v \sqrt{2gh + \psi v^2} - \chi v^2)\frac{Q\gamma}{\chi g},$$

and hence the best velocity is, as in Art. 256,

$$v = \sqrt{\frac{\chi - \sqrt{\chi^2 - \psi}}{\psi \sqrt{\chi^2 - \psi}} \cdot gh}.$$

If, now, we neglect also $\zeta_1$ and take $\delta = 0$, we have $\chi = 1$, and hence the best wheel velocity is

$$v = \sqrt{\frac{1 - \sqrt{1 - \psi}}{\psi \sqrt{1 - \psi}} \cdot g h} = \sqrt{\frac{1 - \frac{r_1}{r}}{\left[1 - \left(\frac{r_1}{r}\right)^2\right] \frac{r_1}{r}} \cdot g h}$$

$$= \sqrt{\frac{g h}{\left(1 + \frac{r_1}{r}\right) \frac{r_1}{r}}}.$$

The efficiency is in the last case,

$$\eta = \frac{\chi - \sqrt{\chi^2 - \psi}}{\psi \chi} = \frac{1 - \frac{r_1}{r}}{1 - \left(\frac{r_1}{r}\right)^2} = \frac{1}{1 + \frac{r_1}{r}} = \frac{r}{r + r_1},$$

and therefore greater the longer the arms with reference to the diameter of the entrance reservoir.

From $v$ we can find

$$v_1 = \frac{r_1}{r} v,$$

as also

$$c_2 = \sqrt{\frac{2 g h - \zeta c^2 + \psi v^2}{1 + \zeta_1}},$$

and

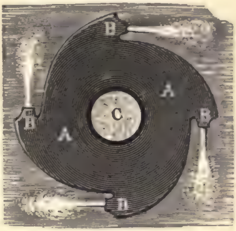
$$F_2 = \frac{Q}{c_2}.$$

In order to make the resistance at entrance as small as possible, we must have $\frac{F_2}{F}$ small or $F$ large; so large that the velocity $c$ at entrance to the wheel shall not be greater than that of the approaching water. In order to effect this, we must have the circular cross-section of the entrance orifices equal to the cross-section of the conduit pipe—*i. e.*, $2 \pi r_1 e = \pi r_1^2$, or $e \doteq$ the half radius of the reservoir. Finally, we have the diameter or width of the exit orifices,

$$d = \frac{F_2}{n e}.$$

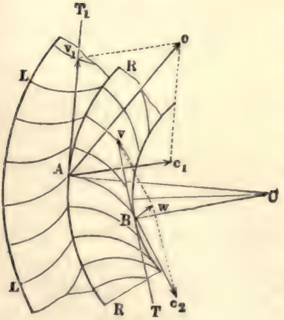
If, as in Fig. 501, we have a single vessel *A A* instead of separate arms, and let the water flow out through well-rounded conical orifices *B B*, the hydraulic resistances become very small, since the motion of the water in the wheel, especially when it is made high, is very slight, and there remains only the loss of work due to passage of the water from the centre pipe *C* to the wheel. The efficiency of such a wheel, exceedingly simple as it is, may easily be as high as $\frac{7}{8}$.

FIG. 501.



§ **268.—Inward-Flow Turbine.**—Turbines with inward flow, as that of Francis's, already described, Figs. 489 and 490, are treated in general precisely like those of outward flow (Fourneyron). We have between these wheels the same relations as between tangential wheels of outward and inward flow (Arts. 235 and 236). If, as there, we represent the radius of that circumference where the water enters by $r_1$, and of that circumference where it departs by $r$, and the corresponding velocities by $v_1$ and $v$, then the formulæ and rules already deduced for the inward-flow turbine apply also to outward-flow turbines as well. If in an inward-flow turbine the water enters along the guide blades *L L*, Fig. 502, with the velocity $c$, so that the angle $c A v_1 = \alpha$, then we have for the relative velocity $c_1$,

FIG. 502.



$$c_1^2 = c^2 + v_1^2 - 2\, c\, v_1 \cos.\, \alpha.$$

If $\delta$ is the angle $T\,B\,c_2$, we have for the relative velocity $c_2$, since during motion from $A$ to $B$ there is a loss of work due to the centrifugal force of

$$\left(\frac{v_1^2 - v^2}{2\,g}\right) Q\,\gamma,$$

$$\left(1 + \zeta_1\right) \frac{c_2^2}{2\,g} = x - h_2 + \frac{c_1^2}{2\,g} - \frac{v_1^2 - v^2}{2\,g}$$

$$= x - h_2 + \frac{c^2}{2\,g} + \frac{v^2}{2\,g} - \frac{2\,c\,v_1\,cos.\,\alpha}{2\,g},$$

or when

$$\left(1 + \zeta\right) \frac{c^2}{2\,g} = h_1 - x, \qquad \text{and} \qquad h_1 + h_2 = h$$

$$(1 + \zeta_1)\,c_2^2 = 2\,g\,h + v^2 - 2\,c\,v_1\,cos.\,\alpha - \zeta\,c^2,$$

precisely as for turbine with outward flow. The velocity already found then for the inner circumference is here the outer velocity, viz.:

$$v_1 = \sqrt{\frac{2\,g\,h}{\frac{2\,sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta - \alpha)} + \zeta\left(\frac{sin.\,\beta}{sin.\,(\beta - \alpha)}\right)^2 + \zeta_1\left(\frac{r}{r_1}\right)^2}}\,.$$

Since, now, $\frac{r}{r_1}$ is less than unity, while for turbines with outward flow $\frac{r}{r_1}$ is greater than unity, we see that, other things being the same, the best velocity for the outer circumference is a little greater for the inward-flow turbine. The difference is, however, so small that we can approximately put them equal. But for equal velocities, the number of revolutions are inversely as the radii. If, then, $u$ is the number of revolutions for an outward-flow and $u_1$ for an inward-flow turbine, and $\nu$ = the ratio of the outer radius to the inner, we have

$$\frac{u_1}{u} = \frac{1}{\nu}, \quad \text{or} \quad u_1 = \frac{u}{\nu}.$$

An inward-flow turbine then, under the above assumptions, makes a less number of revolutions than an outward-flow turbine. Since also we can put $c_2 = v$, and $c_2$ then is less for the first case than for the second, the hydraulic resistances are also less. This advantage is, however, balanced, since, as we see from the formula

$$sin.\ \delta = \left(\frac{r_1}{r}\right)^2 \frac{sin.\ \alpha\ sin.\ \beta}{sin.\,(\beta - \alpha)},$$

the inward-flow turbine requires a greater exit angle, and hence, as we may see from the value for $\eta$, more work is lost.

EXAMPLE.—Given a fall of $h = 5$ ft. and feed $Q = 30$ cubic ft. Required to calculate and arrange the dimensions for a reaction turbine with inward flow (compare example, Art. 264). If, as before, we take

$$\alpha = 30, \quad \beta = 100, \quad \text{and} \quad \nu = \frac{r}{r_1} = \frac{1}{1{\cdot}35},$$

we should have for $\delta$ the value 72¾°, a value much too large. Taking, therefore,

$$(1) \quad \alpha = 20^\circ,$$

$$(2) \quad \beta = 60^\circ,$$

and

$$(3) \quad \nu = \tfrac{4}{5} = 0{\cdot}8,$$

we have

$$sin.\ \delta = \frac{sin.\ \alpha\ sin.\ \beta}{\nu^2\ sin.\,(\beta - \alpha)} = \frac{sin.\ 20^\circ\ sin.\ 60^\circ}{0{\cdot}64\ sin.\ 40^\circ} = 0{\cdot}72,$$

and hence

$$(4) \quad \delta = 46^\circ\ 3'.$$

If, now, we take the outer radius

$$(5) \quad r_1 = 2{\cdot}45 \text{ ft.},$$

the inner radius is

$$(6) \quad r = \nu\, r_1 = 0{\cdot}8 \times 2{\cdot}45 = 1{\cdot}96 \text{ ft.}$$

Without reference to resistances, we should have for the outer velocity

$$v_1 = \sqrt{g\,h\,(1 - tan.\,\alpha\ cot.\,\beta)} = \sqrt{32 \cdot 2 \times 5\,(1 - tan.\,20^\circ\ cot.\,60^\circ)}$$

$$= \sqrt{161 \times 0 \cdot 78986} = 11 \cdot 11 \text{ ft.}$$

With reference to resistances, however, we have, taking $\zeta = \zeta_1 = 0 \cdot 075$,

$$v_1 = \sqrt{\frac{2\,g\,h}{\frac{2\,sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta - \alpha)} + \zeta\left[\left(\frac{sin.\,\beta}{sin.\,(\beta - \alpha)}\right)^2 + \nu^2\right]}}$$

$$= \sqrt{\frac{312 \cdot 5}{\frac{2\,sin.\,60^\circ\,cos.\,20^\circ}{sin.\,40^\circ} + 0 \cdot 075\left[\left(\frac{sin.\,60^\circ}{sin.\,40^\circ}\right)^2 + 0 \cdot 64\right]}}$$

or

$$(7)\quad v_1 = \sqrt{\frac{312.5}{2 \cdot 5321 + 0 \cdot 075 \times 2 \cdot 455}} = \sqrt{\frac{312 \cdot 5}{2 \cdot 7162}} = 10 \cdot 726 \text{ ft.}$$

The inner wheel velocity is then

$$(8)\quad v = \nu\,v_1 = 0 \cdot 8 \times 10 \cdot 726 = 8 \cdot 581 \text{ ft.}$$

The velocity of the water before entrance to the wheel is

$$(9)\quad c = \frac{v_1\,sin.\,\beta}{sin.\,(\beta - \alpha)} = \frac{10 \cdot 726\,sin.\,60^\circ}{sin.\,40^\circ} = 14 \cdot 451 \text{ ft.,}$$

and the relative velocity of the entering water is

$$(10)\quad c_1 = \frac{v_1\,sin.\,\alpha}{sin.\,(\beta - \alpha)} = \frac{10 \cdot 726\,sin.\,20^\circ}{sin.\,40^\circ} = 5 \cdot 694 \text{ ft.}$$

Hence the absolute exit velocity is

$$(11)\quad w = 2\,v\,sin.\,\frac{\delta}{2} = 2 \times 8 \cdot 581\,sin.\,23^\circ\,1\tfrac{1}{2}' = 6 \cdot 712 \text{ ft.}$$

The number of revolutions per minute is

$$(12)\quad u = 9 \cdot 55\,\frac{v_1}{r_1} = 9 \cdot 55\,\frac{10 \cdot 726}{2 \cdot 45} = 41 \cdot 81.$$

The cross-section of orifices is

$$(13)\quad F = \frac{Q}{c} = \frac{30}{14 \cdot 451} = 2 \cdot 076 \text{ sq. ft.,}$$

$$(14)\quad F_2 = \frac{Q}{c_2} = \frac{Q}{v} = \frac{30}{8 \cdot 581} = 3 \cdot 496 \text{ sq. ft.}$$

If we take the ratio $\lambda = \frac{e}{d} = 2$, and the thickness of floats $s = 0{\cdot}02$ ft., we have

$$2\,\pi\, r\, sin.\, \delta = 2\,\pi \times 1{\cdot}96\, sin.\, 46^\circ\, 3' = 8{\cdot}866,$$

and hence for the inner wheel depth,

$$(15)\quad e = \frac{F_2}{2\,\pi\, r\, sin.\, \delta} + \lambda\, s = \frac{3{\cdot}496}{8{\cdot}866} + 2 \times 0{\cdot}02 = 0{\cdot}3943 + 0{\cdot}04$$

$$= 0{\cdot}4343 \text{ ft.} = 5{\cdot}212 \text{ inches};$$

also for the width of the wheel canals at exit,

$$(16)\quad d = \frac{e}{\lambda} = \frac{5{\cdot}212}{2} = 2{\cdot}606 \text{ inches.}$$

The number of wheel floats is

$$(17)\quad n = \frac{\lambda\, F_2}{e^2} = \frac{2 \times 3{\cdot}496}{0{\cdot}4343^2} = 37,$$

and of the guide blades,

$$(18)\quad n_1 = \frac{n\, sin.\, \alpha}{v\, sin.\, \delta} = \frac{37\, sin.\, 20^\circ}{0{\cdot}8\, sin.\, 46^\circ\, 3'} = \frac{37{\cdot}0342}{0{\cdot}8 \times 0{\cdot}720} = 22.$$

Francis makes the number of guide blades equal to that of the floats, and takes $n = n_1 = 40$.

The delivery of this turbine is

$$L_1 = \left(1 - \frac{\zeta\,(c^2 + v^2) + w^2}{2\,g\,h}\right) Q\, h\, \gamma$$

$$= \left(1 - \frac{0{\cdot}016}{5}\,[0{\cdot}075\,(14{\cdot}451^2 + 8{\cdot}581^2) + 6{\cdot}712^2]\right) 9262{\cdot}5$$

$$= [1 - 0{\cdot}0032\,(0{\cdot}075 \times 281{\cdot}46 + 45{\cdot}05)]\, 9262{\cdot}5$$

$$= (1 - 0{\cdot}0032 \times 66{\cdot}16)\, 9262{\cdot}5$$

$$= (1 - 0{\cdot}2116)\, 9262{\cdot}5 = 0{\cdot}7884 \times 9262{\cdot}5 = 7302 \text{ ft.},$$

or somewhat less than for the turbine with outward flow of Art. 103.

**§ 269.—The Shaft.**—In designing a turbine for a given water-power, besides the principal dimensions, the sizes of some of the principal pieces have also to be computed. For instance, the diameter of the shaft of the turbine and that of its pivot, the thickness of the sides of the supply chamber, etc., are to be computed according to the theory of the strength of materials.

The diameter of the shaft is to be determined from the power and number of revolutions according to the rules for strength of torsion. The formula obtained for horizontal shafts (Art. 191),

$$d = 0 \cdot 357 \sqrt[3]{P a} = 6 \cdot 2 \sqrt[3]{\frac{L}{u}} \text{ inches}$$

can be directly applied to this case; $P$ being the load in pounds, $L$ the power in horse-powers, $a$ the radius of the wheel in feet or $r$, and $u$ the number of revolutions per minute.

$d_1$, the diameter of the pivot of the vertical shaft, is usually made from $\frac{2}{3}$ $d$ to $\frac{3}{4}$ $d$, although, by the ordinary rules of strength of materials, it might be smaller. Taking the allowable pressure per square inch of section at 1500 pounds, we have, when the weight of the loaded shaft is $G$,

$$1500 \frac{\pi d_1^2}{4} = G,$$

and, therefore,

$$d_1 = \sqrt{\frac{G}{375 \pi}} = 0 \cdot 02913 \sqrt{G},$$

for which we will take

$$d_1 = 0 \cdot 03 \sqrt{G} \text{ inches.}$$

This formula, however, is applicable only to vertical shafts which revolve slowly—for example, to those run by horse-power; the pivots of turbines, which revolve much more rapidly, must have a greater diameter on account of the greater development of heat. In this case, it is necessary to increase the diameter with $u$, the number of revolutions, and, with a fair degree of accuracy, we can take

$$d_1 = 0{\cdot}03 \sqrt{(1 + 0{\cdot}01\, u)\, G},$$

in which $u$ denotes the number of revolutions of the shaft.

The bosses, or those parts of the shaft on which the disc of the wheel and the driving wheel are set, must be made larger than the rest of the shaft, since they are weakened by the slot for a wedge. The diameter of these bosses is usually made equal to $\frac{5}{4}\, d$, and the thickness of the sides of the naves, by which the disc and driving wheel are supported upon the bosses, is made equal to $\frac{1}{8}\, d$; accordingly, the exterior diameter of each nave is

$$d_2 = \tfrac{5}{4}\, d + 2 \times \tfrac{1}{8}\, d = \tfrac{23}{12}\, d.$$

The disc of the wheel must be of corresponding strength to $P\, a$, the power of the turbine. If $s$ is the thickness of this disc at the shaft where it is fastened to the nave, we have for the area of the cylindrical surface connecting it to its nave $\pi\, d_2\, s$, and denoting, as before, the modulus of rupture by $K$, we have the force tending to wrench the disc from its nave $= \pi\, d_2\, s\, K$, and consequently the moment of the same,

$$P\, a = \pi\, d_2\, s\, K \frac{d_2}{2} = \tfrac{1}{2}\, \pi\, d_2^2\, s\, K.$$

Putting in place of $K$, the modulus of proof strength $T = 1870$ lbs. (see Vol. I., § 264), we have the necessary thickness of the disc,

$$s = \frac{P\, a}{935\, \pi\, d_2^2},$$

or since

$$P\, a = 12 \times 5252 \times \frac{L}{u} \text{ inch lbs.}$$

(see § 191), we have

$$s = 21{\cdot}5\, \frac{L}{u\, d_2^2} = 5{\cdot}84\, \frac{L}{u\, d^2}.$$

In order to give this disc the necessary rigidity, its thickness is made in practice much greater than this expression would give, and, in fact, equal to the thickness of the disc which carries the guides. The thickness of the latter can be computed in the following manner:

Let us suppose this disc to be solid, and assume that it is in danger of being torn into two halves, along its diameter $2r_1$, by the pressure of the water over it. If the height which measures this pressure is $h$, the force pressing upon each half is

$$P = \tfrac{1}{2}\pi r_1^2 h\gamma,$$

and since the distance of the centre of gravity of a semicircle from the centre of the circle is (see Vol. I., § 113)

$$y = \frac{4r_1}{3\pi},$$

the moment of this force is

$$Py = \tfrac{1}{2}\pi r_1^2 h\gamma \,.\, \frac{4r_1}{3\pi} = \tfrac{2}{3} r_1^3 h\gamma.$$

From the theory of strength of materials, since $2r_1$ stands for the breadth and $s$ for the height of the surface of fracture (see Vol. I., § 236), we have also the following expression for this moment,

$$Py = \frac{2r_1 s^2 T}{6};$$

accordingly, equating the two expressions, we obtain the following formula for determining the thickness of the disc,

$$\frac{2r_1 s^2 T}{6} = \tfrac{2}{3} r_1^3 h\gamma, \qquad \text{or} \qquad s^2 = \frac{2r_1^2 h\gamma}{T}.$$

If we introduce $\gamma = 62\cdot5$ lbs. and $T = 7000$ lbs., we shall obtain the necessary thickness,

$$s = r_1\sqrt{\frac{2\times 62\cdot5\,h}{7000}} = r_1\sqrt{0\cdot017857\,h_1} = 0\cdot134\,r_1\sqrt{h} \text{ inches,}$$

in which $r_1$ and $h$ must be expressed in feet.

To obtain the necessary rigidity, we must add 0·34 inch (see Vol. I., § 363), and therefore take

$$s = 0 \cdot 13\, r_1 \sqrt{h} + 0 \cdot 34 \text{ inch.}$$

EXAMPLE.—For the turbine computed in the example in § 264, since we have, in this case, $L = 16$ horse-powers, and $u = 65$, the requisite diameter of shaft is

$$d = 6 \cdot 2 \sqrt[3]{\frac{L}{u}} = 6 \cdot 2 \sqrt[3]{\frac{16}{65}} = 6 \cdot 2 \times 0 \cdot 63 = 3 \cdot 90 \text{ inches,}$$

for which we might take 4·0 inches.

If the weight of the loaded shaft is $G = 3600$ lbs., the diameter of the pivot, according to the formula given above, would be

$$d_1 = 0 \cdot 03 \sqrt{(1 + 0 \cdot 01 \times 65)\, 3600} = 1 \cdot 8 \sqrt{1 \cdot 65} = 2 \cdot 31 \text{ inches,}$$

instead of which we might take $d_1 = 2 \cdot 5$ inches.

The requisite thickness of the guide disc and also of the wheel disc is

$$s = 0 \cdot 13 \times 1 \cdot 8 \sqrt{5} + 0 \cdot 34 = 0 \cdot 86.$$

## § 270.—The Bearings of the Pivots of Turbines.—

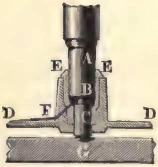
A very important part of a turbine is the pivot and its bearing. The weight of the turbine, which is often considerable, and the high velocity, produce at the base of the pivot or pin so great a moment of friction that the pivot wears away rapidly, unless it is oiled with the greatest care. The majority of turbine-makers have, therefore, always paid especial attention to the insertion of durable pivots. When we consider that the pins of turbines are worn away much faster than those of other vertical shafts, we find the reason for this difference partly in the heating of the pivot, which always accompanies the great velocity of revolution, and partly in the imperfect lubrication, which is also rendered more difficult by the entrance of the water. To obviate this evil as much as possible, a turbine has to be made as light as possible, and the shaft, especially, must not be made unnecessarily long; the rubbing surfaces are to be made very large, and therefore the metal point on which the turbine turns very thick (generally but little smaller than the shaft itself); moreover, we must prevent as far as possible the entrance of the water between the rubbing surfaces, and, finally, let in an uninterrupted stream of olive-oil, or, what is better, nut-oil, between those surfaces which are in contact or on which friction is developed.

Besides the support at the pin or pivot, another bearing is, of course, to be placed at or near the upper end of the shaft.

A very simple bearing, but applicable only to small pressures, is shown in Fig. 484. The point *C* rests in a step of composition, which can be raised or lowered, as may be necessary, within a foot-step bolted to the floor of the wheel chamber by means of tightening keys *L S*. Oil is carried to the pivot by a pipe *R*, which passes through the bottom of the step by the side of the wedges.

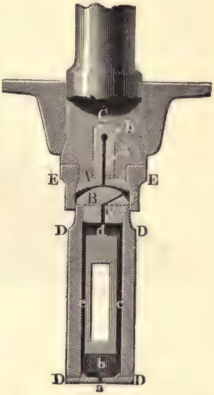
The arrangement of a pivot designed by Cadiat is shown by Fig 503. *A* is the foot of the vertical shaft; *B* a hardened steel pin which is firmly united with *A* either by a screw or by ribs; *C* is the bearing of the pivot, and is also of hard steel; *D E E D* is the step box of cast-iron, firmly fastened to the floor; *E E* is the brass which gives lateral support to the shaft and prevents the access of the water to the pivot; *F* a pipe through which oil is carried into the vacant space between *B* and *E*; finally *G* is the lever or tightening key for raising or lowering the turbine.

FIG. 503.



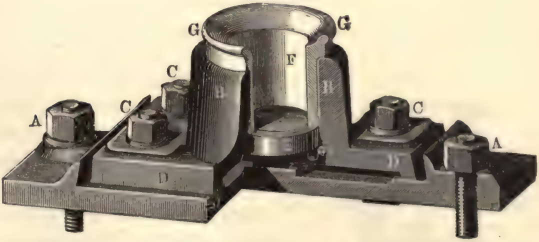
The bearing and arrangement for oiling used by Fourneyron is very complicated. The general arrangement is shown in Fig. 486, but for information as to details, Fig. 504 may be used. Fig. 486 must, however, be used to show the manner in which the step *Z* rests upon a movable lever *O R*, which can be raised or lowered by a rod *R S* by means of a screw *S*. We also see in *V* the pipe for bringing the oil. To secure a better circulation of the oil, it is well to

FIG. 504.



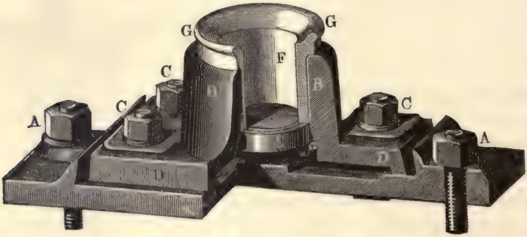
have the mouth of the pipe as high as possible, so that at least it shall be above the surface of the head-water. The rubbing parts *A* and *B*, Fig. 504, are of hardened steel. The upper part *A* is firmly united to the shaft *C*; the lower part *B*, on the other hand, is fastened in a box *D D*, which can be moved up or down in the step box *Z* by means of the lever *O R*, Fig. 486. For security, the bottom *A*, Fig. 504, is hollowed out in the form of a segment of a sphere, and the top of *B* is also made convex; moreover, *A* and *B* are both surrounded by a metal sleeve, which is also intended to retain the oil between the rubbing surfaces. The oil brought in through a pipe enters through *a* into the vacant space *b*, thence through the passages *c*, *c* into the space *d*. From *d*, through three passages *e*, *f*, which at first run up straight and afterwards obliquely, it flows along the surface of the steel step up to the rubbing surfaces, where opportunity is given it to spread by means of those grooves running radially. Finally, a hole *g h* leads from the middle of these surfaces into the shaft, and through this hole the oil can flow off and thus be kept in circulation.

FIG. 505.



A fixed step, complete, is represented in Fig. 505. *A A* is the bed plate, fastened with two screw bolts *A*, *A*; *B B* is the step box with its foot plate *D D* fastened by four screw bolts *C*, *C* . . . . to the bed plate.

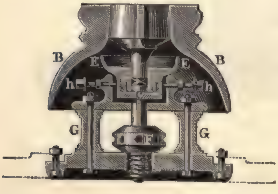
Fig. 506.



Within the step box lies the step *E* of bronze or steel, provided with a circular channel for oil, and held firmly upon the foot plate by a pin *a*, and upon it rests the cylindrical brass sleeve or box *F G* which surrounds the pivot of the turbine. If the turbine runs in the open air and not under water, oil can be carried from the receiver *G G* through vertical channels to the channel in the step; but if the bearing is under water, the unguent must be brought to the pivot through special small pipes and side channels in *B B* . . . . , and carried off in the same way.

If we wish to keep the water entirely away from the pivot of a turbine, we can use the so-called atmospheric lubrication of Laurent. It consists essentially of this: A diving bell is fastened at the foot of the shaft which surrounds the pivot, and the air enclosed in this bell prevents the access of the water to the interior of the bearing. The arrangement of a bearing with atmospheric lubrication is shown in Fig. 507. *A* is the pivot, and *B B* the diving bell;

Fig. 507.



*c* the steel step and *d* the brass surrounding the pivot. The last two are placed in the step box which ends in a bowl *E E* to be filled with oil. This step box rests on the cast-iron seat *G G* by means of the set screw *F*, and can not only be raised and lowered as is needed, but it can also be set in a horizontal direction by other side screws *h*, *h*.

The pivots are also protected from the access of the water by suspending the turbine shafts. This method of suspension we have already become familiar with in § 249, Fig. 488, in the case of the inward-flow turbines, and another method of suspension is used in Fontaine's turbine, the description of which must be deferred till further on.

NOTE.—The use of the pressure of the water to diminish the friction, recommended by Girard, also belongs here. See "Note sur les expériences de surfaces glissantes et sur leurs applications aux pivots des arbres verticaux," in "Comptes rendus de l'Académie des Sciences à Paris," T. 55. Also, Dingler's "Polytechn. Journal," Bd. 167.

§ **271.—Comparison of Turbines.**—From a comparison of the turbines of Fourneyron, Cadiat, and Whitelaw, the following results are obtained: A turbine with guides is always a more perfect mechanical appliance, since by means of these almost all the energy can be extracted from the water (through the equalization of $c_2$ and $v$), which is impossible with turbines without this apparatus. With regard to all the accessory circumstances, all those turbines require about the same velocity of the wheel to produce the maximum effect—viz.:

$$v = 0{\cdot}7\sqrt{2gh} \quad \text{to} \quad \sqrt{2gh},$$

only these maximum powers are different—viz.: with Fourneyron's turbines about 75, with Cadiat's 65, and with Whitelaw's only 50 to 60 per cent of the total power. These proportions vary, however, with the quantity of water; while with one of Whitelaw's turbines, the percentage of efficiency is not essentially altered by a change in the orifices of discharge. With the other turbines, it comes out considerably smaller, when the sluice is set lower for a smaller quantity of water. Moreover, there is this other distinction between the other two turbines, that with an exterior regulating sluice the efflux is always with a full section, while with an inner sluice, if about half the depth of the wheel is covered, the wheel passages are not completely filled with water.

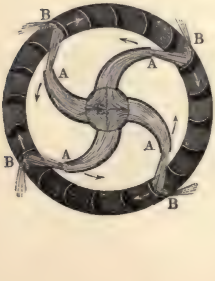
With regard to the loss of water which occurs through the annular crevices between the wheel and the regulating sluice, etc., this is smallest in Fourneyron's turbines, larger in Whitelaw's, and still larger in Cadiat's, because the pressure of the water inside does not greatly exceed that of the atmosphere in the first class of turbines, especially with the better constructions; but with the last class this pressure is generally considerable, and these wheels have also one more joint or crevice than the other turbines. Moreover, turbines without guide blades, and especially Whitelaw's, are always simpler and easier to construct profitably than those of Fourneyron, which are also more liable than the former to be diverted from a good, useful effect, by foreign substances brought in by the water.

In general, we can-assert that the turbines of Fourneyron and Cadiat are especially suitable to be used with small or average falls (under 31 ft.) and large quantities of water, while the Scotch or Whitelaw turbines are more suitable to high falls and small quantities of water.

Tangential wheels can be especially applied to the utilizing of high falls.

NOTE.—In those turbines which are without guide blades—*i. e.*, which have a high fall—the discharged water still possesses a great velocity $w = c_2 - v$ (compare the examples which have been computed), and the wheel is thereby deprived of a considerable part of mechanical power. This loss can, however, be obviated or lessened by using the energy of this discharged water to drive a second wheel. A construction of this sort has been carried out by Ober-Bergrath Althaus in the tanning mill at Vallendar near Ehrenbreitstein. The essential parts of the arrangement can be seen in Fig. 508. $AEA$ is an ordinary reaction-wheel with four curved revolving pipes and a fall of 124 ft., and $BB$ is a larger wheel with floats which is set in rotation by the water issuing from $AA$. Since the two wheels turn in opposite directions, they must be connected together by a special form of

FIG. 508.



wheel-work. The outer wheel affords the additional advantage of serving at the same time as a fly-wheel, and thereby giving a more uniform rate of motion to the whole machinery (see "Inner-österreichisches Gewerbeblatt," Jahrgang 5, 1843).

§ **272.—Experiments on Turbines.**—Experiments on the performances of the reaction-turbines, last considered, with an outward flow, have been published in great numbers, but the necessary confidence cannot be placed in all the statements on this subject. The assertion that with these motors, which are in many respects so superior, efficiencies of 85 to 95 per cent have been attained, must be contradicted at once, and, with charitable judgment, be ascribed to deceptions. Since the efflux of water through the most perfect orifice has a coefficient of velocity $\phi = 0{\cdot}97$ (see Vol. I., § 405), the introduction into the wheel by the arrangement of guide blades produces already a loss of energy amounting to

$$\left(\frac{1}{\phi^2} - 1\right)\frac{c^2}{2g} \cdot Q\gamma = 0{\cdot}06\,\frac{c^2}{2g}\,Q\gamma;$$

moreover, since the friction of the water in a pipe which is, as an average, three times as long as it is wide (according to Vol. I., § 430), consumes the work

$$0{\cdot}019 \times 3\,\frac{v^2}{2g}\,Q\gamma = 0{\cdot}057\,\frac{v^2}{2g} \cdot Q\gamma,$$

and since $\frac{v^2}{2g}$ is nearly equal to $\frac{c^2}{2g} = h$, there is now left from these resistances only 88 per cent of power; if we allow only one per cent for the resistance of the bends, two per cent of loss for the impact at the ends of the floats, and three per cent for the energy which the tail water retains, and if we even leave out of consideration other resistances—such as, for example, that in the guide blades—we have left only 82 per cent of useful work; and we might with certainty regard a turbine as an extremely good one if it had an efficiency of 75 to 80 per cent (compare § 260).

Moreover, the experiments of impartial experimenters—such, *e.g.*, as Morin, Brückmann, and others—give efficiencies for these wheels which come, to be sure, close to 80 per cent, but never fully reach this value.

Morin reports the results of his experiments in the work, "Expériences sur les roues hydrauliques à axe vertical, appelées turbines," Metz et Paris, 1838. He first treats of the experiments he made on a Fourneyron turbine at Moussay. This wheel was 0·85 metre (2·789 ft.) in exterior diameter, 0·11 metre (0·361 ft.) high, had a fall of 7·5 metres (24·607 ft.) and used 0·738 cubic metre of water per second (26·06 cubic ft.), giving an available water-power amounting to 73·8 horse-powers (72·8 English horse-powers). The general result of these experiments was that whether the wheel ran more or less under water, it gave, with 180 to 190 revolutions per minute, the maximum useful work of 69 per cent of the total energy. If the number of revolutions was about 50 per cent larger or smaller, this efficiency fell only 7 to 8 per cent. In these cases, the regulating sluice was raised almost to its full extent, but, if it was lowered to half the depth of the wheel, the efficiency fell 8 per cent. If the wheel had run out of the water, the falling off would certainly have been much greater.

Morin then communicates in this treatise the results of his extended experiments on a turbine in Mühlbach. This turbine had an exterior diameter of 2 metres (6·562 ft.) and a depth of $\frac{1}{3}$ metre (1·094 ft.); its fall amounted to $3\frac{1}{2}$ to $3\frac{3}{4}$ metres (11·483 to 12·303 ft.), and the quantity of water used per second was $2\frac{1}{2}$ cubic metres (88·29 cubic ft.); it therefore afforded an available power of 117 to 125 horse-powers (115 to 123 English horse-powers). With 50 to 60 revolutions per minute and the widest opening of the regulating sluice, it gave a maximum useful effect of 78 per cent, which, however, is perhaps to be put equal to 75 per cent only, because Morin took too small a coefficient of efflux in measuring the water. This high degree of efficiency was diminished by 2 to 4 per cent when the number of revolutions was 40 per cent larger or smaller than the given number. It was not altered when the wheel ran slightly or considerably (1 metre or 3·28 ft.) under water. Likewise no considerable change in the efficiency occurred when the quantity of water varied in the ratio 3 to 5. The efficiency also decreased with the height of the regulating sluice, so that, for example, with an opening of 0·05 metre (0·164 ft.), and with the most advantageous number of revolutions (58), it came out only 0·373. Morin also made special experiments in regard to the ratio $\frac{v}{\sqrt{2gh}}$, and found, in complete harmony with theory, that this ratio increases with $v$ (on account of the centrifugal force), but decreases when the sluice is opened more.

§ **273.**—Redtenbacher, in his work "Über die Theorie und den Bau du Turbinen und Ventilatoren," gives the results of the experiments on a turbine at Siebenen in Switzerland. This turbine had the following dimensions and proportions: $r_1 = 0·938$ metre (3·077 ft.), $r = 1·128$ metre (3·701 ft.), $h = 1$ metre (3·281 ft.), $e = 0·254$ metre (0·833 ft.), $Q = 0·2$ cubic metre (7·06 cubic ft.), $\alpha = 12°$, $\beta = 45°$, $\delta = 10°$, etc. The principal results of the experiments with this wheel were the following: with an opening of the regulating sluice $e_1 = 0·1$ metre (0·328 ft.), the best number of revolutions was 17·5, and the corresponding maximum efficiency was $\eta = 0·464$; if the sluice was opened $e_1 = 0·2$ metre (0·656 ft.), the maximum efficiency was then $\eta = 0·646$, with 21·1 revolutions per minute; and when the sluice was opened $e_1 = 0·254$ metre (0·833 ft.), the maximum effect came out only 0·640, with 20·6 revolutions. These percentages, so comparatively small, Redtenbacher justly attributes to the too great curvature of the floats. This turbine also ran in the air.

Besides other interesting results which Redtenbacher deduces from the workings and relations of the well-known Fourneyron turbines, this in particular may be pointed out, that such a wheel, when giving its maximum effect and with the sluice fully open, makes half as many revolutions as when it runs freely—*i. e.*, without doing any work.

The experiments which Combes made on his reaction-wheels, with and without guide blades, likewise led to smaller percentages of efficiency. On a model wheel without guides, 0·14 metre (0·459 ft.) in exterior diameter and with 25 floats, this percentage amounted in the most favorable case to only 51·1, the number of revolutions being 335 per minute, the fall 0·48 metre (1·575 ft.) and the quantity of water 285 litres (10·07 cubic ft.) per minute. With a model wheel of the same size, which had 20 guide blades and 30 floats, and with the angles $\alpha = 30^\circ$ and $\beta = 90^\circ$, the efficiency $\eta$ proved to be at the most $= 0·566$, and occurred with a head of pressure of 0·81 metre (2·658 ft.), 199 revolutions per minute and 372 litres (13·14 cubic ft.) of water per minute. With a wheel of the true size, which was used for working pumps in Paris, it was also found to be only 0·53. This wheel had an exterior diameter of 0·97 metre (3·183 ft.), a depth of 0·16 metre (0·525 ft.), a fall of 0·91 to 1·83 metres (2·986 to 6·004 ft.), and used between 400 and 85 litres (14·13 and 3·00) of water per second. The number of vanes was 36, while the guide blades were entirely absent, and the number of revolutions per minute, with the maximum power of 117·75 kilog. metres (851·33 ft. lbs.), was 75.

Extensive experiments with two Fourneyron turbines have been made by Morris in Delaware. See the Journal of the Franklin Institute, December, 1843, and the "Polytechn. Centralblatt," 1844, Heft X. The first of the two experimental wheels had an exterior diameter of $4\frac{2}{3}$ ft. and a depth of 8 inches, its fall was about 6 ft., and it used on an average 1700 cubic ft. per minute. The highest efficiency, 70 per cent, occurred with the highest opening of the sluice, 6 inches, and with 52 revolutions, or a velocity of the inner circumference of the wheel $v_1 = 0·46 \sqrt{2gh}$. Moreover, $\eta$ varied, for variations of $v_1$ from $v_1 = 0·5 \sqrt{2gh}$ to $0·9 \sqrt{2gh}$, only between the limits 0·64 to 0·70. The second wheel had an exterior diameter of 4 ft. 5 inches, a depth of 6 inches, a fall of about 4½ ft., and used 14 cubic ft. of water per second. It ran under water, and when the sluice was opened 4½ inches, it gave the following efficiences: When $v_1$ was 25 to 30 per cent of $\sqrt{2gh}$, $\eta$ came out $= 0 \cdot 63$; with $v_1 = 40$ to 50 per cent of $\sqrt{2gh}$, $\eta$ was equal to $0 \cdot 71$; with

$$\frac{v_1}{\sqrt{2gh}} = 0 \cdot 45, \text{ or } u = 49,$$

the maximum efficiency was obtained—viz., $\eta = 0 \cdot 75$; with

$$\frac{v_1}{\sqrt{2gh}} = 0 \cdot 5 \text{ to } 0 \cdot 7,$$

$\eta$ came out equal to $0 \cdot 60$.

NOTE.—Later experiments have been made by Marozeau with a turbine with horizontal divisions. They gave a mean efficiency of 0·60. See "Polytechn. Centralblatt," Jahrg., 1848, or "Bulletin de Mulhouse," 1846, Nr. 101. Capt. M. Ordinaire de Lacolange has also made recent experiments on a Fourneyron turbine. See "Civilingenieur," Bd. III., 1857. Lacolange has published these experiments in a special work under the title, "Théorie de la turbine Fourneyron d'après M. Weisbach, etc., suivie d'expériences, etc.," Bordeaux, 1856.

§ **274.—Hydropneumatic Arrangement of Girard.**—To increase the efficiency of turbines, special means have also been used. Above all, the hydropneumatic arrangement of Girard should be considered here, and also the use of Boyden's diffuser. The essential characteristics of both these expedients may be given in the following:

The hydropneumatic process of Girard consists in this: the wheel chamber of the turbine is surrounded by an air-tight casing, the space so set off is filled with compressed air, and thus an efflux under water is prevented. It is, of course, a matter of fact that a turbine performs less work when running under water than when running in the air; but when the sluice is fully opened, this difference is not great enough to take especial means to obviate it. The case is entirely different, however, when the turbine runs under water with its regulating sluice partly lowered. Though the water may still issue with a full section—and this must always be the case when the turbine runs under water—yet at the entrance of the water from the supply chamber into the wheel there is a sudden change of velocity, and in consequence a considerable loss of pressure. This loss is the greater the more the sluice is lowered, and, therefore, the smaller the height of the sluice openings or $e_1$ is in comparison with $e$ the depth of the wheel. Denoting by $c$ the absolute velocity of the water at its entrance into the wheel, and, therefore, by $\frac{e}{e_1} c$ the velocity of efflux of the water from the sluice, we have the corresponding loss of head of pressure,

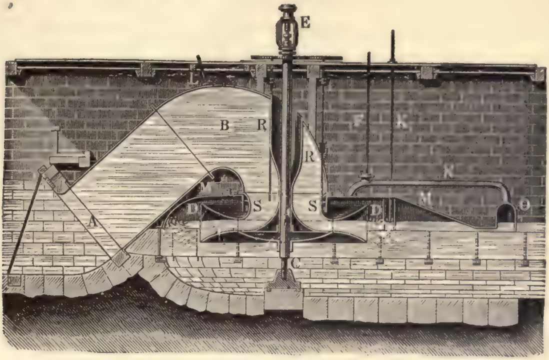
$$\frac{1}{2g}\left(\frac{e}{e_1} c - c\right)^2 = \frac{c^2}{2g}\left(\frac{e}{e_1} - 1\right)^2 \quad \text{(Compare § 257).}$$

This loss entirely vanishes when the water, in its motion through the wheel, does not fill its passages; when, therefore, we are dealing with a pressure turbine. But since this can occur only with efflux into the air, a special advantage is afforded, when the sluice is lowered, by the introduction of air to separate the tail water from orifices of the wheel.

The arrangement of such a turbine with hydropneumatization can be seen in Fig. 509. The turbine here represented uses from 3 to 5 cubic metres (106 to 177 cubic ft.) of water per second; it has the small fall of only 0·45 to 0·60 metre (1·476 to 1·969 ft.), and with an exterior diameter of 3½ metres (11·483 ft.), it makes only 20 revolutions per minute. Girard constructed this turbine for a spinning-mill at Eindhoven, in Holland.

In order that the water might enter the wheel undisturbed, it had to be brought to the supply chamber by a bent pipe *A B*, in the form of a siphon. A peculiarity of this turbine is the gradual flaring of the wheel *D D* (in French, évasement) from the inside out. Since the section $F_2$ of the orifices of discharge of the wheel passages is thereby increased, and consequently the velocity of efflux is diminished, a greater part of the energy of the water can by this means be derived from it than when the passages have everywhere the same height. It should of course be added that to derive much benefit from this increase of width, the water, in its efflux from the wheel, must actually fill the passages, which does not happen in the case of efflux in the air or with a considerable flaring of the wheel from within out, whereby the cross-section $F_2$ is brought very near to the section $F_1$, especially when the sluice is not entirely opened. The hollow shaft *C E* is suspended at *E*, and turns about a slender fixed spindle, the foot of which can be seen at *C*.

Fig. 509



An air compressor, which is itself set in motion by the turbine, by means of a pipe *F*, forces air into the wheel chamber which is surrounded by the casing *M*, and another pipe *K* carries off the air when it becomes too much condensed, in order that the surface of the water in the casing may keep at a certain height. A bell *O* collects the air which is carried off with the water, and by the pipe *N* this air is carried back into the wheel chamber. The arrangement, method of suspension and of moving the regulating sluice *S S* are of the usual sort. The pipe *R R* which surrounds the shaft has an oblong section in order to present as little resistance as possible to the motion of the water. The air which collects in the top of the siphon *A B* can be removed by means of a pipe *L* and a small suction pump.

**275.—Boyden's Diffuser.**—It is not only to the turbines of Girard, but also to the older and the later constructions, as, *e.g.*, to those of Boyden and Francis, that a conical form has been given to the crowns of the wheel, in order to increase the section $F_2$ of the orifice of efflux. The advantage thereby obtained will appear from the following considerations: If $e$ denotes the exterior and $e_1$ the interior depth of the wheel, we must put

$$\frac{c_2}{c} = \frac{v}{c} = \frac{F}{F_2} = \frac{r_1 \sin. \alpha}{r \sin. \delta} \cdot \frac{e_1}{e},$$

so that we have the following expression for $\delta$, the angle of exit,

$$\sin. \delta = \frac{r_1}{r} \cdot \frac{e_1}{e} \cdot \frac{c}{r} \cdot \sin. \alpha = \left(\frac{r_1}{r}\right)^2 \frac{e_1}{e} \cdot \frac{\sin. \alpha \sin. \beta}{\sin. (\beta - \alpha)};$$

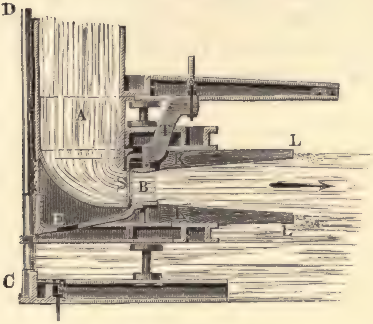
while for turbines with plane crowns, in which $e_1 = e$,

$$\sin. \delta = \left(\frac{r_1}{r}\right)^2 \frac{\sin. \alpha. \sin. \beta}{\sin. (\beta - \alpha)}.$$

Hence we can reduce the angle of exit $\delta$, and therefore the absolute velocity of efflux $w$, and transmit to the wheel more energy from the discharged water, when we make the outer depth of wheel $e$ greater than the inner depth $e_1$.

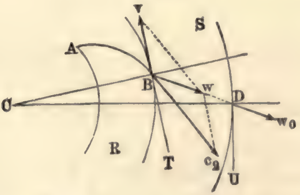
Another expedient to attain the same end is in the use of Boyden's diffuser. This consists of an annular space which likewise gradually widens from the inside out, and which encloses the wheel. Through it, the water is carried out of the wheel into the wheel pit or to the tail water. Fig. 510 shows a section of a part of such a turbine with the diffuser, as designed by Francis. *C D* is the right-hand half of the shaft; *A* the supply chamber; *B E* the wheel; *K L*, *K L* are the crowns which form the diffuser, made of wooden staves. The annular sluice *S* moves between the wheel and the supply chamber, and is kept in place by means of the arms *T*., etc., of the revolving wheel itself.

FIG. 510.



The action of this diffuser will appear from the following: Let *A B R*, Fig. 511, be a part of the wheel, and *B D S* a portion of the diffuser. The relative velocity $c_2$ with which the water issues from the wheel at *B*, and the velocity of revolution *v*, have for their resultant the absolute velo-

FIG. 511.



city $w$ with which the water enters the diffuser. The water passes through this diffuser almost in a straight line $BD$, and then comes out at $D$ with a velocity $w_0$ yet to be determined. Let us put the radius $CB = r$, $CD = r_0$, the inner and outer depths of the diffuser $= e$ and $e_0$, and also the angle of exit $TBc_2$, as before, $= \delta$, the angle $TBD$, at which the water enters the diffuser, $= \theta$, and the angle $UDw_0$, at which it issues from it, $= \theta_0$. We then have, since

$$\frac{sin.\ CDB}{sin.\ CBD} = \frac{CB}{CD},$$

or

$$\frac{cos.\ \theta_0}{cos.\ \theta} = \frac{r}{r_0},$$

and

$$r\,e\,v\,sin.\,\delta = r\,e\,w\,sin.\,\theta = r_0\,e_0\,w_0\,sin.\,\theta_0,$$

the velocity of efflux

$$w_0 = \frac{r}{r_0} \cdot \frac{e}{e_0} \cdot \frac{v\,sin.\,\delta}{sin.\,\theta_0} = \frac{r}{r_0} \cdot \frac{e}{e_0} \cdot \frac{w\,sin.\,\theta}{sin.\,\theta_0}$$

$$= \frac{r}{r_0} \cdot \frac{e}{e_0} \cdot \frac{w\,sin.\,\theta}{\sqrt{1 - \left(\frac{r}{r_0} \cdot cos.\,\theta\right)^2}}.$$

Since $\frac{r}{r_0}$ and $\frac{e}{e_0}$ are proper fractions, $w_0$ is $< w$, and hence $\frac{w_0^2}{2g} \cdot Qy$, the energy of the water at its exit from the diffuser, is less than $\frac{w^2}{2g} \cdot Qy$, the energy of the water when it issues from the wheel.

It should be added, however, that $w$ is greater with the use of the diffuser than without it. If we disregard the resistances to the motion of the water and put the head due to the pressure of the water at its passage from the wheel into the diffuser $= y$, we have

$$c_2^2 = 2\,g\,(h_1 - y) + v^2 - 2\,c\,v_1\,cos.\,\alpha,$$

and

$$w_0^2 = w^2 + 2\,g\,(y - h_2).$$

Assuming also $c_2 = v$, we have $\theta = 90^\circ + \frac{\delta}{2}$, and consequently

$$w_0^2 = w^2 + 2\,g\,(h_1 - h_2) - 2\,c\,v_1\,cos.\,\alpha$$

$$= w^2 + 2\,g\,h - 2\,c\,v_1\,cos.\,\alpha,$$

or inserting

$$w = 2\,v\,sin.\,\frac{\delta}{2},$$

$$c = \frac{v_1\,sin.\,\beta}{sin.\,(\beta - \alpha)},$$

and

$$w_0 = \frac{r}{r_0}\cdot\frac{e}{e_0}\,\frac{v\,sin.\,\delta}{sin.\,\theta_0} = \frac{r}{r_0}\cdot\frac{e}{e_0}\,\frac{v\,sin.\,\delta}{\sqrt{1 - \left(\frac{r}{r_0}\cdot sin.\,\frac{\delta}{2}\right)^2}},$$

we have

$$\left[2\left(\frac{r_1}{r}\right)^2\frac{sin.\,\beta\cdot cos.\,\alpha}{sin.\,(\beta - \alpha)} - \left(2\,sin.\,\frac{\delta}{2}\right)^2 + \left(\frac{r\,e}{r_0\,e_0}\right)^2\frac{sin.^2\,\delta}{1 - \left(\frac{r}{r_0}\,sin.\,\frac{\delta}{2}\right)^2}\right]v^2 = 2\,g\,h,$$

and hence the corresponding velocity of the revolution is

$$v = \sqrt{\frac{g\,h}{\left(\frac{r_1}{r}\right)^2\cdot\frac{sin.\,\beta\,cos.\,\alpha}{sin.\,(\beta - \alpha)} - 2\,sin.^2\frac{\delta}{2}\left[1 - \left(\frac{r\,e}{r_0\,e_0}\right)^2\frac{cos.^2\frac{\delta}{2}}{1 - \left(\frac{r}{r_0}\,sin.\,\frac{\delta}{2}\right)^2}\right]}}.$$

This value has to be introduced into the formula

$$w_0 = \frac{r}{r_0}\cdot\frac{e}{e_0}\cdot\frac{v\,sin.\,\delta}{sin.\,\theta_0},$$

to obtain the velocity of the discharged water.

EXAMPLE.—In the example in § 264, the most advantageous velocity of revolution was found to be

$$v_1 = 13 \cdot 319 \text{ ft.}, \qquad \text{and} \qquad v = 1 \cdot 35 \times 13 \cdot 319 = 17 \cdot 980 \text{ ft.},$$

so that the absolute velocity of discharge is

$$w = 2\, v \, sin. \frac{\delta}{2} = 2 \times 17 \cdot 980 \times sin.\, 8^\circ\ 21' = 5 \cdot 2222 \text{ ft.},$$

and hence the corresponding loss of work proves to be

$$\frac{w^2}{2\,g} \cdot Q\, y = 0 \cdot 0155\, (5 \cdot 222)^2\, Q\, \gamma = 0.423\, Q\, \gamma.$$

But if we surround the wheel with a diffuser whose radius $r_0 = 2\, r_1$, and exterior depth $e_0 = \frac{4}{3}\, e$, since

$$\left(\frac{r}{r_0}\right)^2 = (\tfrac{1}{2})^2 = \tfrac{1}{4},$$

and

$$\left(\frac{r}{r_0} \cdot \frac{e}{e_0}\right)^2 = (\tfrac{1}{2} \cdot \tfrac{3}{4})^2 = \tfrac{9}{64},$$

and

$$sin.^2 \frac{\delta}{2} = sin.^2\, (8^\circ\ 21') = 0 \cdot 02109,$$

we have the following equation :

$$2\, sin.^2 \frac{\delta}{2} \left[ 1 - \left(\frac{r\, e}{r_0\, e_0}\right)^2 \frac{cos.^2 \frac{\delta}{2}}{1 - \left(\frac{r}{r_0}\, sin.\, \frac{\delta}{2}\right)^2} \right]$$

$$= 0 \cdot 04218 \left(1 - \tfrac{9}{64} \frac{1 - 0 \cdot 02109}{1 - 0 \cdot 00527}\right)$$

$$= 0 \cdot 04218\, (1 - 0 \cdot 1384) = 0 \cdot 03634,$$

and hence the corresponding velocity of revolution of the wheel,

$$v = \sqrt{\frac{161 \cdot 0}{\left(\frac{1 \cdot 00}{1 \cdot 35}\right)^2 (0 \cdot 9076) - 0 \cdot 03634}} = \sqrt{\frac{161 \cdot 0}{0 \cdot 4617}} = 18 \cdot 674 \text{ ft.}$$

We therefore obtain the following value for the velocity with which the water issues from the diffuser :

$$w_0 = \frac{3}{8} \cdot \frac{18 \cdot 674 \; \sin. \; 16^\circ \; 42'}{\sqrt{[1 - 0 \cdot 00527]}} = \frac{7 \cdot 003 \times 0 \cdot 2874}{\sqrt{0 \cdot 99473}} = 2 \cdot 018 \text{ ft.},$$

and, finally, the mechanical work lost thereby,

$$\frac{w_0^2}{2g} Q \gamma = 0 \cdot 0155 \, (2 \cdot 018)^2 \, Q \gamma = 0 \cdot 0632 \, Q \gamma,$$

whereas, without the diffuser, this loss with the same turbine comes out

$$\frac{w^2}{2g} \cdot Q \gamma = 0 \cdot 423 \, Q \gamma,$$

and therefore nearly seven times as large.

Since $Q = 30$ cubic ft., we have

$$\frac{w^2}{2g} \cdot Q \gamma = (0 \cdot 423) \, (30) \, (62 \cdot 5) = 794 \cdot 0 \text{ ft. lbs.}$$

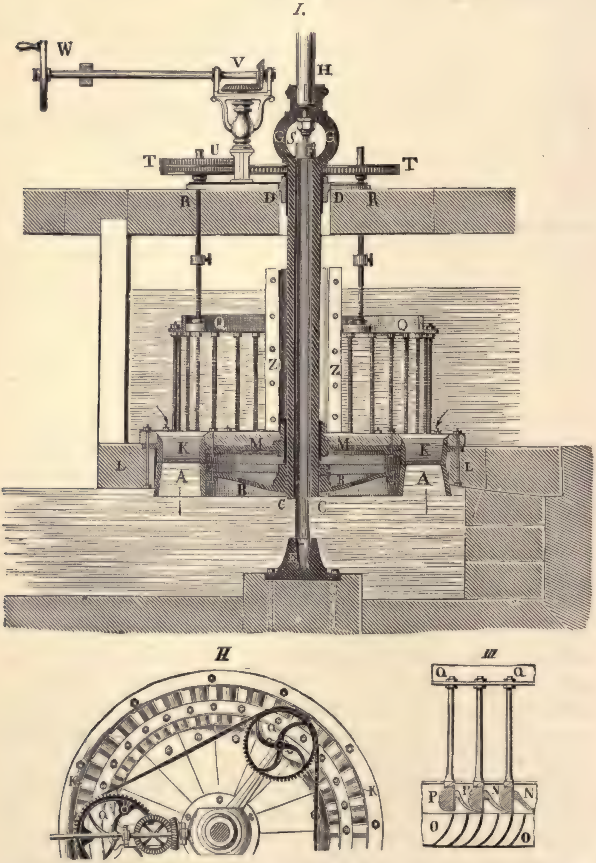
and

$$\frac{w_0^2}{2g} \cdot Q \gamma = (0 \cdot 0632) \, (30) \, (62 \cdot 5) = 118 \cdot 5 \text{ ft. lbs.}$$

§ **276.—Fontaine's Turbine.**—The turbines of Fontaine, Henschel, and Jonval differ from those of Fourneyron in so far as the guide blades of the former are not by the side of the wheel, but above it, and therefore the water is not conducted to the wheel from the centre nor from the circumference, but from above, and it does not issue from the periphery, but at the bottom of the wheel. In the downward motion of the water from above, in passages also made by curved floats, centrifugal force plays but a subordinate part, and gravity takes its place. There is a difference between the turbine of Fontaine and that of Henschel, since in the former the surface of the tail water stands directly above or below the wheel, while in the latter the water issuing from the wheel forms above the surface of the tail water a column of water, which has its influence upon the motion of the wheel, just as if it stood over the latter. Jonval's turbine is an improved form of Henschel's.

The arrangement of a Fontaine turbine can be seen in Fig. 512 (I. and II.), which shows a vertical section and a

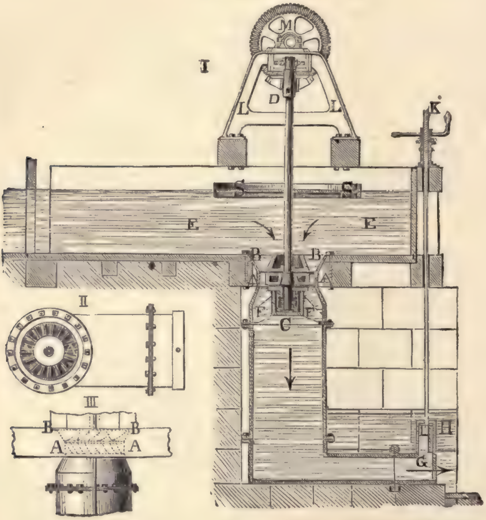
Fig. 512.



plan. *A A* is the wheel, *B B* the centre plate by which, instead of arms, the wheel is united to the hollow shaft *C C D D*. In order that the pivot may not be under water, the shaft *C D* ends in an eye *G G*, in which a steel pin *F S* is put, which can be raised or lowered by the nut *S*, and the shaft revolves in a steel bearing in the head *F* of a fixed spindle *E F*. The revolution is transmitted by a vertical shaft *H* inserted above the eye *G*. To protect the vertical shaft of the turbine from the water, it is surrounded, as in Fourneyron's turbine, by a casing *Z Z*. The drum containing the guides *K K* is bolted to the beams *L L*, and a disc *K M M K* is also fastened to it, which carries a cylindrical bearing *M M*. By this and by the upper bearing *D D*, the shaft of the turbine is maintained in a secure position. The form of a guide blade *N* and of a float or vane *O* can be seen in III. To regulate the quantity of water, a sluice arrangement is used, which consists of as many separate sluices *P, P* . . . as the wheel has floats *N N*. . . . These sluices are fitted with pieces of wood rounded off on their edges, and run in grooves which are let into the cylindrical surfaces of the drum which carries the guides. The sluice rods *P Q, P Q* . . . are rigidly connected by an iron ring *Q Q*, which can be raised or lowered by means of three rods *Q R, Q R*. . . . For this purpose, the ends of these rods *R, R* . . . have screw threads cut upon them *T T*, and pass through the toothed wheels *T, T* . . . , whose naves form the corresponding female screws, and whose circumferences are connected with each other by an endless chain. When, therefore, one of the wheels *T* . . . is set in rotation by means of a crank *W* and gearing *U V*, the other wheels move equally with it, and thus all three rods are raised or lowered uniformly.

§ **277.**—**Jonval's Turbine.**—Fig. 513 contains views of Jonval's turbine. These turbines are also called double-acting, because the water acts in them by pressure and suction at the same time. *A A* is the wheel, which is also united to the vertical shaft *C D* by a centre plate; *B B* is the arrangement of guides, set over the wheel, and with conical openings from the tank or penstock *E E*. The bearing rests in a step box *C*, which is supported and held fast by the brackets *F F*. The position of the guide blades

FIG. 513.

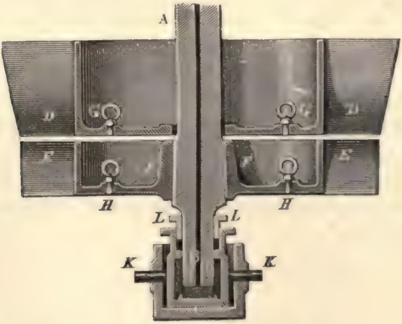


and floats, and also a part of the extension of the pipe in which the wheel is enclosed, are shown in II. and III. To keep the surface of the head-water free from agitation, a wooden float *S S* is placed in it; and in order to regulate the motion of the wheel, a sluice *G* is applied, which can be set higher or lower by a crank and screw. The higher or lower the sluice is, a greater or less quantity of water will, of course, flow down into the tail water *H*, and therefore the wheel will do more or less work. The frame *L L* carries the bearing for the upper pivot of the shaft *C D*, and the bearing of a horizontal shaft, to which the rotation of the wheel is first transmitted by means of the bevel-wheels *M*. The reservoir in which the wheel is enclosed can for small wheels be made of cast-iron, but for larger wheels it must be built of ashlar masonry.

We see, from what has just been given, that the turbines of Fontaine and Jonval correspond perfectly in their principal parts and essential relations, and we can therefore easily infer that they must both be explained by the same theory. In both wheels, the head-water stands at a certain height $h_1$ above the point of entrance into the wheel; but, as regards the tail water, the surface of this stands at a certain height $h_2$ below the wheel in Jonval's turbine, while in Fontaine's it extends up to the wheel, or it even stands above the wheel. As to the means for regulating the motion of these two turbines, it must be observed that Fontaine's are fitted with an inner sluice, while Jonval's turbines have an outer sluice, and that therefore the former is, to this extent, to be compared with Fourneyron's, and the latter with one of Cadiat's turbines.

The Henschel or Jonval turbines have been lately used in many forms and with very good results. The vertical section of a simple wheel of this sort is represented in Fig. 514. The shaft *A B* is bored longitudinally through

FIG. 514.



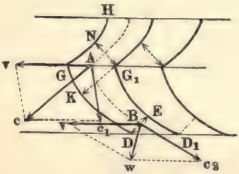
its axis, in order to conduct oil to the surfaces in contact between the pivot *B* and the bearing *C*. *D D* is the ring-shaped piece which carries the guide blades (the guide wheel), and *E E F F* the wheel proper (the running wheel); the discs *G G* and *H H* are provided with holes *G* and *G*, by which impurities, such as sand, slime, etc., can be carried off from time to time. The manner of centring the pivot by screws *K K*, and protecting it from the access of the water by a stuffing-box, can be seen clearly from the figure.

NOTE.—Professor Rühlmann justly mentions in the "Zeitschrift des Hannoverschen Architekten- und Ingenieurvereins," Bd. I., in the article "Beitrag zur Geschichte der horizontalen Wasserräder," that the so-called Jonval or Köchlin turbine is not an invention of Jonval, but of Oberbergrath Henschel, of Cassel. Henschel had designed such a turbine as early as 1837, and in 1841 he set one up in some stone works at Holzminden. Sectionsrath Rittinger calls these wheels "pipe turbines."

## § 278.—Theory of the Fontaine-Henschel Turbines.—

In explaining the theory of the Fontaine-Henschel turbines, we shall make use of the following notation, which will be found to be in complete harmony with what precedes.

FIG. 515.



Let the angle of inclination of a guide *H N G* to the horizon, or the so-called entrance angle $N G G_1 = c A v$, Fig. 515, be $\alpha$, the angle $c_1 A v$, which is included between *A*, the upper part of a float, and the direction of the motion of the wheel, be $\beta$, and the angle $D D_1 E$, at which the foot *D* of the float or vane is inclined to the horizon, be $\delta$; further, let the absolute velocity with which the water enters the wheel be $\overline{A c} = c$; let the velocity of the wheel at the point corresponding to the mean radius

$$r = \frac{r_1 + r_2}{2}$$

be $\overline{A v} = v$, the relative velocity of entrance $\overline{A c_1} = c_1$, and the velocity of discharge $\overline{B c_2} = c_2$. Finally, as before, let *F* be the sum of the areas of all the sections $N G_1$ of the water coming out of the guides, $F_1$ the sum of the upper sections $G_1 K$, and $F_2$ that of the lower sections $D E$ of the wheel passages.

Denoting again by $\zeta$ the coefficient of resistance in the passages between the guides, and by $x$ the hydraulic head of the water as it enters the wheel, we have, as before,

$$(1 + \zeta)\, c^2 = 2 g\, (h_1 - x),$$

and, taking into consideration the atmospheric pressure which is to be measured by the height $b$ (33·9 feet) of a column of water,

$$(1 + \zeta)\, c^2 = 2 g\, (b + h_1 - x).$$

For the relative velocity of entrance, we have, as before,

$$c_1^2 = c^2 + v^2 - 2\, c\, v \cos. \alpha.$$

Moreover, if $a$ is the depth of the wheel, $y$ the height of a column of water which measures the pressure of the water directly under the wheel, and $\zeta_1$ the coefficient of resistance in the wheel passages, we have for the relative velocity with which the water issues from the latter,

$$\begin{aligned}(1 + \zeta_1)\, c_2^2 &= 2\, g\, (a + x - y) + c_1^2 \\ &= 2\, g\, (b + h_1 + a - y) + v^2 - 2\, c\, v \cos. \alpha - \zeta\, c^2.\end{aligned}$$

Assuming again that, to extract as much energy as possible from the water, $c_2$ must be equal to $v$, and, moreover, introducing

$$c = \frac{v \sin. \beta}{\sin\, (\beta - \alpha)},$$

we obtain for $v$, the velocity of revolution of the wheel,

$$\left[\frac{2 \sin \beta \,.\, \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 + \zeta_1\right] v^2$$
$$= 2\, g\, (b + h_1 + a - y),$$

and hence the most economical velocity

$$b = \sqrt{\frac{2g(b + h_1 + a - y)}{\frac{2 \sin. \beta \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 + \zeta_1}}.$$

The pressure head $y$ is equal to the height $b$ which measures the pressure of the atmosphere, when the turbine runs in the air; but when it runs under water, it is equal to $b + h_2$, in which $h_2$ denotes the height of the surface of the tail water above the bottom of the wheel; and, finally, when it runs above the water, as in Jonval's turbine, it is equal to $b - h_2 + z$, in which $h_2$ denotes the depth of the surface of the water in the tail race below the bottom of the wheel, and $z$ the height due to the velocity of the water coming out of the reservoir through the sluice into the tail race. The total fall, when the wheel runs in the air, is

$$h = h_1 + a;$$

when it runs under water,

$$h = h_1 + a - h_2;$$

when it runs above the water,

$$h = h_1 + a + h_2;$$

and hence for the first two cases we have

$$v = \sqrt{\frac{2gh}{2\frac{\sin. \beta \,.\, \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 + \zeta_1}},$$

and for the last,

$$v = \sqrt{\frac{2g(h - z)}{2\frac{\sin. \beta \,.\, \cos. \alpha}{\sin. (\beta - \alpha)} + \zeta \left(\frac{\sin. \beta}{\sin. (\beta - \alpha)}\right)^2 + \zeta_1}},$$

and, also, when the opening $G$, by which the vessel communicates with the tail race, is very large, and therefore the water flows off very slowly, we can put

$$z = \frac{1}{2g} \cdot \left(\frac{Q}{G}\right)^2 = 0.$$

From the velocity $v = c_2$, the absolute velocity of entrance can be computed

$$c = \frac{v \sin. \beta}{\sin. (\beta - \alpha)},$$

and the hydraulic head

$$x = b + h_1 - (1 + \zeta) \frac{c^2}{2g} = b + h_1 - (1 + \zeta) \frac{v^2 \sin.^2 \beta}{2g \, . \, \sin.^2 (\beta - \alpha)}.$$

Without regard to the accompanying resistances,

$$x = b + h_1 - \frac{h \sin. \beta}{2 \sin. (\beta - \alpha) \cos. \alpha},$$

and if we leave out of consideration the pressure of the atmosphere,

$$x = h_1 - \frac{h \sin. \beta}{2 \cos. \alpha \sin. (\beta - \alpha)},$$

$x$ becomes $= 0$, or rather equal to the pressure of the air outside ($b$), when

$$h_1 = \frac{h \sin. \beta}{2 \cos. \alpha \sin. (\beta - \alpha)}.$$

The loss of water caused by the imperfect joint depends upon the difference between the inner pressure ($x$) and the pressure outside, at the point where the water passes over into the wheel, and is different in Fontaine's turbine from what it is in Jonval's. In order that the water may flow in a coherent stream, $x$ must never become equal to zero, and therefore we must have

$$b + h_1 > \frac{h \sin. \beta}{2 \sin. (\beta - \alpha) \cos. \alpha};$$

moreover, that the water may not leave the bottom of the wheel, $y$ also must never become equal to zero, and therefore

$$b - h_2 + z > 0,$$

*i. e.*,

$$h_2 < b + z, \qquad \text{or} \qquad h_2 < b + \frac{1}{2g}\left(\frac{Q}{G}\right)^2;$$

and therefore, when the area of the discharge opening $G$ is very great, $h_2 < b$. Accordingly, the height of the wheel above the surface of the water in the tail race must never amount to the height of the water barometer $b = 33 \cdot 9$ ft.

When the reservoir of a Jonval turbine is high and narrow, so that the water moves in it with a velocity not inappreciable, we must consider some other losses in this reservoir; such as, *e. g.*, friction, the resistance of curves, the impact of the water from a sudden change of velocity, etc. It is advisable to give the reservoir a greater width than the space occupied by the wheel, in order to make these losses as slight as possible.

§ 279.—**Delivery of the Fontaine-Henschel Turbines.**—The mechanical effect of a Fontaine-Henschel turbine can be determined in almost exactly the same manner as that of a Fourneyron turbine by deducting from the total power $Q h \gamma$ the mechanical effects corresponding to the accompanying resistances. First, we have the loss in the passages between the guides,

$$L_1 = \zeta \frac{c^2}{2g} \cdot Q \gamma,$$

and then that in the wheel passages,

$$L_2 = \zeta_1 \frac{c_2^2}{2g} Q \gamma,$$

and, thirdly, the loss which corresponds to the energy which the water retains when discharged out of the wheel,

$$= \frac{w^2}{2g} Q \gamma = \frac{\left(2 v \sin. \frac{\delta}{2}\right)^2}{2g} Q \gamma.$$

In Jonval's turbines there is to be added to these the loss of energy which corresponds to the production of the velocity of efflux of the water through the slide $w_1$, and which must be put equal to

$$= \frac{w_1^2}{2g} Q \gamma = \frac{1}{2g} \cdot \frac{Q^3}{G^2} \cdot \gamma.$$

Accordingly, we can put the total power of the wheel equal to

$$L = \left(h - (\zeta c^2 + \zeta_1 c_2^2 + w^2 + w_1^2) \frac{1}{2g}\right) Q \gamma,$$

and we can also easily infer that the loss of power comes out greater, the greater the angle of exit $\delta$ is, and the greater the velocity of discharge $w_1$, or the smaller the orifice of discharge or sluice opening $G$ is. With the sluice fully open and a wide reservoir, we must put $w_1 = 0$. We see from this that the efficiency of Henschel's turbine decreases when the quantity of water used is smaller or the sluice set lower. As to Fontaine's turbine, in regard to the position of the sluice, the same relations hold in it as in Fourneyron's, since in this case also a lowering of the sluice causes the water to enter the wheel with a shock, and thereby produces a destruction of energy.

From all this, we must conclude that the efficiencies of these turbines of Fontaine and Henschel cannot come out to any considerable extent greater than the efficiencies of Fourneyron's turbines under similar circumstances, as is also fully confirmed by the experiments adduced below. From the experiments of the author, we must in this case also take

$$\zeta = \zeta_1 = 0{\cdot}075.$$

§ **280.—The Designing of the Fontaine-Henschel Turbines.**—We must now give the principal rules for designing and constructing Fontaine and Henschel turbines. First, the angles of the floats, $\beta$ and $\delta$, are chosen arbitrarily, the latter being as small as possible, *i. e.*, 15° to 20°, and the former anywhere from 100° to 120°. From $\beta$ and $\delta$ the angle of guides $\alpha$ is at once obtained, since, to prevent the water entering with a shock, we put

$$c_1 \sin.\beta = c_2 \sin.\delta = v \sin.\delta,$$

and

$$\frac{c_1}{v} = \frac{\sin.\alpha}{\sin.(\beta - \alpha)},$$

and therefore by combination,

$$\frac{\sin.\alpha}{\sin.(\beta - \alpha)} = \frac{\sin.\delta}{\sin.\beta};$$

whence follows the following equation :

$$\frac{\sin.(\beta - \alpha)}{\sin.\alpha \sin.\beta} = \frac{1}{\sin.\delta},$$

or

$$(1) \quad \cot.\alpha = \cot.\beta + \frac{1}{\sin.\delta}.$$

From the angles $\alpha$ and $\beta$ is obtained the mean velocity of the wheel,

$$(2)\quad v = \sqrt{\frac{2gh}{2\frac{\sin.\beta\cos.\alpha}{\sin.(\beta-\alpha)} + \zeta\left(\frac{\sin.\beta}{\sin.(\beta-\alpha)}\right)^2 + \zeta_1}},$$

and the velocity of entrance,

$$(3)\quad c = \frac{v\sin.\beta}{\sin.(\beta-\alpha)}.$$

From these velocities, we obtain the cross-sections

$$(4)\quad F = \frac{Q}{c},$$

and

$$(5)\quad F_2 = \frac{Q}{v}.$$

The width of the wheel or the length of a float, measured in the direction of the radius, may be taken in a convenient ratio,

$$\nu = \frac{e}{r},$$

to the mean radius of the wheel $r$. For small turbines, we can take $\nu = 0{\cdot}4$, but for those of large size $\nu = 0{\cdot}2$. Likewise for the ratio,

$$\lambda = \frac{e}{d},$$

of the length of the floats, or length $e$ of the outlets from the wheel to their width $d$, a certain value $= 2$ to $4$ may be taken; accordingly, if $n$ is the number of floats and $s$ their thickness, we have not only

$$F_2 = 2\pi r e \sin.\delta - n s e = \frac{2\pi e^2}{\nu}\sin.\delta - n s e,$$

but also

$$F_2 = n\,d\,e = \frac{n\,e^2}{\lambda},$$

and hence

$$F_2 = \frac{2\,\pi\,e^2}{v}\,sin.\,\delta - \frac{\lambda\,F_2\,s}{e},$$

from which we obtain the following value for the length of a float:

$$(6)\quad e = \sqrt{\frac{F_2\,v}{2\,\pi\,sin.\,\delta}}\cdot\left[1 + \lambda\,s\sqrt{\frac{\pi\,sin.\,\delta}{2\,F_2\,v}}\right],$$

and for the width of outlets we also obtain

$$(7)\quad d = \frac{e}{\lambda},$$

and for the mean radius,

$$(8)\quad r = \frac{e}{v},$$

and number of floats,

$$(9)\quad n = \frac{F_2}{d\,e} = \frac{\lambda\,F_2}{e_2}.$$

The number of guides is taken equal to or, at most, only one fourth less than the number of vanes, and the depth of the wheel is made about equal to its width or to the length of the vanes.

§ 281.—**Method of Construction of the Floats and Guides.**—The floats and guides are warped surfaces, whose generating line on one side passes through the axis of the wheel at right angles, and on the other side through a directrix which we can imagine drawn on the surface of a cylinder described with the mean radius $r$. Since a rectangle is produced by developing a cylinder upon a plane, we can draw lines on this rectangle which, when the latter is wrapped around the cylinder again, will serve as directrices for the surfaces of the guides and floats. The development of these directrices can be made best with straight lines and arcs of circles. If $L K$, Fig. 516, is the development of the circle of contact of the wheel and the drum which contains the guides, the line $A N D$ of a guide is found by setting off

$$A A_1 = \frac{2 \pi r}{n},$$

and drawing $A N$, $A_1 N_1$, so that the angle of inclination $N A L = N_1 A_1 L \ldots$ shall be $= \alpha$; if we now let fall $A O_1$ perpendicularly upon $A_1 N_1$, and from the point of intersection $O$ of this normal $A O_1$ with a line parallel to $K L$, and which limits the upper portion of the guides, describe a circular arc $N_1 D_1$, and in the same way from another point $O$ describe the arc $N D \ldots$, etc.; $A N D$, $A_1 N_1 D_1$, etc., will be the developments of the directrices of the guides. To find the directrices for the vanes, we draw the straight line $E G$ parallel to $K L$, and at a distance $E L$ below it equal to $a$, the depth of the wheel, make

$$E E_1 = \frac{2 \pi r}{n},$$

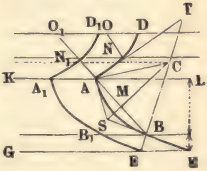
and draw the straight lines $E B$, $E_1 B_1$, etc., so that the angle

$$B E G = B_1 E_1 G$$

shall be equal to $\delta$, the angle of exit. We then let fall $E_1 B$ perpendicularly upon $B E$, and draw $A B$ so as to fulfil the relation

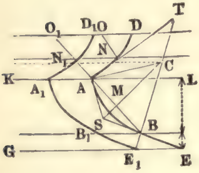
$$A B C = A S C = \frac{\beta + \delta}{2},$$

FIG. 516.



and, finally, we erect at $M$, the middle of the line $A\,B$, the perpendicular $M\,C$, which will intersect $B\,T$ in $C$, the centre of the arc $A\,B$, which constitutes the upper portion of the development of the directrix of a float, while the straight line $B\,E$ makes the lower part.

FIG. 517.



We can easily see that by this method of constructing the guides and vanes, the water will issue from the guides and from the wheel without contraction, and with the sections $A\,N_1$ and $B\,E_1$.

EXAMPLE.—A Henschel turbine has to be designed and computed, for which we have at our disposal a quantity of water $Q = 8$ cubic ft. per second, with a fall $n$ of 12 ft. Taking $\delta = 15^\circ$ and $\beta = 110^\circ$, we obtain

$$cot.\ \alpha = cot.\ \beta + \frac{1}{sin.\ \delta} = -\,cot.\ 70^\circ + \frac{1}{sin.\ 15^\circ}$$

$$= -\,0 \cdot 36397 + 3 \cdot 863704 = 3 \cdot 4997,$$

and consequently

$$(1)\quad \alpha = 15^\circ\ 57',$$

and therefore to be made about 16°. Putting $\zeta = \zeta_1 = 0 \cdot 08$, we find the most advantageous velocity of the wheel in the division circle,

$$(2)\quad v = \sqrt{\frac{2\,g\,h}{2\,\frac{sin.\ \beta\ cos.\ \alpha}{sin.\ (\beta - \alpha)} + \zeta\left(\frac{sin.\ \beta}{sin.\ (\beta - \alpha)}\right)^2 + \zeta_1}}$$

$$= \frac{8 \cdot 025\,\sqrt{12}}{\sqrt{\frac{2\,.\,sin.\ 110\,.\,cos.\ 16^\circ}{sin.\ 94^\circ} + 0 \cdot 08\left(1 + \left(\frac{sin.\ 110^\circ}{sin.\ 94^\circ}\right)^2\right)}}$$

$$= \frac{8 \cdot 025\,\sqrt{12}}{\sqrt{1 \cdot 810992 + 0 \cdot 150987}} = \frac{8 \cdot 025\,\sqrt{12}}{\sqrt{1 \cdot 96198}} = 19 \cdot 847 \text{ ft.},$$

and from this we obtain the corresponding velocity of entrance of the water,

$$(3)\quad c = \frac{v\,sin.\ \beta}{sin.\ (\beta - \alpha)} = \frac{19 \cdot 847\ sin.\ 110^\circ}{sin.\ 94^\circ} = 18 \cdot 695 \text{ ft.}$$

From these velocities, the cross-sections of the orifices are computed,

$$(4)\quad F = \frac{Q}{c} = \frac{8}{18 \cdot 695} = 0 \cdot 4279 \text{ sq. ft.}$$

$$(5)\quad F_2 = \frac{Q}{v} = \frac{8}{19 \cdot 847} = 0 \cdot 40308 \text{ sq. ft.}$$

Taking, now, the ratio $\nu = \frac{e}{r} = 0 \cdot 3$, and the ratio $\lambda = \frac{e}{d} = 3 \cdot 5$, and putting the thickness of the floats $s = 0 \cdot 02$ ft., we obtain the required width of wheel or length of the floats,

$$(6)\quad e = \sqrt{\frac{\nu F_2}{2 \pi \sin. \delta}} \left(1 + \lambda s \sqrt{\frac{\pi \sin. \delta}{2 \nu F_2}}\right)$$

$$= (0 \cdot 2727)(1 \cdot 1283) = 0 \cdot 308 \text{ ft.},$$

and the width of the outlets,

$$(7)\quad d = \frac{e}{\lambda} = \frac{0 \cdot 308}{3 \cdot 5} = 0 \cdot 08791 \text{ ft.},$$

and the mean radius of the wheel,

$$(8)\quad r = \frac{e}{\nu} = \frac{0 \cdot 308}{0 \cdot 3} = 1 \cdot 0256 \text{ ft.},$$

and the number of floats,

$$(9)\quad n = \frac{F_2}{de} = \frac{0 \cdot 40308}{0 \cdot 308 \times 0 \cdot 08791} = \frac{40 \cdot 31}{2 \cdot 7} = 14 \cdot 9,$$

for which we may take 15. The number of guides may be the same. The depth of the wheel is $b = e = 0 \cdot 308$ ft., and the width of the suction pipe is to be made only a few inches over $2r = 2 \cdot 051$, say $2 \cdot 25$ ft.

The absolute velocity of the water issuing from the wheel is

$$w = 2 v \sin. \frac{\delta}{2} = 2 \times 19 \cdot 847 \times \sin. (7^\circ 30') = 5 \cdot 180 \text{ ft.},$$

and the velocity of the water in the suction pipe, since its section is

$$= \frac{(2 \cdot 25)^2 \pi}{4} = 3 \cdot 97608 \text{ sq. ft.},$$

is

$$w_1 = \frac{Q}{3 \cdot 9761} = \frac{8}{3 \cdot 9761} = 2 \cdot 012 \text{ ft.}$$

The effective performance to be expected from the wheel is therefore the following:

$$L = \left(h - [\zeta (c^2 + v^2) + w^2 + w_1^2] \frac{1}{2g}\right) Q \gamma$$

$$= [12 - [0 \cdot 08 (\overline{18 \cdot 695}^2 + \overline{19 \cdot 847}^2) + \overline{5 \cdot 180}^2 + \overline{2 \cdot 012}^2] \, 8 \times 62 \cdot 5$$

$$= [12 - 0 \cdot 0155 [0 \cdot 08 [393 + 350] + 26 \cdot 84 + 4 \cdot 05] \, 500$$

$$= [12 - 0 \cdot 0155 [59 \cdot 4 + 26 \cdot 84 + 4 \cdot 05] \, 500$$

$$= [12 - 0 \cdot 0155 (59 \cdot 4 + 30 \cdot 90)] \, 500 = [12 - 1 \cdot 400] \, 500$$

$$= 5300 \text{ ft. lbs.}$$

By the friction of the pivot and by the resistances in the suction pipe, this performance may be reduced to 4860 ft. lbs. = 8·8 horse-powers. The corresponding efficiency is then

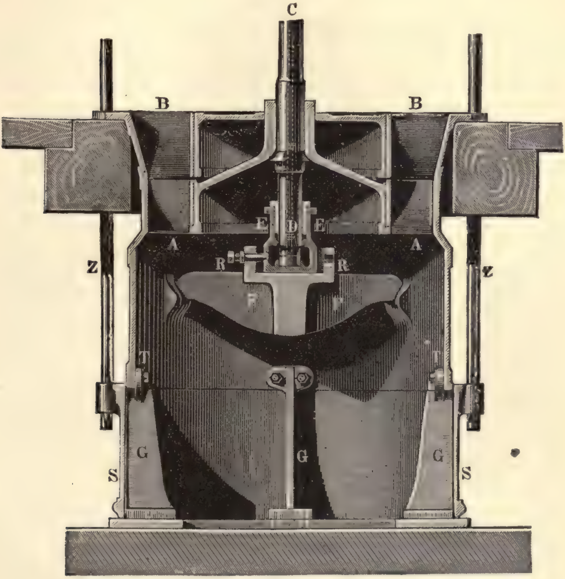
$$\eta = \frac{4860}{8 \cdot 12 \cdot 62 \cdot 5} = \frac{4860}{6000} = 81 \text{ per cent.}$$

When the sluice or valve in the suction pipe is set for a smaller quantity of water, the performance will, of course, come out smaller.

**§ 283.—Regulating Apparatus of the Henschel Turbines.**—To regulate the quantity of water of a Jonval turbine, various other methods have lately been adopted instead of the sluice; for example, a disc valve or throttle valve has been employed in the suction pipe (see Vol. I., § 444), or at the foot of the suction pipe a pipe sluice, or so-called telescopic sluice, has been set. The latter consists essentially of a short pipe *S S*, Fig. 518, which fits on to the lower end *T T* of the suction pipe and surrounds the base *G G* of the latter, so that by means of the rods *Z*, *Z*, it can be drawn up vertically, and the annular orifice below it can be made larger or smaller, according to necessity. The turbine represented in the figure (one of Reichenbach's, of Augsburg) is also notable in the bearing of the pivot *D*. This, as we see, rests in a box *E E*, which can be centred upon a firm frame *F F* by means of the screws *R*, *R*, and which, with a stuffing-box, prevents the access of the water to the rubbing surfaces.

In other turbines of this system (in French, turbines en dessus) the flow of the water is regulated by narrowing or partially contracting the guide passages, in much the same way as in Fontaine's turbines, in Fig. 512, III. Among others, the turbines of Cheneval and of Girard are to be included here. (See "Le Génie industrielle," Tome XII. and XIII.) In the former, each guide has a vertical sluice or valve which can be raised or lowered by a lever operated by means of cams, wheel work, etc. In Girard's turbines, every third passage of the turbine can be covered by a horizontal slide, which is moved by a lever and a train of wheels, etc. In both turbines, the width of the wheel increases from the upper part down, and hence a

FIG. 518.



smaller angle of exit $\delta$ can be used, as in the cylindrical turbines. (§ 275.)

In Girard's turbines, however, this increase of width (in French, évasement) is so great that a full efflux cannot be expected, especially as the process of hydropneumatization is applied to these turbines, and consequently they run in compressed air.

The regulating arrangements here mentioned have the great defect that they can operate only through a loss of energy (compare § 258); the same end can, however, be obtained more perfectly by dividing the whole wheel and drum containing the guides into chambers by cylindrical partitions, and by closing one or more of these chambers at the top, whereby we arrive at exactly the same result as in Fourneyron's turbines with horizontal divisions.

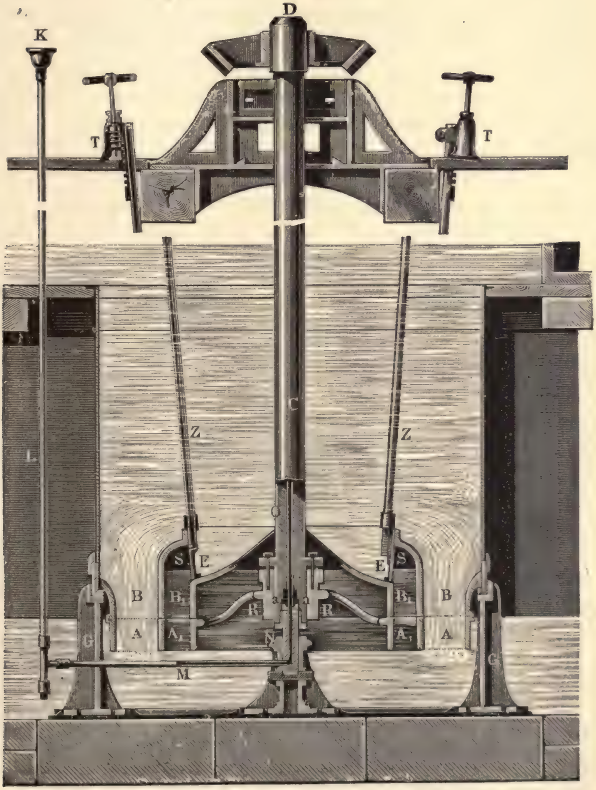
A turbine of this sort, with two divisions, is represented in Fig. 519. $AA$ is the outer and $A_1 A_1$ the inner compartment of the wheel, and $BB$ the outer and $B_1 B_1$ the inner division of the guide wheel. While the wheel, which is here enclosed in a casing, is united to the vertical shaft $CD$ by the arms $A_1 R$, $A_1 R$ and by the nave $RR$, the whole collection of guides rests upon the frame $GG$.

The partition between the guides is bent in at the top, and hence the inner division of the guide wheel is accessible only through an annular opening at the side $EE$.

This opening can be closed at pleasure by slides $S$, $S$, each one of which extends over one, two, or three passages of the guide wheel, the rods $ZZ$ being used in lowering the slides. These rods are hollow, communicate at the top with the open air, and at the bottom with the upper portion of the guide wheel, in order to prevent the water's rising in the inner division of the wheel when the slides are closed.

The rods are raised and lowered by means of endless screws $TT$... Oil is conducted to the pivot $N$ of the turbine through a pipe $KLM$; it rises through a small bore hole in the pivot up to the rubbing surfaces and flows off through a vertical hole in the upright shaft *CD*. In order that the pivots of these turbines may not bind in

FIG. 519.



their bearings, we sometimes give to the latter a ball and socket joint, or one formed of two cylinders crossing at right angles, as, for example, is shown here at *a a*.

§ **284.—Experiments on Fontaine's Turbine.**—Very trustworthy experiments have recently been made on the performances of Fontaine's and Jonval's turbines (see "Comptes rendus de l'Académie des Sciences à Paris," Vol. XXII. and XXIII., 1846, or "Polytechn. Centralblatt," Bd. VIII., 1846). Experiments with Fontaine's turbines had, however, been made before by the civil engineers Alcan and Grouvelle (see "Bulletin de la Société d'encouragement," Vol. XLIV., or "Polytechn. Centralblatt," Bd. VI.) These experiments led to these results, that in Fontaine's turbines also (as in Fourneyron's) the maximum efficiency occurred with the sluice opened highest, and that the performance decreases less with changes in the head of pressure than with changes in the quantity of water. The turbine at Vadeney near Châlons-sur-Marne, whose performance was obtained by Alcan and Grouvelle, had an exterior diameter of 1·6 metres (5·249 ft.), and was 0·12 metre (0·394 ft.) high; the fall amounted to about 1·7 metres (5·577 ft.), the quantity of water to 420 litres (14·8 cubic ft.), and its useful effect to about 8 horse-powers (7·9 English horse-powers). The principal results of these experiments prove to be that with 30 to 50 revolutions the mean efficiency was 0·67. One of Fourneyron's turbines, which, to be sure, had been in use several years, gave, under almost the same circumstances, $\eta = 0·60$ only.

Morin experimented on a turbine in a powder-mill at Bouchet. The wheel experimented upon had a mean diameter of 1·2 metres (3·937 ft.) and a width of 0·25 metre (0·820 ft.); it was fitted with 24 guides and 48 vanes, and had a fall of about 1·5 metres (4·921 ft.), using 0·25 cubic metre of water (8·8 cubic ft.). Experiments were made on it with the sluice raised 2, 3, and 4 centimetres (0·79, 1·18, 1·57 inches), and the following results were obtained: When the sluice was completely raised, and with 45 revolutions per minute, the efficiency came out the highest, viz., 0·69 to 0·70; with a lower position of the sluice, however, when the quantity of water was ¼ smaller, it was 0·57. The efficiency varied but little with the velocity of the wheel, for with 35 revolutions it was still 0·64, and with 55 revolutions it was 0·66. The general result from some experiments made with a fall of 1 metre (3·281 ft.) was that the deviation from the most advantageous velocity might amount to ¼, without causing the efficiency to be more than 4 to 6 per cent smaller. Moreover, it followed, from the experiments, that the greatest force with which the wheel began any irregular motion was almost 1½ times as great as the force giving the maximum performance. In the experiments, the wheel ran a few centimetres under water. From these results, we can conclude that Fontaine's turbine is to be included among the best hydraulic motors. Moreover, a special excellence in this wheel is that the pivot is entirely out of the water. The same end is, however, also attained by the "graissage atmosphérique" of Decker and Laurent, in which the lower part of the shaft is surrounded by a bell which revolves with the shaft. Here the air enclosed in this bell protects the pivot from the access of the water, and is kept at the necessary pressure by a small air pump.

§ **285.—Experiments on Jonval's Turbine.**—The experiments upon the performances of Jonval's turbines have come out not less favorably than those upon Fontaine's turbines. The holders of the patent rights of the Jonval turbine, Andrée Köchlin & Co., have published the results of the experiments upon two wheels from their works, in the "Bulletin de la Société industr. de Mulhouse," 1845 (see Dingler's "Polytechn. Journal," Bd. 94, 1844). Of these experiments, we shall, however, communicate here only the following: A turbine with a diameter of 0·95 metre (3·117 ft.), 0·20 metre (0·656 ft.) high, which was set 0·80 metre (2·625 ft.) below the surface of the head-water, and had a fall of 1·7 metres (5·577 ft.), using 550 litres (19·4 cubic ft.) of water per second, gave, with 73 to 95 revolutions per minute, an efficiency of 0·75 to 0·90. Morin rightly considers these values too large, and thinks that, on account of an incorrect determination of the quantity of water, corrections must be applied which reduce them to 0·63 to 0·71.

Morin himself experimented on a turbine with an exterior diameter of 0·81 metre (2·658 ft.), a width of crown of 0·12 metre (0·394 ft.), and 18 guides and 18 vanes, which worked with a fall of 1·7 metres (5·577 ft.), and with 200 to 300 litres (7·1 to 10·6 cubic ft.) per second. Morin attained the following general results: In the normal condition, with a free influx and efflux of the water, the number of revolutions of the wheel per minute was about 90 and the efficiency 0·72. When pieces were put upon the wheel to contract the openings, the efficiency did not come out much smaller unless the sections of the openings into the wheel were thereby considerably narrowed. The efficiency was not appreciably altered when the velocity was $\frac{1}{4}$ larger or smaller than at the normal rate of the wheel. The efficiency became considerably smaller when the sluice was set lower, whence it follows that the latter is a very imperfect regulator for the wheel. If, for example, the section of the water flowing off into the tail race was reduced by the sluice to 0·4 of its amount under the normal conditions, $\eta$ was then at the most equal to only 0·625.

Redtenbacher also communicates some experiments on a Jonval turbine, and finds the maximum efficiency with the sluice fully open, and when the wheel was not covered with plate-iron sectors, to be = 0·62. At the same time, he found, as in Fourneyron's turbines, that the wheel, when unloaded, makes about twice as many revolutions as under the normal conditions when giving the maximum performance.

§ **286.**—Extended experiments upon the action of three Köchlin-Jonval turbines have been made by Messrs. Hülsse, Bornemann and Brückmann, together with the author, in Fischer's paper manufactory at Bautzen, and have been published by Brückmann in the " Polytechn. Centralblatt," 1849, Lieferung, Nr. 17.

The largest of these wheels had an exterior diameter of 1·4 metres (4·593 ft.) and a width of crown, or length of vane, of $\frac{1}{6}$ 1·4 = 0·233 metres (0·766 ft.); its crown was about 2·3 metres (7·546 ft.) below the surface of the head-water, while the total fall averaged 4·28 metres (14·042 ft.). The number of vanes was 18, and of guides 24. Experiments with a friction brake set directly upon the turbine shaft, gave, with 0·672 cubic metre of water (23·7 cubic ft.) per second, and 80 to 100 revolutions per minute, a performance of about 2115 kilog. metres (15,291 ft. lbs.), which corresponds to the efficiency 0·745. Since, by the friction of the wheel upon the base of its pivot—the wheel weighing 850 kilos. = 1874 lbs., and the pivot being 8·98 centimetres thick (3·53 inches)—there was consumed 234 kilog. metres (1692 ft. lbs.), the performance of the water on the wheel was 2349 kilog. metres (16,983 ft. lbs.), while the potential energy of the water was 672 × 4·28 = 2876 kilog. metres, and hence the efficiency of the wheel was

$$\eta = \frac{2349}{2876} = 0{\cdot}815.$$

The second wheel had an exterior diameter of 0·963 metre (3·160 ft.) and an inner diameter of $\frac{2}{3}$ . 0·963 = 0·642 metre (2·106 ft.), and the number of vanes was 18, while that of the guides was 20. Experiments on this wheel with a dynamometer gave, with a fall of 4·42 metres (14·502 ft.), with 0·370 cubic metre of water (13 cubic ft.) per second, and 115 to 145 revolutions per minute, an effective performance of 1289 kilog. metres (9319 ft. lbs.), and therefore an efficiency of $\frac{1289}{1635} = 0{\cdot}8$, which, by taking into account the friction of the wheel upon the base of its pivot—the wheel weighing 493 kilogrammes (1087 lbs.), and the base of the pivot being 7·62 centimetres (3·00 inches) wide—rises to 0·82.

Finally, the smallest wheel had an exterior diameter of 0·612 metre (2·008 ft.) and inner diameter of 0·393 metre (1·289 ft.), and it had only 12 vanes and the same number of guides. It was only 1·4 metres (4·593 ft.) below the surface of the water in the head race, while the total fall was 4·513 metres (14·807 ft.). With 0·197 cubic metre (7·00 cubic ft.) of water per second and 180 to 220 revolutions per minute, this wheel gave an efficiency of 0·70, which rose to 0·715 by taking into consideration the friction of the wheel (weighing 229 kilog. or 505 lbs.) upon the base of its pivot, which was 6·35 centimetres (2·50 inches) in diameter.

Not less favorable are the results of the experiments with the friction brake which Brückmann made on a Köchlin-Jonval turbine in the spinning-mill of Mattausch at Franzensthal in Bohemia, and which he has also published in the "Polytechn. Centralblatt," for the year 1849, Lieferung 22. This machine, and the former also, came from the works of Escher, Wyss & Co., in Zürich. The wheel had 20 floats, an outer diameter of 4 ft. 6½ inches, and a drum 9 inches high and 9¼ inches broad. The drum carrying the guides widened somewhat towards the top, and had only 15 guides, and its depth was likewise 9 inches. The bottom of the wheel lay 1·4 metres (4·593 ft.) below the surface of the head-water, the total fall was 3 to 3·1 metres (9·843 to 10·171 ft.), and the quantity of water 0·966 to 1·22 cubic metres (34·1 to 43·1 cubic ft.) per second. Instead of a regulating valve, a telescopic sluice, constantly open during the experiments, was placed at the foot of the suction pipe, and there were, besides, coverings at hand by which several of the openings into the guide-wheel could be closed. The experiments of Brückmann have led to the following results: With the passages between the guides fully open and 81 to 91 revolutions of the wheel per minute, the performance of this turbine was 38 horse-powers, corresponding to the efficiency 0·78; if, however, three of the fifteen passages between the guides were covered, the efficiency fell to 0·75, and with five of the passages covered, it fell to as low as 0·65.

**§ 287.—Later Experiments on a Fontaine Turbine.**— Thorough experiments with the friction brake on a Fontaine turbine with two divisions (Fig. 519), which came from the well-known factory of Escher, Wyss & Co., in Zürich, were made in 1852 by Professors Hülsse and Brückmann. The turbine tested furnished the motive power in Grimm's paper factory, etc., at Doberschau, near Bautzen.

The fall of this turbine amounted to 16½ ft., and the normal quantity of water to 16¾ cubic ft. per second. The water first came out of the head race into a supply chamber about 7 ft. wide and 8 ft. deep, and thence into a supply pipe of plate iron, 4⅝ ft. wide; the latter carried it down into the guide-wheel, which joined on to this pipe at the bottom, and which was made of two concentric cylindrical drums in which the guides were fastened; and from this it flowed in an oblique direction into the wheel immediately below, which was also in two parts. The surface of the water in the tail race varied between the level of the head-water and that of the bottom of the drum which held the guides. This motor is therefore a Fontaine turbine running under water. The mean diameter of the outer division of the wheel amounted to 3 ft. 10¼ inches, and its width, radially, to 2·9 inches; the mean diameter of the inner division of the wheel measured 3 ft. 0·85 inches, and the width, radially, was 4 inches. The depth of the wheel was 6½ inches, the distance of the wheel from the guide-wheel ¼ inch, and the thickness of the cast-iron division ring was 1¼ inches. The depth of the guide-wheel was 6·1 inches, the upper width of the outer ring was 4¼ inches, and the lower width 5¾ inches. The number of guides, and also of vanes, was 24. The regulating of the water could be effected in this manner: by having

(1) both divisions of the wheel completely open,

(2) only the outer division of the wheel completely open,

(3) the latter completely and the inner division partially closed.

To close the inner ring of the guide-wheel, iron covers were used having the shape of portions of the ring. These covers were placed opposite each other in pairs, and covered either every passage, every other passage, or every third or every fourth passage of the guide-wheel.

The shaft had a diameter of 6 inches, and weighed 1482 lbs. (1634 English lbs.); it had, at the lower end, a brass step with which it ran on a fixed cast-steel pivot, rounded off on its upper surface, and 3½ inches in diameter.

The power was measured by a friction brake with an arm 6½ ft. long, and the quantity of water by a weir 8 ft. wide. The results of the experiments made with this turbine are these:

(1) When the outer division of the wheel was used, the mean fall was

$$h = 4\cdot93 \text{ metres} = 16\cdot175 \text{ ft.};$$

the mean amount of water was

$$Q = 0\cdot255 \text{ cubic metres} = 9\cdot0 \text{ cubic ft.};$$

the number of revolutions per minute was

$$u = 60 \text{ to } 82;$$

and the efficiency

$$\eta = 0\cdot573 \text{ to } 0\cdot613.$$

(2) When both divisions of the wheel were completely open,

$h = 4\cdot45$ met. $= 14\cdot600$ ft.; $Q = 0\cdot485$ c. met. $= 17\cdot1$ c. ft.

$u = 76$ $\eta = 0\cdot652$

$u = 103$ $\eta = 0\cdot755$

$u = 119$ $\eta = 0\cdot713$.

(3) When half of the inner division of the guide-wheel was closed (12 passages),

$h = 4\cdot51$ met. $= 14\cdot797$ ft.; $Q = 0\cdot359$ c. met. $= 12\cdot7$ c. ft.

$u = 69\cdot5$ $\eta = 0\cdot649$

$u = 86$ $\eta = 0\cdot677$

$u = 100\cdot3$ $\eta = 0\cdot657$.

(4) By closing three quarters (18 passages) of the inner division of the guide-wheel,

$h = 4 \cdot 57$ met. $= 14 \cdot 994$ ft.; $Q = 0 \cdot 300$ c. met. $= 10 \cdot 6$ c. ft.

$u = 57$ to $87\frac{1}{2}$ $\eta = 0 \cdot 576$ to $0 \cdot 640$.

The efficiency of the turbine is a maximum when both divisions of the wheel are completely uncovered, as would also follow from theoretical considerations; and the efficiency comes out the smaller, the more passages of the inner division of the guide-wheel are covered (see "Polytechnisches Centralblatt," Jahrgang 1852, Lieferung 14).

Experiments upon Fontaine's turbines with the hydropneumatization, etc., of Girard, have been made on a wheel of that sort in the paper mill at Egreville by Messrs. Girard, Dufay, Callon, and others, in the year 1851 (see "Comptes rendus de l'Académie des Sciences à Paris," T. 33). The result of these experiments was that a turbine of that description with a fall $h = 1 \cdot 65$ to $1 \cdot 69$ metres ($5 \cdot 413$ to $5 \cdot 545$ ft.), a volume of water $Q = 1 \cdot 75$ to $2 \cdot 22$ cubic metres per second ($61 \cdot 8$ to $79 \cdot 0$ cubic ft.), a number of revolutions $u = 20$ to $24$, and an effective performance of 27 to 38 horse-powers ($26 \cdot 6$ to $37 \cdot 5$ English measures), has an efficiency of $0 \cdot 69$ to $0 \cdot 76$. Later experiments upon a turbine of this sort in the spinning mill at Haudrecy, when $h$ was $= 1 \cdot 66$ to $1 \cdot 78$ metres ($5 \cdot 446$ to $5 \cdot 840$ ft.), $Q = 0 \cdot 54$ to $1 \cdot 09$ cubic metres ($19 \cdot 1$ to $38 \cdot 5$ cubic ft.), and $u = 23$ to $27$, gave $\eta = 0 \cdot 70$ to $0 \cdot 84$, or, as a mean, $\eta = 0 \cdot 75$ (see "Le Génie industrielle," Mars 1855).

Experiments made in the Conservatoire des arts et métiers at Paris upon a turbine of the same sort have led to the efficiency $\eta = 0 \cdot 61$ to $0 \cdot 76$. (Vid. "Le Génie industrielle," Tome XII., 1856.)

§ **288.—Comparison of Turbines with each other.**—If we compare the Fontaine-Jonval turbines with those of Fourneyron, we find, of course, that in some respects they are to be preferred to the latter, while in other respects they are inferior to them. In the first place, a Fontaine turbine, and others of that sort, have an advantage over a Fourneyron turbine, since, in the former, the water at its entrance into the guide-wheel is not diverted so much from the original direction of its motion as in the Fourneyron turbine; accordingly, when the velocity of entrance is the same, the resistance to the entrance of the water is smaller in the former than in the latter turbine; or a greater velocity of entrance can be used in the former wheel than in the latter, and therefore the Fontaine wheel can be made smaller than the Fourneyron. Next, this turbine possesses the other advantage that its guides conduct the water more in parallel filaments than is the case with the Fourneyron turbine, in which a divergence of the streams entering the wheel is unavoidable.

On the other hand, the old or Fourneyron turbines also offer their advantages. In the first place, the pressure on their pivots consists almost entirely of the weight of the wheel, while, in the later turbines, it consists of a water pressure besides, which increases with the force with which the wheel turns. Other things being equal, a greater amount of friction at the pivot must therefore be expected in these turbines than in the Fourneyron turbines. In the second place, the particles of water move side by side in the Fourneyron turbine, with equal velocities of rotation; while in the Fontaine-Jonval turbines, the molecules of water, flowing down beside each other, have very unequal velocities of revolution, the outer ones having the greater and the inner ones the smaller velocities. Hence there arises in those wheels a shock, though a slight one, of the water as it enters the wheel, a greater amount of friction of the water in the wheel passages, and especially a certain irregularity of motion of the water flowing through the wheel, since the centrifugal force constantly urges it from the centre. Finally, an advantage in the older turbines consists in greater facility with which the arrangement of guides and vanes can be made.

NOTES.—1. The Fontaine turbines are well adapted to utilizing the force of the flood and ebb of the tide. If we put a wheel of this sort in a canal opening into the sea, and shut off, by two sluices, on the one hand the lower and on the other hand the upper part of the wheel, then the water standing higher on one side is constrained to pass down through the wheel and set it in rotation. When the current changes from flood to ebb, or the reverse, the position of the sluice must, of course, be reversed.

2. Among the advantages of Jonval's turbines must be counted the fact that we can set them at any height we please above the surface of the water in the tail race (of course not over 33·90 ft.), without losing any considerable amount of power, and that they are, therefore, easily inspected and repaired, and, moreover, no loss arises through a change in the height of the tail water. From the experiments of Marozeau (vid. the work cited at the end), and also from the theory given above and from the special investigations of Morin, it follows that the height of the turbine above the water in the tail race must not exceed a certain limit, since otherwise the water directly under the wheel loses its continuity, whereby, as is easily inferred, a smaller effect is obtained.

**§ 289.—Comparison of Turbines with other Water-Wheels.**—We must now sum up the advantages and defects of turbines, and especially of reaction turbines, in comparison with vertical water-wheels, and weigh their respective merits.

In the first place, turbines possess one great advantage over vertical water-wheels since they can be used with almost any fall from 1 to 500 ft., while vertical water-wheels can, at the most, be used with a water-power of only 50 ft. fall. Of course the efficiency of a turbine is different with different falls—*i. e.*, it comes out smaller with small wheels and high falls than with average and small falls, since the hydraulic resistances are proportionally greater than in large wheels with average falls. On the other hand, with high falls of 20 to 40 ft., an efficiency can be attained by overshot-wheels which cannot be reached by turbines. It is only with mean falls of 10 to 20 ft. that we can expect the same performance from both kinds of wheels; if, however, the fall is small, then in every case turbines will give a greater useful performance than undershot-wheels set in their place. Poncelet wheels can be put on a par with turbines under falls of 3 to 6 ft. at the highest. Turbines have this other great advantage over vertical water-wheels, that they work with almost the same efficiency under different falls, and especially that they are not disturbed in their motion by back-water, since they run under water with hardly any less advantage than in air, and, in some cases, they work with even more utility. Vertical water-wheels, it is true, always lose efficiency when the fall varies, but not to any considerable extent unless the fall itself is small, or the drowning of the wheel comes in. But, on the other hand, alterations in the quantity of water cause far less loss of energy in vertical wheels than in horizontal water-wheels. This circumstance redounds to the advantage of the former wheels in respect to the economy of the water-power. When it is necessary to raise the performance of a vertical water-wheel which has hitherto been running at its normal rate, and especially of a wheel in which the water acts chiefly by pressure, we can let in a greater quantity of water, and, to diminish the performance of such a wheel, we have only to give it less water; in either case, the efficiency does not become appreciably larger or smaller. The case is wholly different, however, with a reaction turbine. A turbine of this sort runs most economically with the sluice fully open, and, therefore, with the maximum quantity of water also; if a smaller amount of work is required, and, therefore, a smaller volume of water is needed, and the sluice is lowered for this purpose, then the performance of the wheel is only partially diminished by diminishing the quantity of water, and partially by the destruction of the *vis viva* of the water, or by the diminution of its pressure, and hence the efficiency is reduced. This destruction of power can be compared to the application of the brake to a wagon or the locking of its wheels in descending a mountain when there is an excess of living force. While, therefore, in vertical water-wheels, by lowering the sluice, we only shut off from the wheel all the superfluous water, and can use this water as we please for other purposes, in reaction turbines, only a part of the excess of water is shut off, and the energy of the other part is destroyed in the wheel.

In pressure turbines, unless they run under water, and unless, therefore, the wheel passages are filled by the water flowing through them, the relative performance is more favorable, since, in these turbines, the water flows through the wheel passage in every position of the sluice without forming an eddy.

§ **290.**—In regard to changes in the velocity of rotation, no great difference is found between horizontal and vertical water-wheels. In both sorts of wheels, the normal velocity may be increased or diminished by about one fourth part of its amount, without appreciably diminishing the performance. But as to the magnitude of this velocity, there is, of course, a great difference. With the exception of undershot-wheels, and especially of Poncelet wheels, all vertical water-wheels generally revolve with a surface velocity of 4 to 10 ft., while turbines have velocities of revolution very different, and generally much greater, depending upon the fall. For these reasons, and also because turbines have smaller radii than vertical water-wheels, they generally make many more revolutions than the latter. Hence, according as the machinery to be driven requires a great or a small number of revolutions—*i. e.*, a rapid or a slow rate of motion—a horizontal or a vertical water-wheel is more suited to giving it its motion. But the rapid motions of a machine are rather prejudicial than otherwise, since the unavoidable resistances, such as friction, and especially shocks, etc., come out in this case larger; and for these reasons, it is often better, by means of intermediate machinery, to increase rather than to diminish the number of revolutions of a wheel, and hence to employ a vertical instead of a horizontal water-wheel.

If the load of a machine is variable, as, *e. g.*, in a forge shop or rolling-mill, the use of a vertical wheel is also to be preferred, since the latter, by its greater mass, will act more as a regulator than a turbine, in using which a fly-wheel is often necessary in order to equalize the changes of motion. With constant loads, however, it must be conceded that turbines have one advantage in this respect, since vertical water-wheels—*i. e.*, when made of wood—often have a so-called heavy quarter; in other words, they are not equally heavy in all parts of their circumference.

In economical relations, turbines can be placed upon a par with vertical water-wheels, while with high falls, and even with mean falls and a large quantity of water, they are indeed to be preferred to vertical water-wheels on account of cheapness. Even in regard to durability, the advantage must be conceded to turbines over vertical water-wheels.

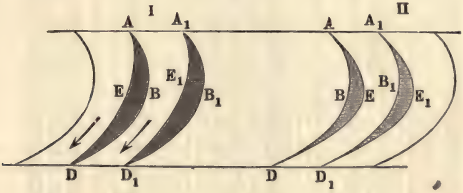
On the other hand, we must not leave out of consideration the fact that turbines require water that is comparatively pure, and that their performance is liable to be reduced to an extraordinary degree by the introduction of sand, mud, moss, weeds, leaves, pieces of ice, twigs, etc.—a contingency which need not be apprehended with vertical wheels. Finally, we must also consider that turbines, and especially those fitted with guides, are more difficult to construct than vertical wheels, and that deviation from the mathematical rules of their construction is much more injurious in its results than in the case of vertical water-wheels. These are the reasons why so many turbines formerly failed of success; and they have not yet received that extended use which they deserve.

**§ 291.—Hänel's Turbines with Double Vanes.**—It is well known that as water flows through a bent pipe of a constant section, it separates from the convex wall of the pipe and therefore does not entirely fill its section; we also know that the water completely joins the wall of the pipe again, in an eddy which reduces the pressure, only when the efflux of the water from the pipe is opposed by some resistance—*e. g.*, by contraction.

This is just the case with the motion of the water through the passages of a turbine. In order that the water may pass through these passages with a full section, it is necessary that the section of these passages should not be the same throughout their entire length, neither should it increase, but it should gradually become smaller and smaller from the entrance to the point of discharge. To attain this end, we must, as a rule, when the angle of entrance $\beta$ is acute, use contracted wheel passages, or fit the floats with double walls.

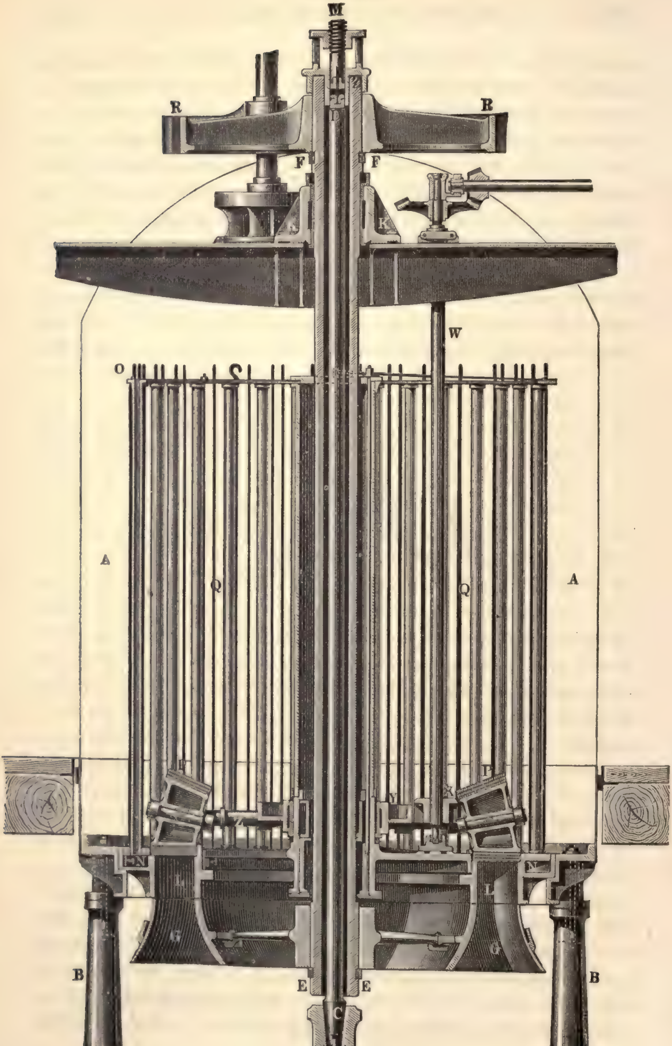
To the floats $A\,B\,D$, $A_1\,B_1\,D_1$, Fig. 520, of a Henschel turbine, which are constructed in the ordinary manner, we can add, for this purpose, the floats $A\,E\,D$, $A_1\,E_1\,D_1$, which run along either on the concave sides, as in I., or, as in II., on the convex sides of the vanes $A\,B\,D$, $A_1\,B_1\,D_1$. The wheel passages, so formed, as $A_1\,B\,D_1$ in I. and $A_1\,E\,D_1$ in II., gradually decrease in width going from $A$ to $D$, while the passages between $A\,B\,D$ and $A_1\,B_1\,D_1$ are wider at $B\,B_1$ than at $A\,A_1$, and hence cause the production of eddies.

FIG. 520.



Turbines with floats having double walls of this sort, and more particularly with rear floats (II., Fig. 520), were first brought into use by Maschinendirector Hänel, in laying out a large mill at Rothenburg on the Saale, and have there proved to be of advantage. The vertical section of a turbine of this kind is shown by Fig. 521. The following is essentially its arrangement: The supply chamber $A\,A$, together with the guide-wheel, rests upon four cast-iron columns $B$, $B$, and the spindle $C$, which, by means of the hollow shaft $E\,F\,F\,E$, carries the running-wheel $G\,G$, rests in the head $H$ of a pillar which is supported with the

FIG. 521.

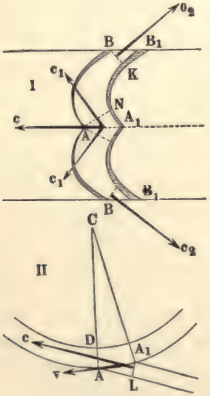


columns *B*, *B* by a cross-shaped bed piece. The hollow shaft *EFFE* passes at *KK* through a collar, carries at *FF* the driving-wheel, and ends in a nut *M*, whose screw forms the pivot *Z* by which the shaft rests upon the column *CD*. To reduce the friction as much as possible between the pivot and its bearing, a plate of bell-metal is loosely inserted, and to the rubbing surfaces oil is carried through a longitudinal hole. To remove the air which collects in the passages between the guides, an annular space *NN*, divided into chambers, is made on the outer circumference of the drum which carries the guides, and this space communicates by holes on the side with the passages in the guide-wheel, and by vertical pipes, as *NO*, with the outer air. To keep foreign bodies which are carried along with the water from entering the wheel, a cylindrical grating *QQ* is made by 64 vertical rods and 32 air-pipes, and this grating surrounds the whole guide-wheel. The sluice apparatus by which the motion of the wheel is regulated consists of two conical rollers *PP* and two annular gutta-percha strips, whose ends are fastened two to the guide-wheel and the other two to the rollers. These rollers can turn not only about their geometric axes, but also about the axis of the turbine, so that the gutta-percha strips can be wound up on the rollers and off of the annular opening, and the reverse. For this purpose, the vertical shaft *WX*, etc., is used with the pinion *X*, which engages the toothed sector *Y* on which the arms *Z* are fastened; and these arms with their forked ends take hold of the rollers *PP*. The strength necessary to sustain the pressure of the water is given to the gutta-percha strips by a number of iron plates placed close together and running crosswise of the strips.

The principal dimensions of such a turbine are the following: The ordinary fall $h = 4 \cdot 12$ to $6 \cdot 18$ ft.; the volume of water $Q = 5 \cdot 46$ to $62 \cdot 23$ cubic ft. per second; number of floats and guides $= 32$; mean diameter of wheel, $5 \cdot 149$ ft.; width of ring above $= 7 \cdot 72$ inches, below $= 15 \cdot 45$ inches; depth of wheel, $1 \cdot 030$ ft. The angle of approach $\alpha = 22\frac{1}{2}°$, of entrance $\beta = 45°$; the mean angle of exit $\delta = 24\frac{3}{4}°$. The normal number of revolutions was $u = 33$. It follows, from the very extensive experiments of Maschinendirector Hänel, that with the guide passages more or less open ($\frac{1}{4}$ to $\frac{4}{4}$), with the wheel from 0 to 1·545 ft. under water, and with a volume of water $Q$ from 5·79 to 62·23 cubic ft., this turbine gives an efficiency of 0·64 to 0·70. Further information can be obtained in Bd. V. (1861) "Der Zeitscrift des Vereins deutscher Ingenieure."

§ **292.**—**Schiele's Turbines.**—When a jet of water strikes almost tangentially upon the middle circumference of a cylinder which is furnished with floats, as $B\,A\,B$, $B_1\,A_1\,B_1 \ldots$, Fig. 522, and which is enclosed in a casing, the water flows off in two parts along the halves of the floats $A\,B$, $A\,B$, and is discharged at the two bases of the cylinder at $B$, $B$, etc. If this cylinder is held fast only in its geometric axis, it will be set in rotation by the water gliding along its floats, and accordingly will form a horizontal water-wheel, and, in fact, the Schiele turbine.

FIG. 522.



If $A$, the middle of the wheel, is at a certain depth $h_1$ below the surface of the head-water, and if the water at its entrance into the wheel has the pressure measured by the height $x$, then the velocity of entrance of the water will be

$$c = \sqrt{2g(h_1 - x)},$$

and if the wheel has the velocity $v$, we have, under the supposition that the direction of $c$ is almost tangential, for the relative velocity of the water as it begins to move in the wheel,

$$c_1 = c - v = \sqrt{2g(h_1 - x)} - v.$$

If the middle of the wheel is also at a depth $h_2$ below the water in the tail race, we have for the relative velocity of discharge $c_2$,

$$\begin{aligned} c_2^2 &= c_1^2 + 2g(x - h_2) \\ &= (c - v)^2 + 2g(x - h_2), \end{aligned}$$

or putting in $c^2 = 2g(h_1 - x)$, and in place of $h_1 - h_2$, the total fall $h$,

$$\begin{aligned} c_2^2 &= 2g(h_1 - x) - 2cv + v^2 + 2(x - h_2) \\ &= 2gh - 2cv + v^2. \end{aligned}$$

In order that the water may flow off from the wheel with the least possible amount of energy, *B*, the end of the float, must be made almost tangential to the circumference of the wheel, and $c_2$ must be made equal to $v$. With this assumption, we have

$$2gh - 2cv = 0, \qquad \text{and hence} \qquad cv = gh.$$

If $F$ denotes the section of the water approaching with the velocity $c$, and $F_1$ the section of the jet in the wheel immediately after its entrance, when it has the velocity $c_1$, we have

$$F_1 c_1 = Fc, \qquad \text{or} \qquad F_1(c - v) = Fc,$$

and hence

$$c = \frac{F_1 v}{F_1 - F}$$

and

$$\left(\frac{F_1}{F_1 - F}\right) v^2 = gh;$$

whence follows the best velocity for the wheel,

$$v = \sqrt{\frac{F_1 - F}{F_1} \cdot gh}.$$

If $\alpha$ denotes the angle of approach $A_1 A L$ (II.), $\beta$ the angle of the floats at the point of entrance $A A_1 N$ (I.), and

$\delta$ the angle of the floats at the point of exit $B B_1 K$; if, moreover, $a$ is the depth of the entering jet, and $e$ the width of the floats $A D$ (II.), we have

$$\frac{F}{F_1} = \frac{a \sin. \alpha}{2 e \sin. \beta},$$

and therefore also

$$v = \sqrt{\left(1 - \frac{a \sin. \alpha}{2 e \sin. \beta}\right) g h}.$$

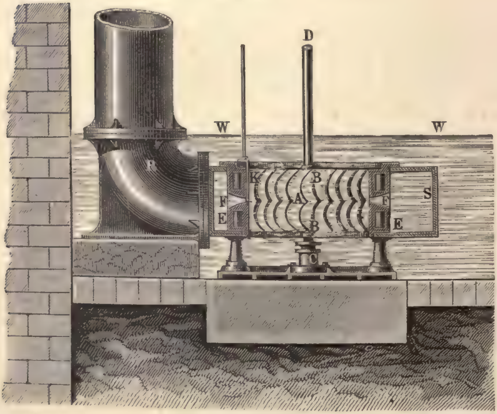
In the same way, we have

$$a c \sin. \alpha = 2 e c_1 \sin. \beta = 2 e c_2 \sin. \delta,$$

whence we get for the angle of exit $\delta$,

$$\sin. \delta = \frac{c_1}{c_2} \cdot \sin. \beta = \frac{c - v}{v} . \sin. \beta.$$

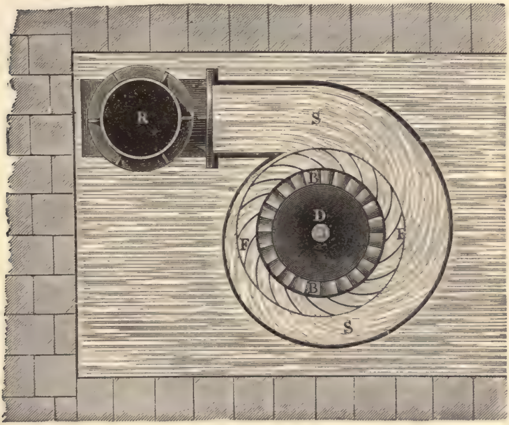
FIG. 523.



In Figs. 523 and 524 are represented a vertical and horizontal section of a Schiele turbine. The wheel proper $B A B$ rests upon the shaft $C D$, and is enclosed in a casing

*E E*, which contains in the middle the annular entrance passage *F F*, provided with guides. This casing is again surrounded by a spiral supply chamber *S S*, which joins directly on to the supply pipe through which the water is brought. The water is carried through the guide passages *F F'*. . . to *A*, the middle of the wheel, flows thence in two streams along the floats *A B*, *A B*, and is discharged under the water *W W* at the top and bottom of the wheel. To regulate the supply of water, slide valves as *K* are applied to the outlets of the supply passages, by which the latter can be closed. Since the water passes through the wheel passages in opposite directions, it exerts no pressure upon the wheel in the direction of its axis, and since the wheel is also cast hollow, so that it almost floats in the water, the friction upon the pivots of these wheels comes out extraordinarily small. These wheels are also worked with suction pipes, and are also run on a horizontal axis. (See Dingler's Journal, Bd. 164, 1862.)

FIG. 524.



§ **293.—Screw Turbines.**—The screw turbine (Fr. turbine-hélice; Ger. Schraubenturbine) is not essentially different from the Henschel and Fontaine turbine. In this turbine also the water flows in the wheel passages down from above; but these passages are made of only from two to four very long vanes, which are curved like the threads of a screw, running around the shaft. The vertical section of such a screw turbine is shown in Fig. 525. This turbine was built by Plataret, and works in a spinning-mill at Saint-Maur, near Paris. *A*, the wheel of this motor, is of cast-iron, and consists essentially of two helicoidal vanes, which are fastened to a nave forced upon the turbine shaft *CD*, and make just one turn about this shaft. The depth of this wheel is 0·52 metre (1·706 ft.), the exterior diameter is 1·04 metres (3·412 ft.), and the inner diameter or that of the nave is 0·25 metre (0·820 ft.); consequently the pitch of each screw is = 0·52 metre (1·706 ft.), and the inclination of the surface outside, or at the extreme distance from the axis, is

$$tan.\ \alpha = \frac{0 \cdot 52}{\pi \times 1 \cdot 04} = \frac{1 \cdot 706}{3 \cdot 412} = \frac{1}{2\,\pi} = 0 \cdot 15915, \text{ hence } \alpha = 9^\circ\ 3',$$

while the inclination inside is

$$tan.\ \alpha_1 = \frac{0 \cdot 52}{\pi \times 0 \cdot 25} = \frac{1 \cdot 706}{\pi \times 0 \cdot 82} = \frac{2 \cdot 08}{\pi} = 0 \cdot 6621, \text{ hence } \alpha = 33^\circ\ 30'.$$

The area of the section of both wheel passages, after deducting the thickness of the iron, amounts altogether to 0·14 square metre (1·507 square ft.). The wheel moves in a casing *EE*, carefully bored out, with a clearance of 1 millimetre (0·04 inch). The turbine shaft is suspended, like the Fontaine turbine, Fig. 512, and turns about a cylindrical spindle which rests upon the block *F*. *HH* is a collar for the shaft, and is supported by a three-armed cross-piece *KK*. To prevent any eddying of the water *WW*, the vertical wooden doors *T*, *T* are suspended inside, which divide the whole wheel chamber above the wheel

FIG. 525.

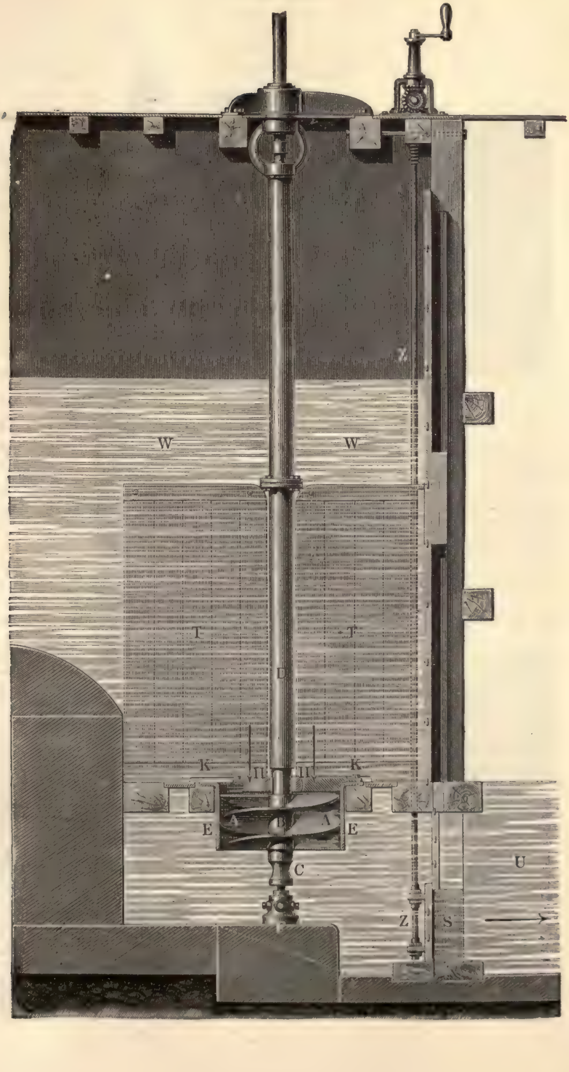
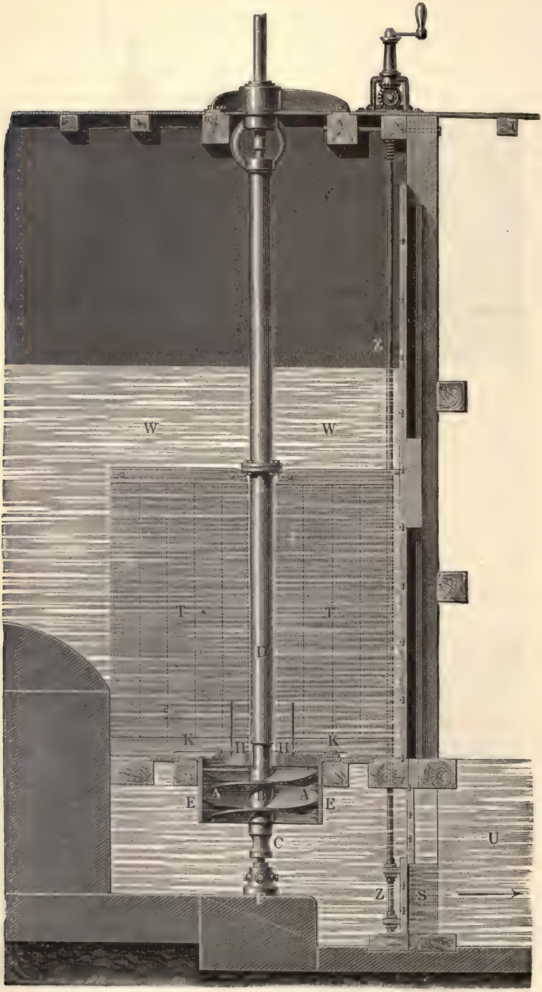
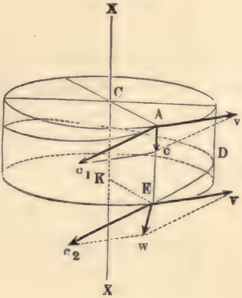


FIG. 526.



into two parts. To regulate the volume of water, a sluice *S* is used, which stands beneath the water of the tail race, and which can be moved by means of a rod *ZZ*. The performance of this wheel, obtained with a friction brake, is 20 to 28 horse-powers, with a fall of about three metres (9·843 ft.), and a volume of water of about 0·85 cubic metre (30·0 cubic ft.) per second. A computation based upon a gauging of the water, not very precise, led to the efficiency $\eta = 0 \cdot 70$. The following brief statement will suffice to convince one that the action of the water in screw turbines is not altogether without advantage.

FIG. 527.



Since this turbine has no arrangement of guides, we can assume that the water enters the wheel *BD* with a vertical velocity *c*, Fig. 527, and accordingly the velocity of the wheel must fulfil the relation

$$v = c \cot. \alpha.$$

If the angular velocity of the wheel is $w = \frac{\pi u}{30}$, we have the velocity of rotation at the distance $\overline{CA} = \overline{KE} = z$,

$$v = \omega z,$$

and if the pitch or depth of the wheel $\overline{AE}$ is denoted by *a*, we have for the angle of inclination of the helicoidal vane *ABDE* at this distance *z*,

$$\tan. \alpha = \frac{a}{2 \pi z};$$

and we can put

$$\omega z = c \cot. \alpha = \frac{2 \pi z c}{a},$$

and hence follows the angular velocity ω, which the wheel must have in order that the water may everywhere enter it without shock,

$$\omega = \frac{2 \pi c}{a}.$$

For the relative velocity $c_1$ with which the water begins its motion in the wheel, we have

$$c_1^2 = c^2 + v^2,$$

and, on the other hand, for the relative velocity $c_2$ with which it comes out of the wheel, we have

$$c_2^2 = c_1^2 + 2 g (x - y),$$

in which $x$ denotes the hydraulic head at entrance, and $y$ that at the exit of the water from the wheel, and the accompanying hydraulic resistances are neglected. Now, since $c^2 = 2 g (h_1 - x)$, where $h_1$ denotes the height of the water above the wheel, it follows that

$$c_2^2 = c^2 + v^2 + 2 g (x - y) = v^2 + 2 g (h_1 - y),$$

or, finally, since $h_1 - y$ is equal to $h$, the total fall of the wheel, we have

$$c_2^2 = v^2 + 2 g h.$$

To obtain the maximum of useful work, $c_2$ would have to be equal to $v$, which, according to this formula, is possible only for $v = \infty$. Accordingly the screw turbine is like every other reaction wheel without guides (see § 243 and § 255). If, however, we assume $v$ to be very large only, we obtain

$$c_2 = v = w z,$$

and consequently the relative velocity of efflux, as well as the velocity of revolution, is proportional to the distance $z$ from the axis of the wheel.

The absolute velocity of discharge of the water from the wheel is

$$w = 2 v \sin. \frac{\alpha}{2} = 2 \omega z \sin. \frac{\alpha}{2},$$

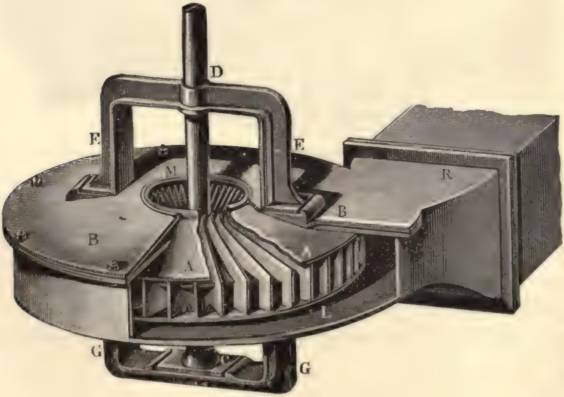
approximately

$$= 2\,\omega\,z\,tan.\,\frac{\alpha}{2} = \omega\,z\,tan.\,\alpha = \omega\,z\,\frac{a}{2\,\pi\,z} = \frac{\omega\,a}{2\,\pi},$$

and hence the same over the whole base of the wheel.

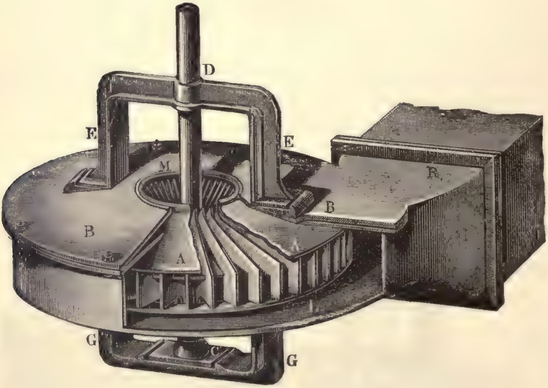
§ **294.**—**Thomson's Turbines.**—In the reaction turbines of Fourneyron, Fontaine, Francis, and others, the water approaches the wheel chamber with so slow a motion that we can leave entirely out of consideration its living force; but there are also turbines in which the water is brought to the wheel with a velocity which is wholly or nearly equal to the velocity of revolution. For example, the case water-wheel of Thomson is a wheel of this sort, which is represented partly uncovered in Fig. 528.

FIG. 528.



The wheel *A A* consists of radial vanes which are placed between conical crowns, and increase in height from the outside towards the centre. The shaft *C D* rests in a frame *E E G G* which is firmly united to a case *B B* which surrounds the whole wheel. This case fits on to the orifices of the wheel *M*, *M*, with a tolerably water-tight joint, while it surrounds the outer circumference of the wheel eccentrically, and on one side is joined to the pipe *R* through which the water is supplied. In consequence of setting this casing which encloses the wheel, with its centre not coinciding with the centre of the latter, an annular passage *L* is formed, which has its greatest width at the opening of the supply pipe, and passes round the wheel with a gradually decreasing width. In this passage, the water moves with a velocity $v_1$ which is but little greater than that of the circumference of the wheel. With a volume of water $Q$, the initial or greatest section of this passage is

FIG. 529.



$$F = \frac{Q}{v_1}.$$

If $x$ is the hydraulic head of the water which is supplied at a velocity $v_1$, $h_1$ the hydrostatic head, and $\zeta$ the coefficient of resistance to the motion of the water in the annular passage, we can put

$$2g(h_1 - x) = (1 + \zeta)v_1^2.$$

Denoting also the velocity of the inner circumference of the wheel by $v$, and the relative velocity of the water issuing from the wheel passages by $c_2$, and using the other notations as in inward-flow tangential and reaction wheels, we have

$$\begin{aligned}(1 + \zeta_1)c_2^2 &= 2g(x - h_2) + v^2 - v_1^2 \\ &= 2g(x - h_2) - \left[1 - \left(\frac{r}{r_1}\right)^2\right]v_1^2 \\ &= 2gh - \left[2 + \zeta + \left(\frac{r}{r_1}\right)^2\right]v_1^2,\end{aligned}$$

and hence the velocity of the outer circumference of the wheel is

$$(1)\quad v_1 = \sqrt{\frac{2gh - (1 + \zeta_1)c_2^2}{2 + \zeta - \left(\frac{r}{r_1}\right)^2}}.$$

The relative velocity of discharge $c_2$ is arbitrary, but it must be taken as small as possible (4 ft. at the most); $\frac{r}{r_1}$, the ratio of the radii, is also to be small (*e.g.*, $\frac{1}{5}$ to $\frac{1}{4}$).

Hence from the last formula (1), we have first the velocity of the exterior of the wheel $v_1$, and then that of the interior,

$$(2)\quad v = \left(\frac{r}{r_1}\right)v_1,$$

and the section of the supply opening,

$$(3)\quad F = \frac{Q}{v_1},$$

and also that of the orifices of discharge,

$$(4)\quad F_2 = \frac{Q}{c_2}.$$

By putting also

$$F_2 = 2\,\pi\, r\, e = 2\,\pi\, r^2,$$

and, therefore, the depth of the wheel inside $e = r$, the inner radius of the wheel is obtained,

$$(5)\quad r = \sqrt{\frac{F_2}{2\,\pi}},$$

from which the exterior radius $r_1$ can now be easily determined. If $e_1$ is the outer depth of the wheel, we have for the relative velocity of entrance,

$$(6)\quad c_1 = \frac{r\,e}{r_1\,e_1} \cdot c_2.$$

The performance of the wheel comes out,

$$L = \left[h - \left(\zeta \frac{v_1^2}{2\,g} + \frac{c_1^2}{2\,g} + (1 + \zeta_1)\frac{c_2^2}{2\,g} + \frac{v^2}{2\,g}\right)\right] Q\,\gamma$$

$$= \left[h - \left[\zeta + \left(\frac{r}{r_1}\right)^2\right]\frac{v_1^2}{2\,g} + \left[1 + \zeta_1 + \left(\frac{r\,e}{r_1\,e_1}\right)^2\right]\frac{c_2^2}{2\,g}\right] Q\,\gamma.$$

EXAMPLE.—The design and computation of a Thomson turbine is to be made, the fall being 10 ft., and the volume of water 12 cubic ft. Putting the relative velocity of discharge $c_2 = 4$ ft., we have the section of the discharge openings,

$$F_2 = \frac{Q}{c_2} = \frac{12}{4} = 3 \text{ sq. ft.},$$

and hence the inner radius of the wheel is

$$r = \sqrt{\frac{F_2}{2\,\pi}} = \sqrt{\frac{3 \times 7}{2 \times 22}} = \sqrt{\frac{21}{44}} = \sqrt{0 \cdot 4773} = 0 \cdot 691 \text{ foot},$$

for which we might put $r = 0 \cdot 70$.

Taking $\frac{r_1}{r} = 4$, we obtain the outer radius of the wheel,

$$r_1 = 4 \times 0 \cdot 7 = 2 \cdot 8 \text{ ft.}$$

The depth of the wheel inside is $e = r = 0 \cdot 7$ ft., whence the depth outside might be taken $e_1 = 0 \cdot 6$ ft., so that the velocity of entrance into the wheel comes out,

$$c_1 = \frac{r\,e}{r_1\,e_1} \cdot c_2 = \tfrac{1}{4} \cdot \tfrac{7}{6} \cdot 4 = \tfrac{7}{6} = 1 \cdot 167 \text{ ft.}$$

The velocity of the outer surface of the wheel, when we take $\zeta = 0 \cdot 5$ and $\zeta_1 = 0 \cdot 2$, is

$$v_1 = \sqrt{\frac{2gh - (1 + \zeta_1)c_2^2}{2 + \zeta - \left(\frac{r}{r_1}\right)^2}} = \sqrt{\frac{644 - 1\cdot 2 \times 16}{2\cdot 5 - \frac{1}{16}}} = \sqrt{\frac{624\cdot 8}{2\cdot 4375}}$$

$$= 16\cdot 0102 \text{ ft.},$$

while the velocity of the inner surface is

$$v = \frac{r}{r_1} v_1 = \frac{16\cdot 010}{4} = 4\cdot 0025 \text{ ft.},$$

and consequently the number of revolutions of the wheel per minute is

$$u = \frac{30\,v}{\pi\,r} = \frac{30 \times 4\cdot 003 \times 7}{22 \times 0\cdot 7} = \frac{1200\cdot 9}{22} = 54\cdot 59.$$

From these we determine the maximum or initial section of the annular supply passage,

$$F = \frac{Q}{v_1} = \frac{12}{16\cdot 01} = 0\cdot 7495 \text{ sq. ft.},$$

and hence the width of the same,

$$d = \frac{F}{e_1} = \frac{0\cdot 7495}{0\cdot 6} = 1\cdot 249 \text{ ft.}$$

Since

$$\left[\zeta + \left(\frac{r}{r_1}\right)^2\right] v_1^2 = 0\cdot 5625 \times \overline{16\cdot 01}^2 = 144\cdot 18,$$

and

$$\left[1 + \zeta_1 + \left(\frac{r\,e}{r_1\,e_1}\right)^2\right] c_2^2 = 1\cdot 285 \times 16 = 20\cdot 56,$$

we have the following for the fall, utilized by the wheel,

$$h_1 = h - \left[\zeta + \left(\frac{r}{r_1}\right)^2\right]\frac{v_1^2}{2g} - \left[1 + \zeta_1 + \left(\frac{r\,e}{r_1\,e_1}\right)^2\right]\frac{c_2^2}{2g}$$

$$= 10 - 0\cdot 0155\,(144\cdot 18 + 20\cdot 56) \doteq 10 - 2\cdot 553 = 7\cdot 447 \text{ ft.}$$

The efficiency of the wheel is

$$\eta = \frac{7\cdot 447}{10} = 0\cdot 745,$$

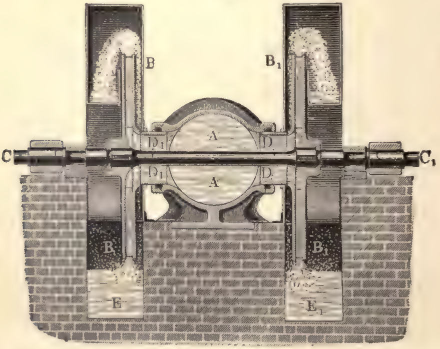
and its performance

$$L = Q\,h_1\,\gamma = 7\cdot 447 \times 12 \times 62\cdot 5 = 5585 \text{ ft. lbs.} = 10\cdot 2 \text{ horse-powers.}$$

§ **295.—Turbines with a Horizontal Axis.**—The practice has lately begun of building vertical water-wheels according to the principles of reaction turbines, but little is as yet definitely known as to their utility. The Jonval and Whitelaw wheels, in particular, are set upon a horizontal shaft (compare § 238). That this arrangement can be of advantage only with high falls, is easily inferred, since it is only with high falls that an unavoidable loss of fall accompanying the discharge of the water from the wheel can be disregarded. A wheel of this sort always has the advantage over a turbine that it can be set on lighter foundations more securely, and with greater protection from the access of the water than an ordinary turbine. According to Jonval and Redtenbacher, we can with advantage set two wheels opposite each other upon the same horizontal shaft, since the pressure of the water in each wheel in the direction of the axis is thereby balanced without having any effect upon the pivots.

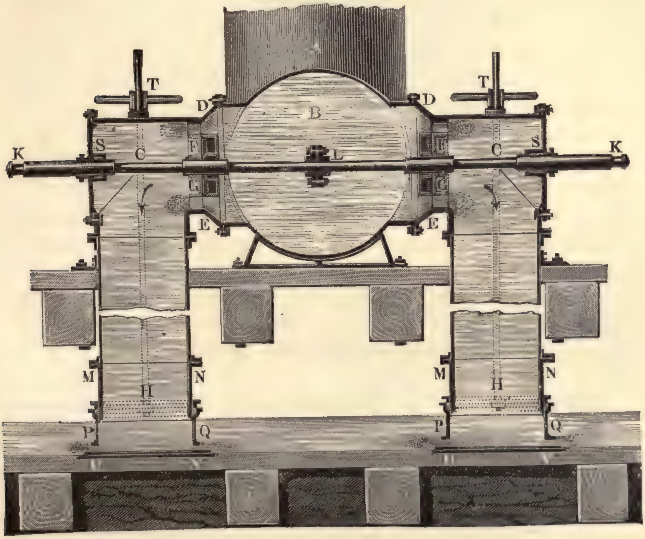
The arrangement of a vertical double turbine with separate revolving pipes, as given by Redtenbacher, is shown in Fig. 530. $A\ A$ is the supply pipe opening at the side, $B\ B$ one wheel and $B_1\ B_1$ the other, $C\ C_1$ the horizontal shaft, $D\ D$ and $D_1\ D_1$ are the packing rings (see Vol. II., § 246), and finally $E$ and $E_1$ are the tail races. We can also easily imagine how in a similar manner a Combes or Fourneyron turbine is to be set up. They would have in addition an arrangement of guides before each wheel, and under similar circumstances would, of course, be much smaller. A throttle-valve inserted with supply pipe is best suited to regulate the motion of the wheel.

FIG. 530.



Upon the same principles, we can also combine two Jonval turbines on an ordinary horizontal shaft. The two wheels opposite each other are supplied from a common reservoir, but discharge the water into separate waste pipes. A wheel similarly constructed with 31 ft. fall and 6396 cubic ft. of water per minute, drives a cotton-mill at West Springfield, Massachusetts; it has a diameter of 40 inches, and at its normal rate makes 220 revolutions per minute, giving thereby an efficiency of 0·65. According to the Journal of the Franklin Institute, there are said to be several such turbines in the United States, of 15 to 140 horse-powers, with falls of 9 to 26 ft., and which work with advantage in furnishing the motive power for spinning-mills, paper-mills, rolling-mills, etc. (See also the "Polytechnisches Centralblatt," Jahrgang 1850, Lieferung 9, or the "Civil Engineer and Architects' Journal," 1850, Febr., page 68.

FIG. 531.



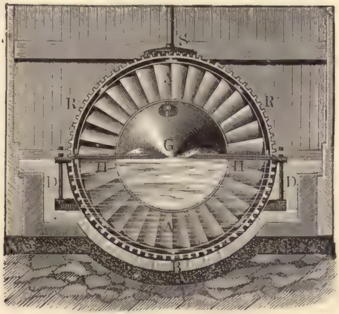
Similar double turbines have been made by Roschkoff, lieutenant-colonel in the imperial corps of mining engineers at Catharinsburg, Russia. A vertical longitudinal section of a wheel of this sort is shown in Fig. 531. The supply pipe *A* opens into the horizontal reservoir *B*; on this fit at both sides the casings of the turbines *D E S*, *D E S*, and the casings end in the vertical suction pipes *H U V*, *H U V*. The water brought to the wheel chamber through the supply pipe is led to the wheels *F G*, *F G*, by means of the guide-wheel, and after having accomplished its work, flows off through the suction pipes into the tail race. To regulate the discharge, an annular sluice *P Q* is used, which can be raised or lowered by means of a worm-wheel *T* and rods (compare Fig. 518).

The shaft *K L K*, which carries the wheels *F G*, *F G*, comes out of the casings of the wheels through the stuffing boxes *S*, *S*, receives on the outside the driving wheels, and is supported near them upon firm bearings. It might also be well to support the shaft upon another bearing inside the reservoir. This turbine has one great advantage over other turbines with horizontal shafts, since it uses the fall equally at all points of the circumference of the wheel (vid. the "Civilingenieur," Bd. III., 1857).

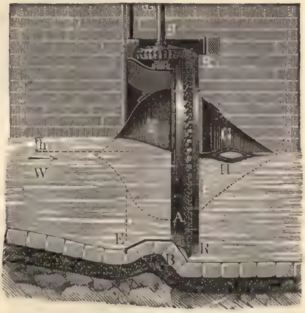
§ **296.—The Screw Water-Wheel.**—The screw water-wheel (Fr. roue-hélice; Ger. Schraubenrad) is essentially different from the screw turbine. This wheel is essentially a Burdin turbine with a horizontal axis, without guides, and with only a part of the wheel exposed to the action of the water at one time (see § 234). It is, however, different from the Burdin turbines, since the water is brought directly to the wheel by the head race and in the direction of the axis of the wheel. The arrangement of a screw water-wheel of this sort can be seen in Fig. 532, I. and II. I. shows a rear view and II. a vertical longitudinal section of the whole machine. *A A*, the wheel proper, is made with helicoidal vanes, like an ordinary Fontaine turbine; it hangs in a stone sill *D B D*, which surrounds the lower half of the circumference of the wheel and is concentric with it. To conduct the water *W* to the wheel in the proper direction, not only is the head race in front of the sill surrounded by a plate metal casing *E* which gradually contracts towards the wheel, but a pear-shaped plate metal covering *F* is also set in, the base of which is directed towards the inner part of the wheel into which the floats do not extend, while the point is directed towards the current. Moreover, in order that the water may flow out into the tail race after performing its work on the wheel, without causing any whirls or eddies, a conical cover *G* is also placed behind the wheel. The two covers *F* and *G* are firmly fastened to the side walls of the canal by cross-pieces *H*, *H*, having a lens-shaped section, and they are at the same time used to support the horizontal shaft of the wheel. In order that $\delta$, the angle of exit of the water, may be reduced as much as possible, the width of the wheel passages gradually in-

FIG. 532.

I.



II.

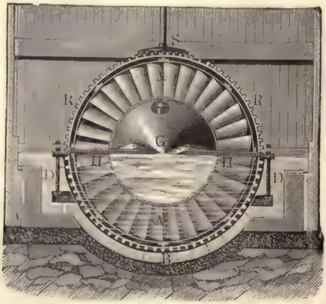


creases from the front to the rear, and consequently the two convex surfaces of the wheel have a conical form. The power is transmitted by the bevel-wheel *R R* which surrounds the outer convex surface of the wheel close to the rear, and engages the pinion *S* of a vertical shaft which continues the rotation. We can easily infer that a screw water-wheel of this sort is especially suited to utilizing a water-power of small fall and a large quantity of water.

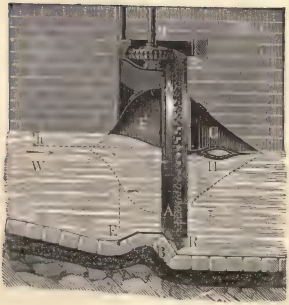
Since, in this case, the efflux of the water coming out of the wheel takes place under water, the effective head due to pressure or to velocity is therefore the same for all the particles of water flowing through the wheel—namely, the fall or distance *h* between the surface of the water above and below the wheel, and consequently the action of the water, is the same at all points of the wheel (compare § 152).

533.

I.



II.



For this reason, the theory of the Fontaine turbines explained above (§ 278) finds its immediate application to these screw water-wheels, especially when, as is generally the case, the velocity of approach and discharge of the water is very small (3 ft. at the most).

Since the depth of the water has no influence upon the way in which it acts in the wheel, this wheel will work just as well with a high as with a lower state of the water, and can consequently be used with advantage instead of the ordinary undershot-wheels when the height of the water in the race is very variable.

Girard has built a wheel of this sort for supplying the motive-power for a chocolate factory at Noisiel (sur Marne), the fall being 0·5 metre (1·640 ft.) on an average, and the volume of water about 3 cubic metres (105·9 cubic ft.) per second (see the work "Nouveau Récepteur hydraulique, dit Roue-Hélice à axe horizontal, ou Turbine sans directrices, par Girard," Paris, 1855).

CLOSING REMARK.—It is not until recent times that the literature of turbines has been greatly extended. Since we have already cited a great number of treatises in the course of this article, we shall now mention only the principal ones, and especially the original works on reaction turbines. The first treatise on Fourneyron's turbine is found in the "Bulletin de la Société d'encouragement," for the year 1834, translated into German in Dingler's "Polytechnisches Journal," Bd. LIII. Next, Morin experimented and published the results of these experiments in the work, "Expériences sur les roues hydrauliques à axe vertical, appelées Turbines," Metz et Paris, 1838, and there appears also the first well-grounded theory of these wheels in the article of Poncelet in the "Comptes rendus des séances de l'Acad. de Paris," entitled, "Théorie des effets mécaniques de la Turbine-Fourneyron," Paris, 1838. In the second edition of D'Aubuisson's Hydraulics, these wheels are treated briefly and without any special theoretical views. Combes' work, "Recherches théoretiques et expérimentales sur les roues à réaction ou à tuyaux," Paris, 1843, is, to be sure, by no means comprehensive, but is worthy of considerable notice in so far as one finds the hydraulic resistances here considered for the first time in the development of the theory, which Poncelet and even Redtenbacher did not do. The work of the last-named author, "Theorie und Bau der Turbinen und Ventilatoren," Mannheim, 1844, is elaborated chiefly according to Poncelet's theory, but, as to the rest, is the most complete and best treatise upon this subject. In regard to the newer turbines, the following treatises deserve notice: "Rapport sur un Mémoire de M. M. A. Koechlin, concernant une nouvelle turbine (Jonval) construite dans leurs ateliers, par Poncelet, Piobert et Morin," and

"Note sur l'application de la théorie du mouvement des fluides aux expériences de M. Marozeau, par Morin," in the XXII. Volume (1846) of the "Comptes rendus," etc., etc. An extract from this is found in the "Polytechnisches Centralblatt," Bd. VIII., 1846; also, "Expériences et note sur la turbine de M. Fontaine-Baron, par Morin," in the XXIII. Volume (1846) of the "Comptes rendus," etc., etc.; an extract in German likewise in the "Polytechnisches Centralblatt," Bd. VIII. Respecting Jonval's and Fontaine's turbines, reference is also made to the "Bulletin de la société d'encouragement" for the year '43 and '44; Paris, 1844 and 1855. Good drawings and description of the turbines of Cadiat, Callon, Fourneyron, and Gentilhomme are also found in Armengaud's "Publication industrielle." For Porro's turbine, examination must be made of the "Polytechnisçhes Centralblatt," Bd. VII., 1846. The arrangement of a Nagel turbine can be learned from Dingler's Journal, Bd. XCV., and that of one of Passot's turbines from the same Journal, Bd. XCIV. Bourgeois' screw water-wheel (Fr. turbine-hélice) is a turbine with helicoidal passages (vid. "Polytechn. Centralblatt," Bd. I., 1847). So also Plataret's screw turbine, at St. Maur, near Paris, is described in the "Polytechnisches Centralblatt," 1849. Thomson's turbines—*i. e.*, the patent case water-wheel and the patent suction wheel—stand by themselves. These two wheels are described in the "Mechanics' Magazine," January, 1851. The turbines of Girard, etc., are discussed in "Le Génie industrielle, par Armengaud Frères," Tome XII. and Tome XIII., 1856 and 1857. See also the "Notizblatt des Architekten- und Ingenieurvereins zu Hannover," Bd. III., 1853. The theory of Fourneyron's inward-flow turbines is discussed by Prof. Zeuner, in Vol. II. of the "Civilingenieur." Plates for graphical methods for finding the most important elements of the construction of turbines are communicated by Bornemann in Vol. IV. of the "Civilingenieur." The turbines of Francis, etc., are treated in the work, "Lowell Hydraulic Experiments," by James Francis, Boston, 1855. The treatise on "Die Turbinen oder horizontalen Wasserräder," by Harzer, Weimar, 1851, is principally a translation of the first edition of the above work. A more recent work is Peter Rittinger's "Theorie und Bau der Rohrturbinen," Prag 1861 and 1865. Turbines are treated in a peculiar manner in Rankine's "Manual of the Steam-Engine and other Prime Movers," London and Glasgow, 1859. In regard to the turbine at the London Industrial Exhibition, 1862, and in particular concerning Thomson's vortex water-wheel, examination is to be made of a discussion by Bernhard Lehmann in the "Journal of the Society of German Engineers," Vol. VII., 1863, Vol. II. (1858); this journal also contains a new theory of horizontal water-wheels by R. R. Werner. A general theory of the construction of floats for turbines is communicated by F. K. H. Wiebe, in "Civilingenieur," Bd. 5

# CHAPTER VI.

## WATER-PRESSURE ENGINES.

§ **297.—Water-Pressure Engines.**—Water-pressure engines (see § 170) are put in motion by the pressure of water confined in an upright or nearly upright pipe. This motion, however, is not, as in the case of water-wheels, a constant rotary motion, but is reciprocal. The principal parts of such a machine are, as may be seen from Fig. 534, I. and II., the following: *A* is the reservoir or supply cistern, *A B* the supply pipe (Fr. tuyau de chute; Ger. Einfallröhre), *C* is the working cylinder (Fr. cylindre principal; Ger. Stiefel or Treibcylinder), in which the water performs

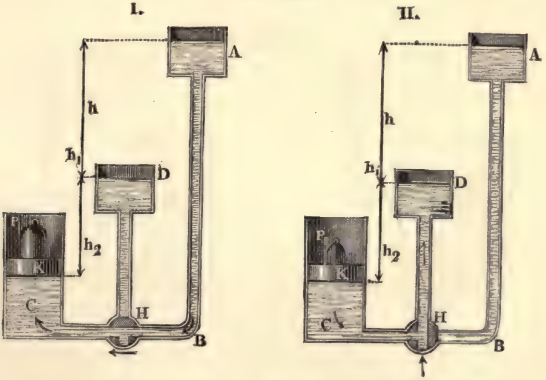
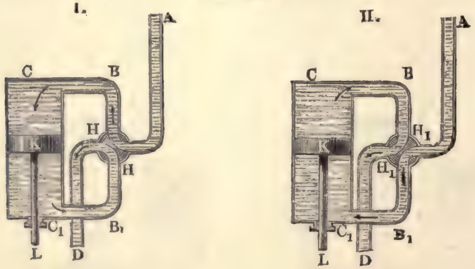


Fig. 534.

its work by forcing upwards the loaded driving piston $K$ (Fr. piston moteur; Ger. Treibkolben), and $H\,D$ is the discharge pipe (Fr. tuyau de décharge; Ger. Austrageröhre). In the connecting pipe $B\,C$, which joins the working cylinder with the supply pipe, is situated the regulator (Fr. régulateur; Ger. Steuerung), which in this case is a cock (Fr. robinet; Ger. Hahne) with a T-shaped channel, which serves to unite or to disconnect the supply pipe and the working cylinder. In the first case, the water forces the piston, with its load $P_1$, upward, and, in the second case, the water beneath the piston, cut off from the supply pipe, returns through the cock and is discharged by the pipe $H\,D$, while the piston, now unloaded, descends. There are single and double acting engines, and also engines with one and with two *cylinders.* In the single-acting engine (Fr. machine à simple effet; Ger. einfachwirkende Maschine), which is shown in Fig. 534, the piston is moved by the water, in one direction only; in the opposite direction, its own, or an added weight $P_2$, is the moving force. In the double-acting engine (Fr. machine à double effet; Ger. doppeltwirkende Maschine), on the contrary, the downward as well as the upward motion is occasioned by the force of the water. Fig. 535, I. and II., shows the arrangement of such an engine. From this figure is to be seen how the motive water passes first (I.) through $A\,B\,C$, forces

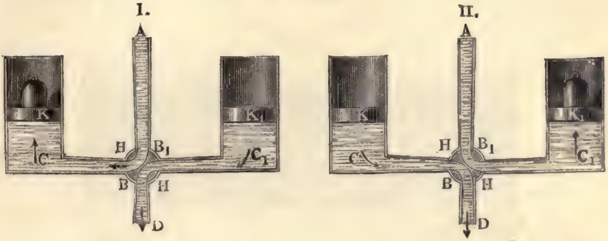
FIG. 535.



the piston *K* downward, and thus drives the water beneath through the passage $C_1$ $B_1$ $D_1$, and, second (II.), through *A* $B_1$ $C_1$, forcing the piston upward, and the water above out through *C B D*.

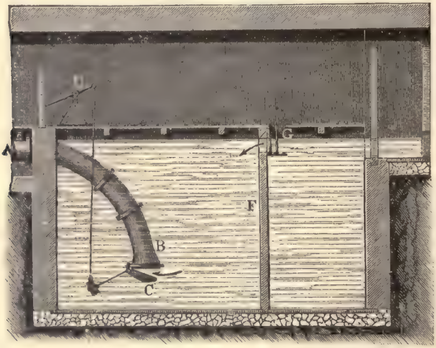
The water-pressure engines, thus far described, have had but one cylinder; Fig. 536 represents an engine with two cylinders. Here, while the motive water *A B C* forces the piston *K* upward (I.), the piston $K_1$ descends, and the dead water beneath passes off through the passage $C_1$ $B_1$ *D*; and, again (II.), while the motive water causes $K_1$ to ascend, *K* descends, and the dead water passes off through the discharge pipe *D*.

FIG. 536.



**§ 298.—Supply Pipes.**—The principal parts of a water-pressure engine will now be described more in detail. The motive water for the engine is collected, in the first place, in the so-called supply cistern or feeding reservoir. It is advantageous to construct this basin as large as may be, so that the water may have more chance to drop its impurities, and so that the variation of level in the basin may be small. It is, furthermore, necessary to place in this reservoir, gratings or lattices, to hold back foreign substances, such as chips, leaves, etc., and, if necessary, to place partitions therein, so that the water is forced to travel through a circuitous route, and is given more opportunity to deposit its impurities. The opening of the supply pipe should be at least 1½ feet from the bottom of the basin, and 3 to 5 feet below the water level, to prevent as well the entrance of heavy matter and the formation of whirlpools. To this end also, it is well to introduce the pipe into the basin by curves, so that it may open downward. Furthermore, a flap valve or a conical plug may be used to close the opening and prevent the entrance of the water. Such a feeding apparatus is represented in Fig. 537. *A B* is the curved entrance piece of the supply pipe, *C* the flap valve, *D* the lever to move the flap valve, *F* a partition, and *G* two lattice gratings to retain floating bodies.

FIG. 537.



The supply pipe is usually made of cast-iron, in lengths of 5 to 8 feet, and having a width of ⅓ to ½ that of the working cylinder. The thickness of the pipe should be ¾ to ⅘ inches; the first thickness is for the upper, the second for the lower pipes. More accurately, the thickness *e* is obtained from the formula,

$$e = 0{\cdot}0024\, p\, d_1 + 0{\cdot}75 \text{ inches},$$

in which $d_1$ is the inner diameter in inches and $p$ the pressure in atmospheres (at 34 feet). The formula in Vol. I.,

§ 363, gives. for simple conducting pipes, a smaller thickness; these are not, however, applicable here, since the water acts with varying force, and sudden shocks, due to the shutting off. It is, moreover, necessary to test the pipes before using. For this purpose, the pipe is closed at both ends, filled with water, and this water connected with an hydraulic press by a small pipe. By repeated strokes of the piston of this press, a pressure will be exerted which should exceed (four to five times) the pressure designed to be sustained by the pipe. If the pipes do not allow any leakage, they may be used. Many pipes do not stand this first test, but are, perhaps, serviceable, since, by the formation of rust, their porosity may be destroyed; this may be proven by another test, some weeks later. For the water-pressure engine at Huelgoat, described below, boiled linseed oil was used for the hydraulic test, and the pipes were thus given an internal coating, which protected them from the chemical action of the water.

The lengths are fastened together either by simple sleeves or by flanges and bolts (see § 164). Between the flanges is to be placed a layer of lead or putty, which is compressed between them strongly by the bolts. To make the joint more secure, the space between the flanges is filled with molten lead, which also fills up the ring-formed channels in the faces of the flanges. The putty is prepared with powdered chalk, linseed oil, and chopped hemp. On the inside, the joint is often strengthened by sleeves of copper plate, in the same way as the coupling of wooden pipes. A pipe connection with flanges and bolts is shown in Fig. 538, partly in elevation, partly in section. The connection of the flanges *A A* and *B B* by the bolts *A B*, *A B* is essentially the same as in the usual conducting pipes, § 164; the sleeve *C C* has in the middle of its exterior surface a ring *d*, which fits into the joint between the pipes.

Fig. 539 shows a simple bell joint. In order to attain complete tightness by lead, etc., the bell *A*, as well as the lower end of the pipe *B*, is furnished with ring-shaped grooves. To distribute the weight of the supply pipe, some of the lengths, say every 50 ft., are furnished with lugs or rims *C C*, by which they rest in the slots *D D*.

FIG. 538.

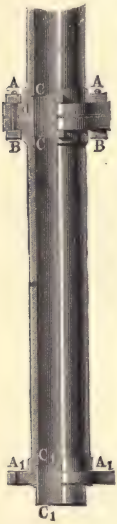


FIG. 539.

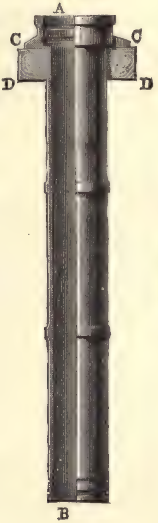
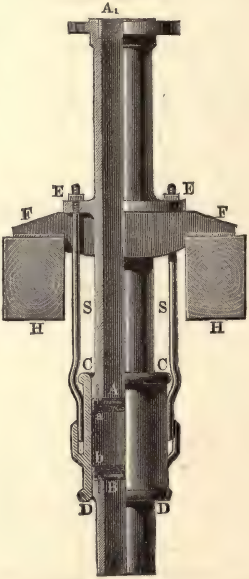


FIG. 540.



Besides these couplings, a sliding sleeve connection is also necessary, in order to place with advantage the whole supply pipe, as well as to allow for the changes of length by changes of temperature (see the compensation pipes, Fig. 343, § 164). In the movable connection represented in Fig. 540, the pipes *A*, *B*, placed about 1 ft. apart, are furnished with a leathern rim *a*, *b* on each end, and surrounded by an accurately fitted sleeve *C C D D*. The upper pipe $A A_1$ has, in the middle, the supporting ring *E E*, to which are fastened the rods *S S* carrying the sleeve, and which rests directly upon the cast-iron lugs *F F*, and indirectly upon the bearings *H H*.

§ **299.—Working Cylinder.**—The working cylinder is constructed either of cast-iron or, on account of the greater capability of gun metal to be polished, of this latter substance. In order that the number of strokes shall not be great (3 to 6 per minute), and therefore the loss of work be small, the cylinder is made much longer than wide, so that the piston stroke $s$ shall be $2\frac{1}{2}$ to 6 times the diameter $d$ of the piston. The mean velocity $v$ of the piston is made about 1 ft., so that the mean velocity $v_1$ of the water in the supply pipe, and consequently the hydraulic resistance, shall not be too great. It is advisable not to exceed 10 ft. for this velocity, and, better still, to limit it to 6 ft. If we assume $v = 1$ and $v_1 = 6$ ft., we obtain for the ratio of the diameter $d_1$ of the supply pipe to the diameter $d$ of the cylinder, since the water used is $= \frac{\pi d^2 v}{4} = \frac{\pi d_1^2 v_1}{4}$,

$$\frac{d_1}{d} = \sqrt{\frac{v}{v_1}} = \sqrt{\tfrac{1}{6}} = 0 \cdot 408,$$

or about $0 \cdot 4$.

If the quantity of motive water used per second $= Q$, we can put, for a double-acting engine, or for a single-acting engine with two cylinders,

$$Q = \frac{\pi d^2}{4} . v,$$

and hence the necessary diameter of the working cylinder is

$$d = \sqrt{\frac{4 Q}{\pi v}} = 1 \cdot 13 \sqrt{\frac{Q}{v}};$$

therefore for $v = 1$, $d = 1 \cdot 13 \sqrt{Q}$ ft.

For a single-cylinder engine,

$$Q = \tfrac{1}{2} \frac{\pi d^2}{4} v,$$

and therefore

$$d = 1 \cdot 60 \sqrt{\frac{Q}{v}};$$

therefore for $v = 1$, $d = 1 \cdot 60 \sqrt{Q}$ ft.

If, now, the stroke $s$ of the piston has been made $= 2\frac{1}{2}\,d$ to $6\,d$, the time of a simple stroke (ascent or descent) is to be found by the formula

$$t = \frac{s}{v};$$

or for $v = 1$,

$$t = s \text{ seconds},$$

and therefore the number of half strokes per minute

$$n_1 = \frac{60''}{t} = \frac{60 \,.\, v}{s};$$

or for $v = 1$, $$n_1 = \frac{60}{s},$$

and the time of a stroke,

$$n = \frac{n_1}{2} = \frac{30\,v}{s};$$

or for $v = 1$, $$n = \frac{30}{s}.$$

It is furthermore well to make the rate of ascent somewhat slower and the rate of descent somewhat faster than the mean velocity in a single-acting one-cylinder engine, since the hydraulic resistance is greater during the ascent.

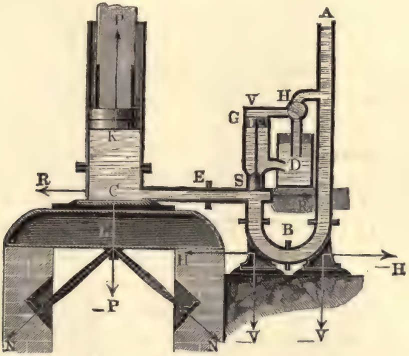
The working cylinder should be exactly turned and polished inside, so that the piston may play easily therein. The thickness, on account of the continual wearing it suffers, is made relatively very great (in existing machines it is from 2 to 3 inches); it however depends upon the head of water and the diameter of the cylinder, and is properly calculated from the formula

$$e = 0{\cdot}0024\, p\, d + 1{\cdot}25 \text{ inches.}$$

The cylinder may be strengthened by iron hoops or rings.

The driving piston is forced upward, or in the direction of its motion, by a force $P$ which is measured by the weight $F h \gamma$ of a column of water, having a base $F$ equal to the area of the piston, and a height $h$ equal to the perpendicular distance from the piston to the water surface in the supply reservoir; and an equal force $(-P)$ in the opposite direction is brought to bear upon the bottom of the cylinder. Generally, the height $h$ is some hundred feet; the force of the water is therefore very considerable, and it is necessary to make the support of the cylinder correspondingly strong. Since this kind of machine is used mostly to raise water from mines, it is usually found in shafts, and, therefore, cannot be placed directly upon an unyielding foundation, but it becomes necessary to support it by arches or girders of iron, or by strong oak beams. In some machines, the cylinder is placed directly upon cast-iron arches.

FIG. 541.

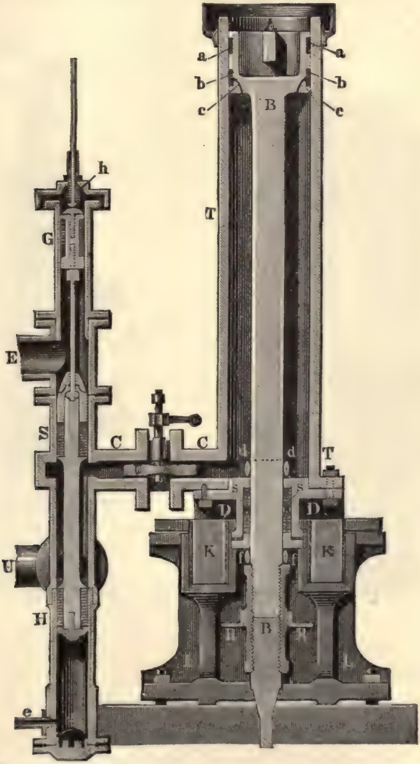


In the machine sketched in Fig. 541, the working cylinder rests upon a couple of iron beams $L$, strengthened in the middle by cast-iron braces. The force $-P$ is in this case partly taken up by these braces, which, therefore, exert the inclined, downward-directed thrusts $N$, $N$ against the foundation wall, and, finally, against the solid rock.

In the same way, the supply pipe exerts a downward-acting force $(-V)$ proportional to its area $F_1$ which renders a special support necessary. Besides these, the working cylinder has to withstand the horizontal force $R = F_2 h \gamma$, which increases with the area of the connecting pipe, and the supply pipe has to withstand the force $(-H)$, which is proportional to the cross-section $F_1$, or $= F_1 h \gamma$. These forces counteract the equal and opposite forces $(-R)$ and $H$, so that the machine as a whole has no tendency to move sideways; but a tendency to burst asunder exists, which must be counteracted by the joints $E$ and $B$ and by the bed plates. In the machine represented above, the curved connecting pipe $B S$ has also to sustain a vertical pressure $(-V)$, and therefore should have a firm foundation.

§ **300.—Driving Piston.**—The driving piston, upon which the water acts directly, consists essentially of a cylinder fitted into the working cylinder by turning the outer rim. In order to make it perfectly tight, without sustaining too much resistance to motion, the so-called *packing* (Fr. garniture; Ger. Liderung) is used; this may be applied either to the piston itself or to the cylinder. In the first case, the piston is a short cylinder only $\frac{1}{3}$ to $\frac{1}{2}$ the diameter in height, while in the second case it has a length equal to that of the working cylinder, and is then usually called a *plunger* (Fr. plongeur; Ger. Mönchskolben, Bramahkolben). The packing of the driving piston is generally made of strips or rings of leather, more rarely of leather disks or metal rings; it must be always fastened to the inner surface of the cylinder, with reference to the water pressure, or to the outer surface of the piston, so that, on the one hand, no water may pass, and, on the other, the friction may not be too great. For these reasons, the hydrostatic packing, where the leather or material forming the packing is pressed against the polished surface by the water itself, is to be preferred. Usually three or four such rings of leather are sewn or riveted together, after having been soaked in fat, and laid in grooves turned in the circumference of the piston, or fastened by screws with a metal ring on the base of the same.

Fig. 542.



In Fig. 542 is represented a driving piston (of a water-pressure engine at Clausthal) with packing laid in grooves. *A* is the piston proper, and *B B* the piston rod. Further, *a a* and *b b* are the packing rings, and *c c* the small passages by which the water pressure is applied to the inner side of these rings.

Fig. 543 shows the piston packing of a Freiberg machine.

FIG. 543.

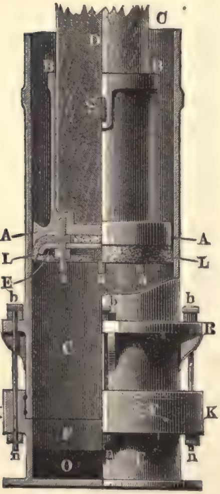
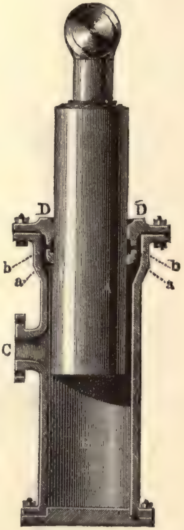


FIG. 544.



*A A B B* is the cast-iron piston stock into which the foot *D* of the piston rod fits, and is fastened by the key *S*. The foot plate *A A* of this piston stock is covered with the leather *h h*, and this again by the iron plate *E*. Both the foot plate and the covering plate are curved at the outer edges, in order to serve as supports for the leather. Four bolts *a*, *a* . . . serve to press the plate *E* against the leather, and to fasten both to the foot plate. The figure also shows how the working cylinder is fastened to the foot piece *F* by the bell *KK* and the bolts *b n*, *b n*. . . . This foot piece forms, at the same time, a part of the connecting pipe which enters at *O*.

A plunger piston may also be furnished with hydrostatic packing, as may be seen from Fig. 534. Here *A* is the piston, *B* the cylinder, *C* the connecting pipe, *DD* the packing box, fastened on with screws, *a a* the packing rings, and *b b* the passages for the water. It is also easier to place and maintain this packing in a box for the purpose than when it is fastened to the piston. The use of this piston without packing is to be recommended also, because it is easier to turn the outside of a cylinder perfectly true than the inside. Finally, a peculiar advantage of this arrangement is that, by changing the piston and the packing box, it is possible to change the power of the engine according to requirement.

§ **301.—Piston Rod and Stuffing Box.**—The piston rod (Fr. tige du piston; Ger. Kolbenstange) is directed, from the piston itself, either towards the opening of the cylinder or towards the bottom. In the first case, no special elaboration is required, and it may then be made of wood, as in Fig. 543; in the second case, on the contrary, it must pass through a stuffing box, is therefore to be turned off to allow this, and must be made of iron or gun metal. The strength of such a rod may be found from the theory of elasticity.

If $d$ is the diameter of the driving piston, and $p$ the water pressure per square inch of the piston, we have

$$P = \frac{\pi d^2}{4} . p;$$

if, now, $d_2$ is the diameter of the piston rod, and $T$ the modulus of proof strength of the material, we have for the strength of the rod

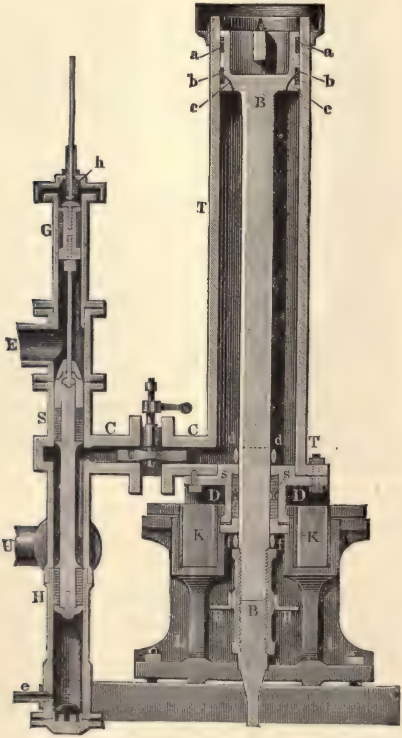
$$P = \frac{\pi d_2^2}{4} T;$$

we obtain therefore by equating these forces.

$$d_2 = d\sqrt{\frac{p}{T}}.$$

In this formula, $T$ is to be taken from the table in Vol. I., § 212, while we have $p = \frac{h\gamma}{144}$, in which $h$ is the head in feet.

FIG. 545.



For a piston rod of wrought-iron which is only subjected to tension, we can put $T = 10000$ pounds, and, consequently,

$$d_2 = 0{\cdot}01\, d \sqrt{p} = 0{\cdot}00655 \sqrt{h} \text{ inches.}$$

When subject to compression, they should be made twice as strong (compare Vol. I., § 269).

The stuffing box (Fr. boîte à garniture; Ger. Stopfbüchse) is a box situated at one end of the cylinder and filled with leather rings or cut hemp, so that the piston rod will move through it readily, and yet the water, steam, air, etc., as the case may be, cannot pass. In water-pressure engines, the stuffing boxes are usually packed with leather rings, on which account they are also called leather boxes (Fr. boîtes à cuir; Ger. Lederbüchse). In Fig. 545, *B B* is the piston rod and *D D* the stuffing box, the packing of which is pressed in by the cover *B*. Sometimes a metal ring, with small holes for the passage of oil, is placed between the layers of leather, as *s s*, Fig. 545. If the piston rod goes through the cover of the cylinder, the cover of the stuffing box is furnished with a hollow for the reception of the oil, but if it passes through the foot plate the oil must be forced in artificially.

In the machine at Clausthal, oil presses are also used, which force the oil through small passages, by means of a small piston, into the X-formed brass ring above mentioned.

The lubricant is composed of 6 parts of lard, 5 of tallow, and 1 of olive oil, or, better, of pure olive or neat's-foot oil.

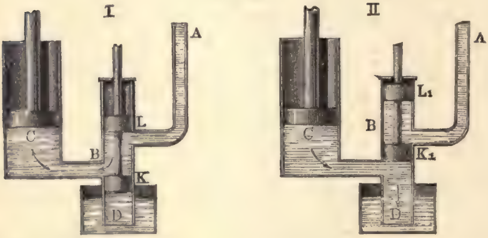
§ **302.**—**Regulator.**—The regulator is, as it were, the soul of a water-pressure engine; by it the machine is enabled to perform its work without interruption. It is composed essentially of two principal parts, one of which, alternately, shuts the water from, or admits it to, the working cylinder, and the other is necessary to connect the first with the engine proper (with the piston rod), so that no outside force will be necessary to work it. We can very well call the first mechanism the *inner* and the second the *outer* gear or regulator.

As regards the *inner* regulator of a water-pressure engine, we have to speak especially of the *piston regulator*. Old machines are furnished with cocks, and some new water-pressure engines are furnished with clack and slide valves, like a steam-engine.

The manner in which a water-pressure engine is regulated by a cock is explained above (§ 297), and the way in which a piston regulator works is to be seen from the following:

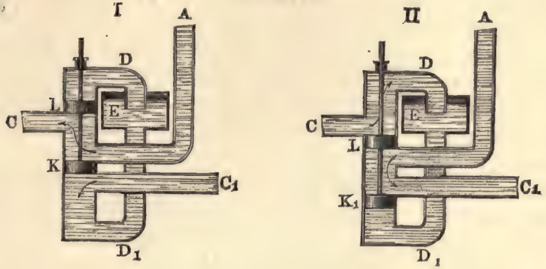
*Piston Regulator.*—Fig. 546, I. and II., shows the arrangement of a piston regulator for a single-acting single-cylinder engine. *A* is the supply pipe, *C* the working cylinder, *B* the cylinder enclosing the regulator piston or the regulator cylinder, *D* the discharge pipe, *K* the regulator piston, and *L* the so-called counter piston, which only serves, by creating a counter pressure, to make the movement of the regulator piston or rod easier. When the regulator piston *K* is in its lowest position (I.) the working cylinder is in connection with the supply pipe, and the driving piston can only ascend. When, on the contrary, it is in its highest position (II.) the regulator piston $K_1$ shuts off the water in the supply pipe, and that which is beneath the piston is forced out at *D*.

FIG. 546.



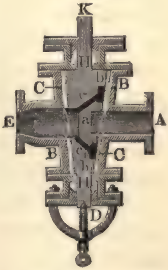
The arrangement of the regulator piston for a double-acting or for a double-cylinder engine may be seen from Fig. 547, I. and II. Here $A$ is the supply pipe, $C$ the connecting pipe for one cylinder and $C_1$ that for the other, $D$ the discharge pipe for the one and $D_1$ that for the second. It is seen from $I$ how the piston, in its upper position, admits the water to $C$, and the dead water from $C_1$ flows through $D_1$ into $E_1$; whereas, at the lower position of the piston, the water is turned into $C_1$, and the water shut off in $C$ may flow through $D$ into $E$.

FIG. 547.



**303.—Regulating Cock.**—A stop-cock is used as a regulator in the old water-pressure engine at Bleiberg in Kärnthen, and in the engine at Schemnitz in Ungarn, constructed by Schitko. It has the form of a frustrum of a cone, and fits into a box similarly shaped; in order to be easily turned, it terminates in smaller cylinders, which are surrounded by stuffing boxes. On account of the great wear, the box is furnished with a lining of hard metal, which is readily replaced. In Fig. 548, $H\,H$ is the cock, $B\,B$ its box, and $C\,C$ the lining; $K$ is the head to which is applied the turning force, and $D$ is a screw to raise or lower the cock $H\,H$ in the box $B\,B$, according to requirement. The boxes or passages of the cocks differ for single-acting machines

FIG. 548.



from double-acting, or from single-acting double-cylinder engines, as has been seen above.

If the direction of the water is changed by the cock 90°, it will be pressed against the box diagonally by the water, and this pressure for great heads and a medium bore causes great friction and a rapid wearing of the parts. This evil, however, Schitko has remedied by an arrangement as shown in Fig. 548. Opposite the main bore $a$, two cuts, $b$ and $b_1$, are made in the cock, and these connected with the bore by small holes $c$, $c_1$, so that a counter pressure is formed to counteract that in the main bore.

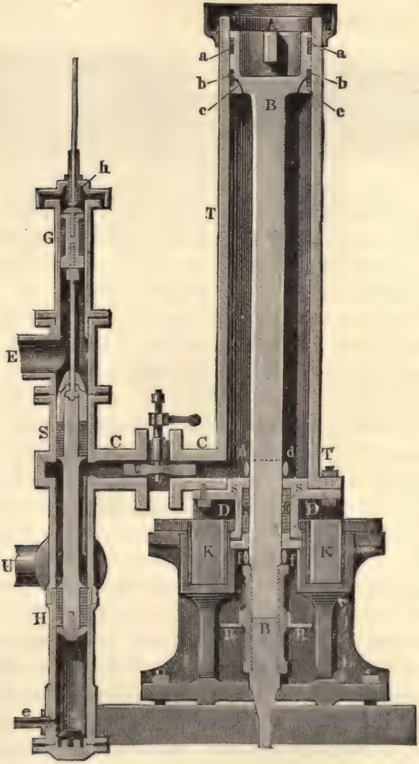
To diminish the wearing away, or at least to prevent unequal wear, it is of advantage to turn the cock in the same direction, and not merely 90° and back, since in the first case every part of the circumference of the cock is brought into contact with every part of the surface of the casing. Cocks were first used by Brendel, and are still used in some machines constructed by him. The details of Brendel's regulator will be explained further on (§ 314).

§ **304.—Regulator Piston.**—For the regulator piston, a piston is generally used with packing consisting of rings of leather, similar to those described above (§ 301). In the Huelgoat machine the gun-metal regulator piston had run seven years without packing; but in 1839, in the presence of the author, having worn away 1 millimeter, it was replaced by an exactly-turned packing, consisting of twenty-four thicknesses of leather, making 5 inches in height when pressed together.

Reichenbach has used also a piston with packing of tin rings, and of late, in some Bavarian machines, a combined leather and tin packing has been found advantageous.

When, at the end of the stroke of the driving piston, the regulator piston $S$, Fig. 549, rises and gradually shuts off the water column from the cylinder $T$ $T$, and therefore prevents the motion of the water along $E$ $C$, the piston is pressed by the water on one side, and a great wearing of the same is occasioned; to prevent this, the end of the connecting pipe *C D*, Fig. 550, is enlarged, so as to surround the regulator cylinder, and make the water press equally on all

FIG. 549.



sides of the moving piston. In this case, however, the packing suffers somewhat, since it can swell in the passage *C D*, while above and below it is again compressed; and on this account the leakage between the valve cylinder and the working cylinder, through holes, shown in Fig. 551 in cross-section, is increased. This is advantageous for some reasons, but is a disadvantage in that it increases the hydraulic resistance.

FIG. 550.

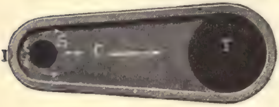
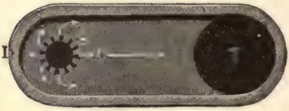


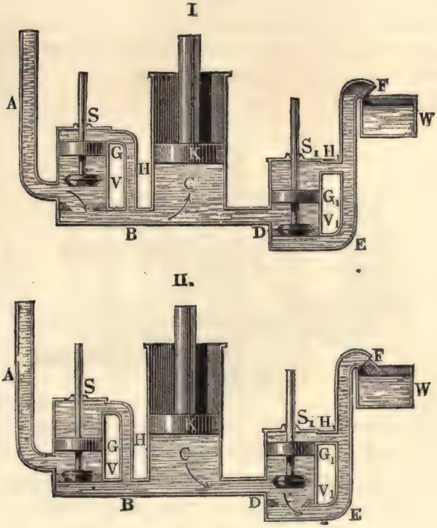
FIG. 551.



Of great consequence to the movement of a water-pressure engine is also the form of the regulator piston *S*, Fig. 549. The communication between *C* and *E* must not be suddenly closed, and the movement of the water column in the supply thus be arrested at once, since the engine would then be subjected to a considerable shock, manifested generally by noise, and which would occasion the supply pipe to spring or to give at the joints. In order to prevent this shock or so-called *pounding* of the machine, it is, of course, only necessary to shut off the water gradually. This is effected by a slow movement and by a peculiar form of piston. In the following will first be shown how to produce a slow motion of the regulator. As regards the form of the piston, it is only necessary to make the head, or rather that part of the piston which first effects the shutting off, a cone, or to add a conical top to it. This will leave between *C* and *E* a ring-formed opening, which will be gradually closed by the ascent of the piston. Besides this, notches are sometimes made in the piston stock itself, which, running from the top down, become smaller, and finally stop altogether, so that at first a slight connection between *C* and *E* is left, which is only entirely closed at the last part of the stroke. In the engine at Clausthal, both these expedients are used. In the Huelgoat machine, on the contrary, the otherwise nearly barrel-shaped piston is provided with ten notches.

§ **305.—Valve Regulators.**—The manner in which a water-pressure engine is regulated by valves is shown in Fig. 552, I. and II. *V* is the inlet and $V_1$ the outlet valve, each working in its own cylinder. By the ascent of the driving piston (in I.) *V* is opened and $V_1$ closed, so that the water has free entrance through the valve opening, from the supply pipe *A*, and passes into the working cylinder, through the connecting pipe *B*. By the descent of the driving piston, on the contrary, *V* is closed and $V_1$ opened, so that now the water in the cylinder can flow out through the connecting pipe *D*, the valve opening $V_1$ and the discharge pipe *E F*, into the cistern *W*. In order to make the moving of the valves as easy as possible, counter pistons *G* and $G_1$ are used, which are placed on the same stems with the valves; the space above *G* is connected with the pipe *B*

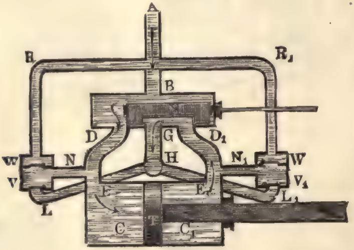
FIG. 552.



by another pipe $H$, and in the same way the space over $G_1$ is connected with $E\ F$ by $H_1$. If the cross-section of such a counter piston is nearly equal to the cross-section of the corresponding valve, the upward pressure upon the whole mechanism is nearly equal to the downward, and only a small force is needed to move it.

The way in which the regulation is effected by the *slide valve* is shown in the horizontal water-pressure engine represented in Fig. 553. During the movement of the driving piston $T$ from left to right, the water enters the valve chamber $B\ D\ D_1$ and passes into the cylinder through $D\ E$. At the completion of this part of the stroke, the position of the slide valve $C$ is reversed, so that the channel $S$ therein stands over the opening $D$ of the connecting pipe $D\ E$, and also over the opening $G$ of the discharge pipe $G\ H$; the motive water enters the other part $C_1$ of the cylinder, forces the driving piston from right to left, and the water in the part $C$ of the cylinder out through the valve channel into the discharge pipe $G\ H$. When the driving piston reaches the extreme left position, the slide valve is moved again to the right, and a new stroke is commenced.

FIG. 553.

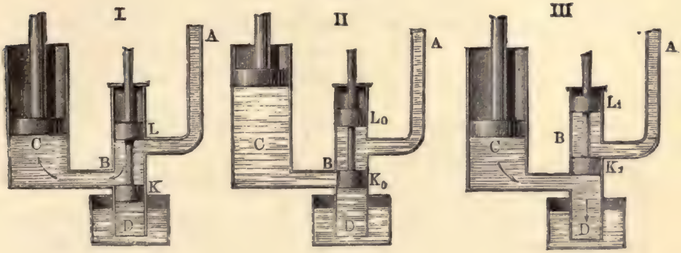


The remaining arrangements of the regulator will be explained farther on (§ 307).

### § 306.—Characteristics of the Regulator of a Water-Pressure Engine.

—The arrangement for moving the regulator of a water-pressure engine is quite complicated, and, indeed, more so than in steam-engines, since in this case we have to deal with a substance almost incapable of either compression or extension, and which loses its living force immediately when it is cut off from the pressing column. At the moment when the regulator piston $K_0$, Fig. 554, during its ascent, shuts off the motive water *A B* from the cylinder *C*, the pressure upon the driving piston ceases, and the latter, by virtue of its inertia, traverses a very small distance, through which water cannot follow it. A vacuum is thus formed, and the only force acting on the piston is the atmospheric pressure on its upper surface. Let *h* be the head of water before the shutting off by the regulator piston, *b* the height of a column of water sustained by atmospheric pressure, *F* the area of the driving piston and $\gamma$ the density of the water.

FIG. 554.



The force acting upon the piston, and which is equal to its weight and load, is

$$P = F h \gamma.$$

The additional force acting upon it, after shutting off the water, is

$$P_1 = F b \gamma,$$

and therefore the whole weight of the driving piston, which tends to bring it to rest, after the motive water is shut off, is

$$P + P_1 = F(h + b)\gamma.$$

Denote now by $M = \frac{G}{g}$ the mass of the piston and rod, and by $v$ the velocity of the same at the moment of shutting off the water; we have, then, $\frac{Mv^2}{2} = \frac{Gv^2}{2g}$ = the living force of the mass. The distance $s_1$ traversed by the piston, after the water is cut off, and before coming to a state of rest, is

$$s_1 = \frac{work}{force} = \frac{G}{F(h+b)\gamma} \cdot \frac{v^2}{2g}.$$

Since $v$ is small, generally less than one foot, and therefore $\frac{v^2}{2g}$ not more than 0·015; since, also, the ratio $\frac{G}{F(h+b)}\gamma$ has a moderate value, the path $s_1$ of the piston, during its retarded movement, must be very small.

If, then, the regulator piston were connected directly with the working machinery, and the motion of the same depended on that of the driving piston, this latter could not accomplish, during the portion of the stroke $s_1$, the movement of the regulator piston to the position $K_1$ (III.), so as to allow the water beneath to pass out through the discharge pipe $D$ in order that the driving piston may be free to descend.

This circumstance is still more unfavorable when the regulation is to be effected by a downward motion of the regulator piston, at the end of the driving-piston stroke. If, for example, the regulator piston has attained the position $K_0$ (II.), the passage through the regulator cylinder is wholly closed by the piston $K_0$, and the motion of the descending driving piston is suddenly arrested. Not only does this almost momentary stopping of the inert mass of the driving piston, rod, etc., occasion a most injurious vibration of the machine, but the regulator piston cannot now move downward any farther, and the whole operation of the engine is at an end.

This deficiency, moreover, is not peculiar to the piston regulator, but is common to all sorts of regulators. It therefore becomes necessary to remove this difficulty by especial mechanical expedients.

§ **307.—Contrivances to Supplement the Regulator.**—The mechanical contrivances to induce the regular action of water-pressure engines are various, according as these engines are:

(1) Merely reciprocal-acting—*i. e.*, when they move up and down or forward and back, in a straight line, or

(2) If, besides the original rectilinear motion, they possess a constant rotary motion, which of course is effected from the first by intervening mechanism.

The change of motion, however, from reciprocal to circular, is not easily accomplished in a single-acting, single-cylinder engine, but requires at least a double-acting engine. With two double-acting engines, one of which acts half a stroke before the other, this end is perfectly attained.

In those water-pressure engines having a constant circular motion, the regulator piston rod is so connected with the revolving mechanism that it shall move up and down or forward and back in the same time that the driving piston occupies for the whole stroke. In order that the motion of the driving piston may not be interrupted or destroyed, the following contrivances are made use of:

(1) The height of the regulator piston is made such that it cannot completely close the opening of the connecting pipe into the regulator cylinder, and therefore, either above or below the piston, the working cylinder is connected with the regulator cylinder. In this case, at the middle position of the regulator piston, a small quantity of water escapes from $A$ into $D$, or is lost from the motive water.

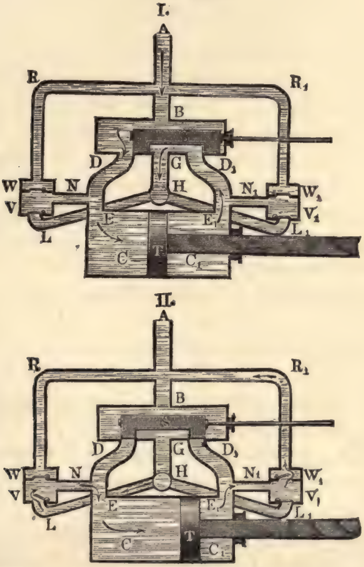
(2) A side pipe is led from the connecting pipe into the discharge pipe, and the opening into the first is closed by

an inward opening valve (a suction valve); and another side pipe is led from the connecting pipe into the supply pipe, and this opening closed by a valve (lifting valve), which opens outward—*i. e.*, into the side pipe. When the regulator piston (II., Fig. 554) attains the position $K_0$, and consequently prevents the entrance of the water from $A$ into $C$, the first-named valve opens, and enough water is sucked from the discharge pipe to fill the space traversed by the driving piston, while the motive water remains shut off by the regulator piston; when, on the contrary, the regulator piston, in its descent, attains the same position $K_0$, and the discharge of the water from $C$ into $D$ is stopped, the second or lifting valve is opened by the pressure of the driving piston on the water beneath it, and the connection between the supply and connection pipes is again established.

While the *vis viva* of the driving piston is greatly changed by this process, it does not give rise to a shock, but only to a notable variation of the velocity of the driving piston.

The working of a regulator, with suction and pressure valves, is shown in Fig. 555, I. and II., which represents a horizontal, double-acting water-pressure engine, having, finally, a rotary motion, and in which the system of regulator pistons is replaced by the *slide valve* (Fr. tiroir; Ger. Schiebventil). At the position $S$ of the slide valve (in I.), the motive water flows from the supply pipe $A\,B$ into the valve chamber $B\,D\,D_1$, and thence, through the connecting pipe $D\,E$, into the working cylinder $C$, and the driving piston is forced by it from left to right, while the water which has performed its work is driven out, through the connecting pipe $E_1\,D_1$, into the valve channel $S$, and thence into the discharge pipe $G\,H$. Towards the end of the stroke of the driving piston, the slide valve $S$ (II.) has moved so far towards the left that it covers both connecting pipes in the valve chamber, and, consequently, neither the water from the supply pipe $A\,B$ can enter the working cylinder, nor that in the cylinder flow out through the discharge pipe $G\,H$. The succeeding movement of the driving piston opens the left suction valve $V$, and thus establishes the connection between the discharge pipe $H$ and the left chamber $C$ of the cylinder, and the water flows from $H$, through the pipe $HL$, the valve $V$ and the passage $NE$, into the cylinder; at the same time, the right pressure valve $W_1$ is opened, which establishes the connection between the right cylinder chamber $C_1$ and the supply pipe $AB$, and enables the water to pass from $C_1$, through the pipes $N_1$ and $R_1$, into the supply pipe. The slide valve afterwards moves farther to the left, and the opening $D_1$, of the connecting pipe $E_1 D_1$, into the valve chamber, is freed while the valve channel $S$ is brought over the openings $D$ and $G$. The motive water now pressing upon the right surface of the piston, the latter is forced from right to left, and the water in the left chamber $C$ of the cylinder passes out through $EDGH$. The slide valve now begins the opposite

FIG. 555.



movement, and covers, for a second short time, both openings $D$ and $D_1$ of the connecting pipes, so that now the right suction valve $V_1$ and the left pressure valve $W$ are opened, and the driving piston completes its stroke without further disturbance.

§ **308.—Kinds of Regulators.**—In single-acting engines, and especially in those which have only a rectilinear motion, it is not possible to connect the regulator directly with the motive mechanism, or to make the motion of the regulator piston rod depend directly upon that of the driving piston rod, since then, at the moment when the regulator piston or valve closes the connection between the working and the regulator cylinders, not only the driving piston, but also the regulator piston connected with it, come to a state of rest. In order that the regulator piston may be enabled to move through the rest of its stroke after the driving piston has stopped, it is necessary to use an intermediary apparatus, which only acts upon the regulator piston when the driving piston is at rest. This apparatus may consist of the following essentials:

(1) Of a *weight*, which is raised by the driving piston during its up-stroke, and is let fall at the moment it stops, after making the stroke; or,

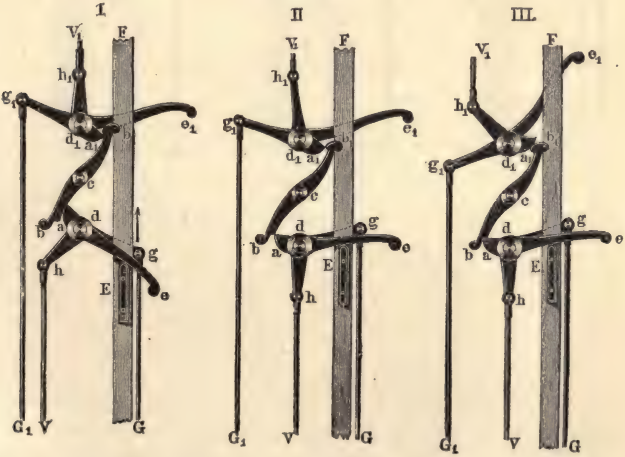
(2) Of a *spring*, which is brought into tension during the movement of the driving piston, and let go at the end thereof; or, finally,

(3) Of a second or *auxiliary water-pressure engine*, which is regulated directly by the motive mechanism, and whose driving piston moves the regulator piston rod of the principal machine, while the driving piston of the latter is traversing the last portion of its stroke and for a short time comes to rest. We have, therefore, to distinguish from each other the *weight regulator*, the *spring regulator*, and the *water-pressure regulator*.

The *weight regulator* consists chiefly of a mechanism by means of which the motive engine during its movement raises a *weight*, which, by falling at the moment when the working cylinder is closed by the regulator cock or piston, etc., moves this regulator through the second part of its prescribed stroke, and completes in this way the regulation. The weight regulator is used in the older and imperfect water-pressure engines under the names drop, hammer, balance, pendulum regulator, etc. In modern times, weights are also used with valve regulators, in such a manner that the motive engine opens the one while the falling weight closes the other valve. The arrangement of such a weight regulator is quite the same as in the case of steam-engines with valve regulators. This system consists, essentially, of several levers in combination with a pawl or ratchet, whence it is also sometimes called a *lever* or *spring-catch regulator* (Fr. encliquetage; Ger. Hebelsteuerung or Sperrklinkensteuerung).

§ **309.**—**Pawls or Ratchets.**—The most essential component of the lever regulator is the *pawl and ratchet*. This

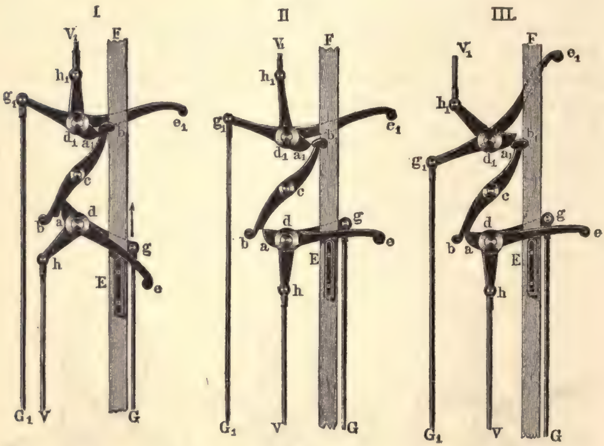
Fig. 556.



is necessary to enable the opening of the valve to be performed directly by the machine and the closing of the same by *falling weights*. The description of Fig. 556, I., II., and

III., will show clearly how this is possible. The ratchet itself is $b\ c\ b_1$; it may be turned about the horizontal axis $c$, and terminates in hooks $b$ and $b_1$. Beneath it is the horizontal arbor $d$, with the tooth $a$ and the three arms $e$, $g$, and $h$, while above it is a similar arbor $d_1$ with the tooth or catch $a_1$ and the three arms $e_1$, $g_1$, and $h_1$. In I. the tooth $a_1$ is engaged with the hook $b_1$, while $a$ stands above $b$; in II. the interlocking of $a_1$ and $b_1$ is destroyed, and in III. the tooth $a$ is engaged with the hook $b$, while $a_1$ is above $b_1$. If in I. $a$ descends, $b\ c\ b_1$ suffers a small rotation, and $a_1$ is unhooked from $b_1$, as in II.; but if $a_1$ descends, as in III., a contrary rotation of $b\ c\ b_1$ takes place, and $a$ is unhooked from $b$. If, now, on the arms $d\ g$ and $d_1\ g_1$ of the two journals $d$ and $d_1$, weights $G$ and $G_1$ are hung, these will cause a rotation of the journals, as the teeth $a$ and $a_1$ are freed from the catches; and if, also, the arms $d\ h$ and $d_1\ h_1$ are connected by the rods $h\ V$ and $h_1\ V_1$ with the regulator valves, these will be opened by the falling of the weights. The arms or claws $d\ e$ and $d_1\ e_1$ serve to turn the journals in the opposite direction; if

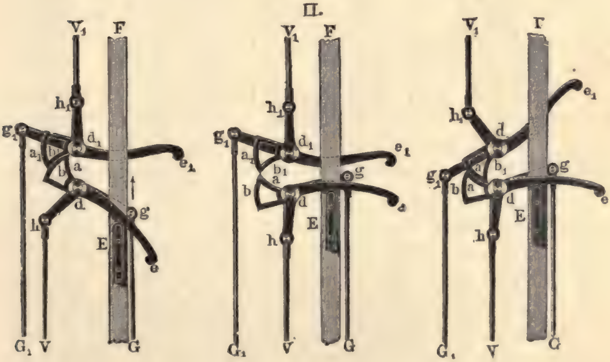
FIG. 557.



$d\,e$ (I.) is raised, $h\,V$ is lowered, the valve $V$ is closed, and the tooth $a_1$ is freed; $g_1\,G_1$ now falls and thereby opens $V_1$. If, on the contrary, $d_1\,e_1$ (III.) is lowered, $h_1\,V_1$ rises, $V_1$ is again closed, and $a$ is unhooked, $G$ falls again, draws up $h\,V$, and so opens the valve $V$ in connection with it. The raising and lowering of the arms $d\,e\;d_1\,e_1$ is performed by the so-called *regulator rod* $E\,F$, which rises and falls at the same time as the driving piston. To this end the so-called tappets (Fr. taquets; Ger. Knaggen) $E$ and $F$ are screwed upon the opposite sides of the rod, one of which ($E$) carries the claw $d\,e$ upward, near the end of the ascent of the piston, while the other ($F$), near the end of the descent, carries the claw $d_1\,e$ downward.

A somewhat simplified lever regulator is shown in Fig. 558, I., II., and III. Here the ratchet is replaced by two circular sectors $a\,b$ and $a_1\,b_1$, which alternately retain and liberate each other. For the rest, this regulator is exactly like the one represented in Fig. 557, and the remaining letters distinguish the same parts. If the regulator rod $E\,F$ ascends with the driving piston, the tappet $E$ (I.) strikes the lever $d\,e$ and lifts it; at the same time $G$ also ascends, and,

Fig. 558.

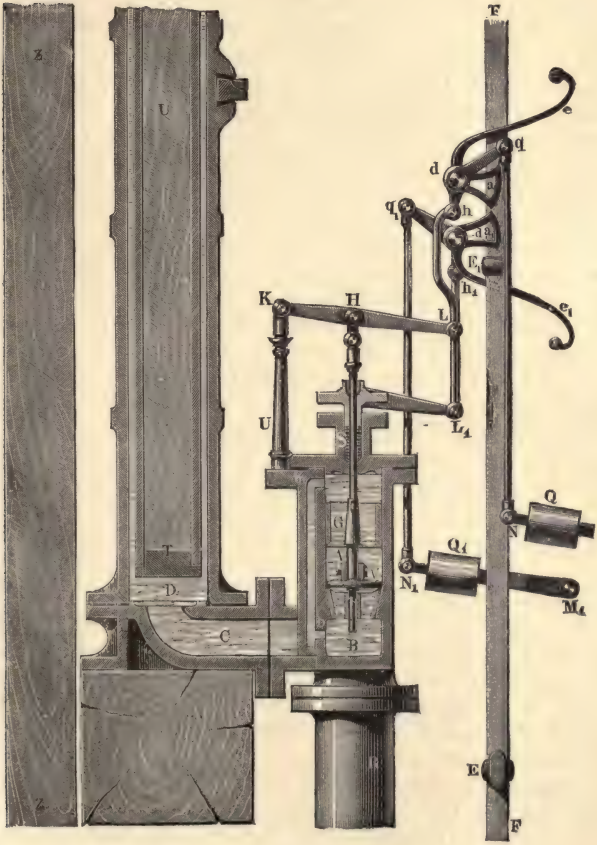


on the contrary, the valve *V* is closed; *b* is also drawn back, and $b_1$ disengaged, as shown in II. If, now, $G_1$ falls, $a_1$ plays in *a*, and the valve $V_1$ is opened, as may be seen from III. The now descending driving piston carries the rod *F G* downward with it, and another tappet upon the rear side of this rod strikes $d_1$ $c_1$ near the end of the descent, and pushes it down, so that the mechanism is again in the position II., and $G_1$ will be raised and $V_1$ closed; also, $a_1$ is disengaged from *a* and *G* falls without hindrance; *b* plays in $b_1$, and *V* is opened, so that now the motive water enters below and drives the piston upward, and the preceding stroke is repeated.

§ **310.—Water-Pressure Engine with Weight Regulator.**—The arrangement and working of a water-pressure engine with weight regulator may be seen from Fig. 559. It is mainly a sectional representation of an engine constructed by Harvey & Co., of Hayle in Cornwall, for a fall of 60 metres (197 ft.).

The supply pipe (not seen in the figure) opens into the first regulator cylinder in front at *A*, and the discharge pipe behind the cylinder at *B*, the working cylinder being connected with the regulator cylinder by the connecting pipe *C*. After the opening of the admission valve (Fr. soupape d'admission; Ger. Eintrittssteuerventil) *V*, the motive water *A* passes through the valve opening into *B*, and thence into *C* and *D*, and forces the driving piston upwards. The latter is a so-called plunger piston (see § 300), and consists of a cylindrical pipe turned on the outside, which is surrounded at the upper end of the working cylinder (not shown in figure) with a stuffing box. The wooden piston stock *T U*, which is firmly fastened to the piston hood, is connected by an ordinary beam connection with a shaft *Z Z* at the left, which lifts the load, and also by a cross-arm with the regulator rod *F F* at the right; these, therefore, follow the motion of the driving piston. Behind the first regulator cylinder *A B* is a second regulator cylinder (not visible here), in which is situated the eduction valve (Fr. soupape d'émission; Ger. Austrittssteuerventil). This valve com-

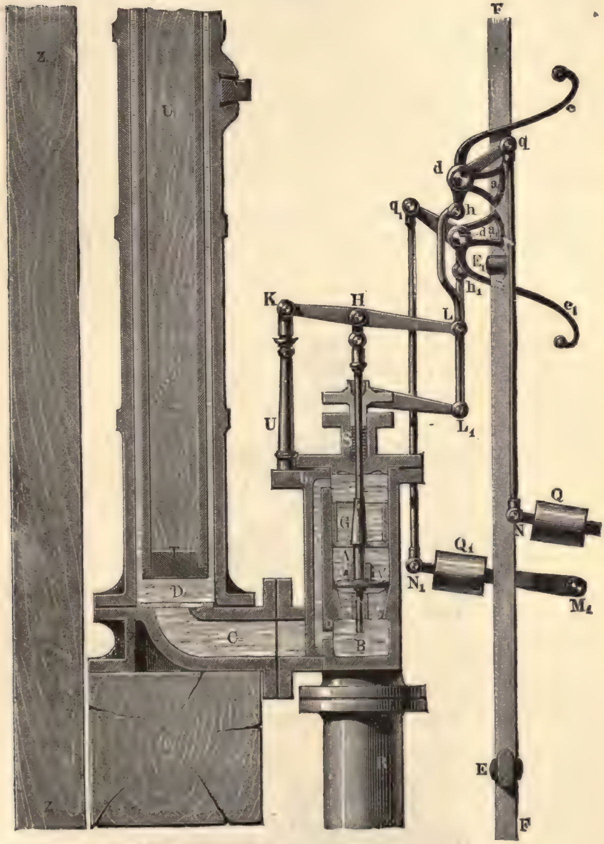
FIG. 559.



municates above with the channel *B* and below with the discharge pipe *R* (comp. § 305, Fig. 552), and allows, when open, the water, pressed by the descending cylinder, and flowing through *C* into *B*, to enter the discharge pipe.

Since the valve *V* is forced against its seat by the whole pressure of the water column in the supply pipe, the opening of the same would require a great expenditure of force if the counter piston *G* were not used. This counter piston is situated on the valve rod, and the space *S G* above the counter piston is connected with the space below the valve piston *B* by the channel $b\,b_1$. By this arrangement, the counter piston *G* and the admission valve *V* are pressed upwards with nearly the same force as they are pressed downwards, and, therefore, the power required to lift this valve is reduced to a minimum. The same arrangement is used for the eduction valve. The rod of the admission valve *V* goes through a stuffing box at *S*, in the cover of the first regulator cylinder, and is connected at *H* with a simple regulator lever *K L*, whose fulcrum is the top *K* of the column *K U*. This lever is fastened by the rod *L h* to the arm *d h* of the journal *d* of a ratchet *a* (see Fig. 558), and may, therefore, by the turning of this journal (*d*), be moved up and down. In the same way, the eduction valve is furnished with a lever, partly hidden in the figure, which is connected, by a rod $L_1\,h_1$, and the arm $d_1\,h_1$, with the journal $d_1$ of a second ratchet $a_1$. To the first journal is also attached the weight *Q* by the rod *q N* and the arm *d q*, and to the other, the weight $N_1$, by the rod $q_1\,N_1$ and the arm $d_1\,q_1$. The weight $N_1$ is movable about the fixed point $M_1$. Finally, upon these journals are placed the arms or levers *d e*, $d_1\,e_1$, which are moved upwards or downwards by the tappets *E*, $E_1$ on the regulator rod *F F*, and thus turn the journals *d* and $d_1$ in one direction. They are turned in the other direction by the counter weights *Q* and $Q_1$. In the position of the engine represented in the figure, the driving piston has reached the lowest position; the regulator rod *F F*, which follows the movement of this piston, has, by means of the tappet $E_1$, pressed down the regulator lever $d_1\,e_1$, and thus closed the eduction valve. Further, $a_1$ is disengaged from $a$, and the journal $d$ may now revolve under

FIG. 560.



the action of the weight $Q$. This will open the admission valve *V*, and the motive water, now entering the working cylinder, will force the driving piston, together with the rods *ZZ* and *FF*, upwards; and when, towards the ends of the ascent, the tappet *E* strikes the lever *dc*, the valve *V* will be closed, the driving piston come to rest, and the ratchet *a* be disengaged from $a_1$, so that now the weight $Q_1$ turns the journal $d_1$ from right to left, and the eduction valve is thus opened. The driving piston, relieved from the pressure of the motive water, begins its descent, and a new stroke is thus commenced.

§ **311.—Auxiliary Water-Pressure Engines.**—The manner of regulating a water-pressure engine by an auxiliary

FIG. 561.

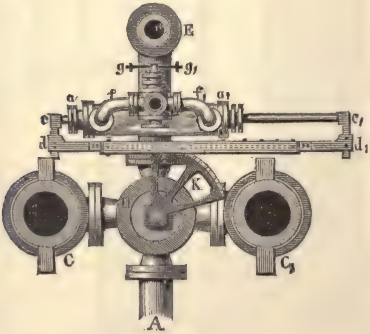
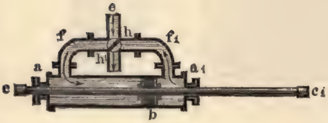


FIG. 562.



engine is to be seen from the plan in Fig. 561, and the section, belonging thereto, in Fig. 562, which show the regulator of the large water-pressure engine in the Leopold mines at Schemnitz. This machine is a double-cylinder engine; $C$ is one and $C_1$ the other cylinder, $E$ the supply and $A$ the discharge pipe; $H$ is the regulating cock (see Fig. 548), and $K$ a quadrant fastened to the head of the same.

The auxiliary water-pressure engine consists of the horizontal working cylinder $a\,a_1$, the driving piston $b$ and its piston rod $c\,c_1$. This is fastened by cross-arms with the peculiar regulator rod $d\,d_1$, so that it forms with these a rectangular frame.

Finally, this last rod is connected, by two chains with flat links, running in opposite directions, with the quadrant-formed cock key, so that the movement of the piston $b$ forward and backward causes the cock to revolve 90° and back. The regulation of the auxiliary engine is effected by the horizontally placed cock $h\,h_1$, which has two bores like the main cock $H$. The motive water is led from the supply pipe $E$, through the small tube $e$, to the cock $h\,h_1$, and thence through the connecting pipes $f$ and $f_1$, now to the one, now to the other side of the piston $b$, so that the latter is set in motion, and the water used for one half the stroke is forced out through a downward-directed discharge pipe during the other half. The turning of the small cock $h\,h_1$ is effected by the double-armed key $g\,g_1$, which is connected by small chains with the double-armed lever parallel to it; this latter sits upon the same arbor with the beam which connects the driving pistons.

The whole play of the regulator is now easily understood; during the ascent of one driving piston and the descent of the other, the cock $h\,h_1$ is turned by the lever $g\,g_1$, and so admits the water to one side of the cylinder $a\,a_1$, while it shuts it off from the other side. In this way a force is generated which moves the piston $b$, together with the cock $H$, into the opposite position, so that now the first working cylinder is cut off from the supply

FIG. 563.

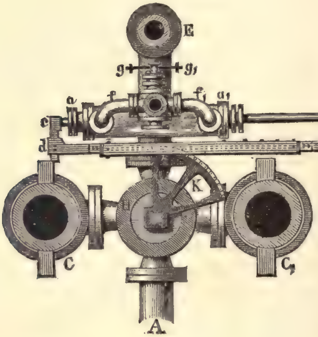
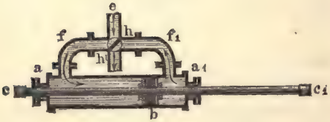


FIG. 564.

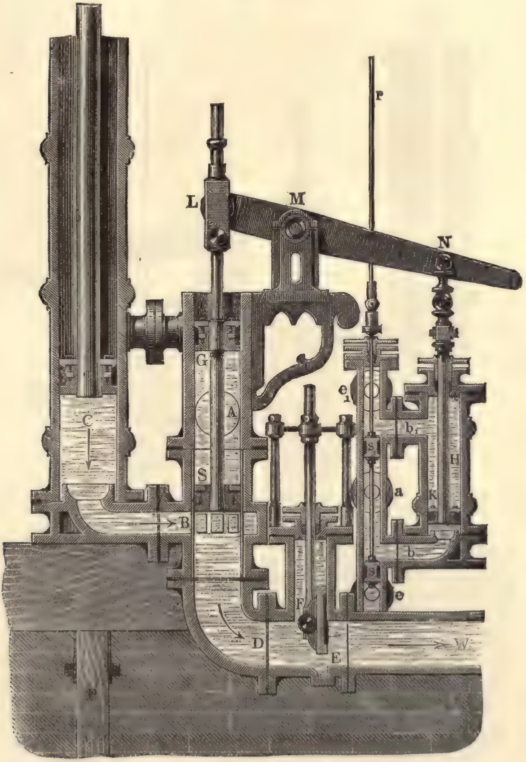


pipe, and the motive water turned into the second, which causes the opposite piston stroke.

REMARK.—The machine at the Leopold mines has the remarkable fall of 710 Austrian ft., or 748 English ft., a length of stroke of 8 ft., and a piston diameter of 11 inches; each piston makes three strokes per minute.

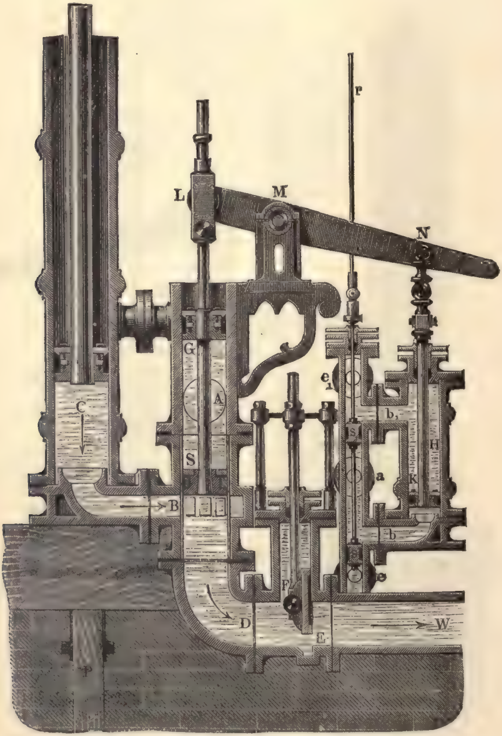
§ **312.**—The method of regulation by an auxiliary engine is to be seen from the sectional representation in Fig. 565, of the water-pressure engine constructed by Darlington for the Alport mines in Derbyshire. As represented in the figure, the driving piston *T* has nearly finished the down-stroke, and completed the regulation of the auxiliary engine. During this descent of the driving piston, the water flows from the working cylinder *C*, through the connecting pipe *B*, into the regulator cylinder *A D*, and thence, through the bend *D* and the opening *E*, beneath the gate *F* into the water beneath, *W*. The auxiliary machine is double-acting; its working cylinder *H* is connected with its regulator cylinder *e a e*₁ by the pipes *b* and *b*₁, while the latter is connected with the motive water by the pipe *a* and with the

FIG. 565.



discharge water $W$ by the pipes at $e$ and $e_1$. Both regulator pistons $s$ and $s_1$ of the auxiliary machine sit upon the rod $r s$, which is connected with the driving piston rod $R T$, and is moved up and down thereby. In this way, during

FIG. 566.



the descent of the driving piston, the two pistons $s$ and $s_1$ also descend and attain the position represented in the figure; at this position, the motive water can pass from $a$.

through *b*, into the chamber beneath the driving piston *K* of the auxiliary engine, and the dead water above can flow out of the chamber above *K*, through $b_1$ and $e_1$. The now ascending driving piston of the auxiliary engine pushes downwards, by means of its rod *K N* and the lever *L M N*, pivoted at *M*, the regulator pistons *S* and *G* of the principal engine, so that not only is the communication between *B* and *D* closed, and the downward movement of the driving piston *T* arrested, but the motive water is admitted at *A* to the regulator, and, therefore, also to the working cylinder. After the completion of the up stroke of *K* and the down-stroke of *G S*, the water in the supply pipe exerts its full force to drive the piston *T* upwards, with which also move upwards the two pistons $s\,s_1$; as the driving piston approaches the end of its up-stroke, $s\,s_1$ attain their highest position, and the motive water is conducted, through $a\,b_1$, into the chamber above the driving piston *K* of the auxiliary engine, which causes this piston to descend. The two regulator pistons *S*, *G*, of the principal machine are thus pulled up again, and not only the entrance of the motive water prevented, and therefore the piston *T* brought to a stop, but the communication with the discharge pipe is also opened, so that the water which caused the piston to ascend may now flow out through *E* into *W*.

The English translation of the first edition of this work* contains a short description of this engine, with plates. According to this, it consists of two adjacent working cylinders of 24 inches diameter and 20 ft. in height, which are fed by a fall of 130 ft., through a 24-inch supply pipe. The driving piston rods of the two cylinders are connected above by a strong cross-head moving in vertical guides; to this is attached the pump rod *P*, which is situated between the driving piston rods, and moves with these up and down. The regulator cylinder is 18 inches, and the working cylinder of the auxiliary engine about 12 inches in diameter. The entrance of the water is regulated by a sluice valve, similar to that used to regulate the discharge.

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* London, 1848, Hippolyte Bailliere, Pub.

§ **313.—Regulator Cylinder.**—In the larger machines of modern construction, the regulator and counter pistons of the principal engine, and the driving piston of the auxiliary engine, are placed in one and the same pipe, the so-called *regulator cylinder*, after the pattern of Reichenbach's engine in Bavaria; and in some machines even, the counter piston

FIG. 567.

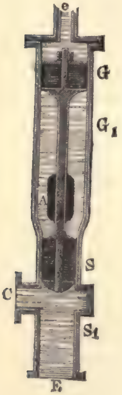
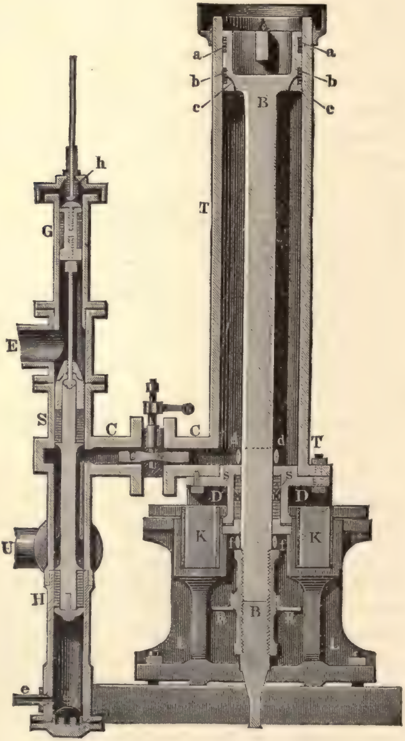


FIG. 568.



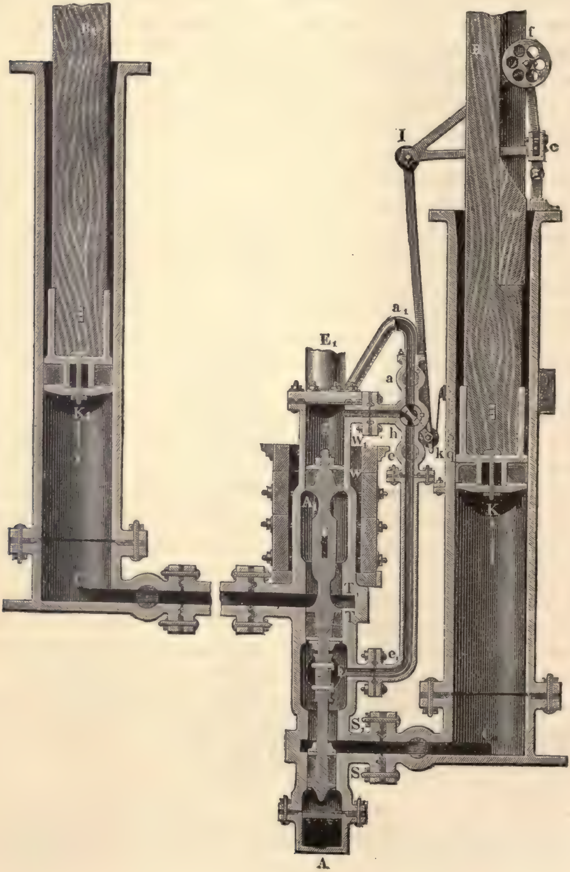
performs the function of driving piston of the auxiliary engine, which is a great simplification. The simplest construction is that shown in Fig. 567, and used in several Freiberg machines. *S* is the main regulator, and *G* the counter and auxiliary driving, piston, *C* the connection with the principal working cylinder, and *E* that with the supply pipe, and *A* the discharge opening for the motive water; finally, at *e* is the connection with the regulator of the auxiliary machine, which here consists of a cock. The piston *G* is larger than *S;* therefore the regulating mechanism *S G* descends as soon as the motive water is admitted through *e;* and, on the contrary, it ascends, under the action of the upward force on *S*, as soon as *e* is closed. By this a certain quantity of regulating water is consumed for each stroke from the motive water, which quantity depends on the space traversed by *G* in its ascent or descent, and which, for this construction, is not very small, since the piston *G* should have a section at least as large again as that of the piston *S*, which again is not made smaller than the supply or connecting pipes.

In the regulator of the machine at Clausthal, shown in Fig. 568, this consumption of regulating water is smaller, as here there are three pistons—namely, the main regulator piston *S*, the counter piston *G*, and the auxiliary driving or reversing piston *H*, the last being smaller than the first. The regulating water is here brought into the regulating cylinder by the pipe *e*, and the regulation of this water is effected by a small cock, through which the water passes before reaching *e*, and through which it is also drawn off after the complete revolution. The movement of this cock is brought about by a plate fastened to the driving piston rod, which turns now to the one side, now to the other, by means of two curved knee-formed arms, an arbor connected with the cock.

REMARK.—The water-pressure engine at Clausthal has a fall of 630 ft., a piston diameter of 17 inches, and a length of stroke of 6·21 ft., and makes four strokes per minute.

**§ 314.—Water-Pressure Engine in the "Alte Mordgrube."**—The arrangement and working of a double-cylinder water-pressure engine is shown clearly in Fig. 569,

FIG. 569.

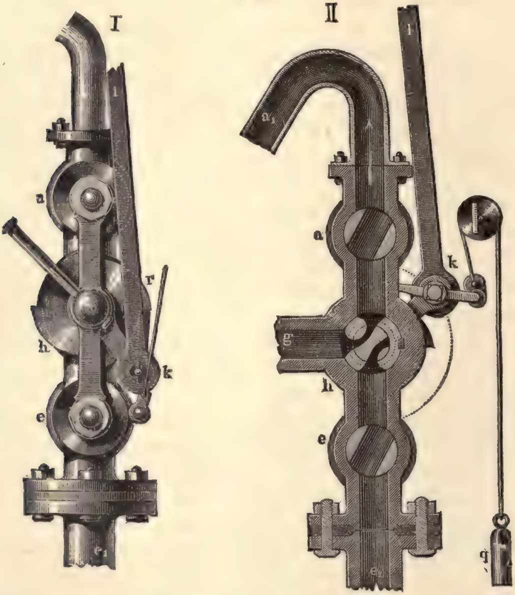


which is a vertical section of the machine. at the "Alte Mordgrube," at Freiberg. Here $C K$ and $C_1 K_1$ are the two working cylinders, $K$ and $K_1$ the driving pistons, $S$ and $T$ the two regulator pistons, and $W$ the reversing or auxiliary piston. $S_1$ $T_1$ and $W_1$ denote the positions which these pistons assume at the change of the motion of the driving pistons. Further, $E$ is the opening of the supply pipe $E_1 E$ into the regulator cylinder, $C S$ the connecting pipe for the first, and $C_1$ $T$ that for the second, $A$ the discharge opening for the first, and $A_1$ (almost hidden by the regulator piston rod) that for the second working cylinder. The two piston rods $B K$ and $B_1 K_1$ are connected by an equal armed lever or so-called *walking beam* (not shown in the figure), so that the ascent of one causes the descent of the other. It is easily seen, therefore, that at the lower position of the regulator piston, as represented, the motive water passes through $E S_1 C$ and drives the piston $R$ upwards; the piston $K_1$, on the contrary, descends, and the dead water passes out through $C_1$ $T_1$ $A_1$.

The regulation of the auxiliary machine is effected by the cock with double bore, already described (§ 303), which is represented at $h$ in Fig. 570, in elevation in I. and in section in II. This cock is in connection with the supply pipe by the pipe $c c_1$, and with the regulator cylinder by the pipe $g h$. At one position of the cock $h$, the motive water passes through $E$ $c_1$ $c$ $h$ $g$ $W$, and presses the reversing piston $W$ downwards, and, at the opposite position, the motive water is shut off from $W$, and, therefore, the ascent of $W$, the return of the regulation water through $g$ $h$ and its discharge through $a$ $a_1$ is possible. In order that, during the shutting off of the motive water from $W$, the regulator piston and connection may ascend and descend when it is again admitted, it is necessary, however, that the regulating piston $T$, upon which the water acts from below, shall have a greater area than the piston $S$, upon which the water presses downwards, and that the reversing piston $W$ shall have a large enough cross-section, so that the water pressures on

*W* and *S* together shall be greater than the opposite pressure upon *T*.

FIG. 570.



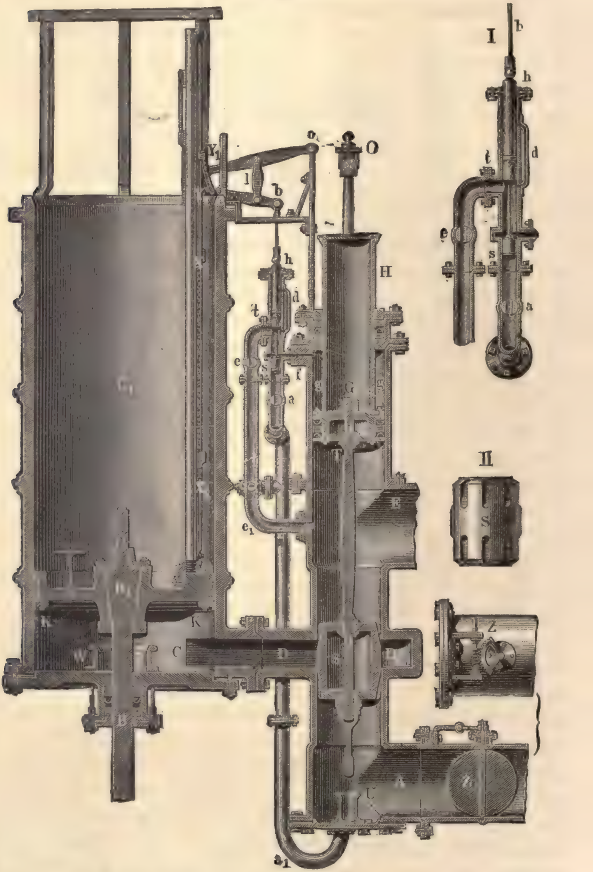
Finally, for the outer regulator of this machine, we have the following mechanism: *r* is a regulator wheel having four teeth, *r k* a ratchet, *k l* a rod, *l c f* an angle lever with the friction wheel *f* (see Fig. 569), and *m* and $m_1$ (last not shown in figure) are two oppositely placed wedges fastened to the driving piston rod *B K*. The ratchet *r k* is, moreover, connected with the axis of the cock by arms, and is supported in the teeth of the little wheel *r* by a small counterpoise *q*. When the driving piston *K* has nearly reached the end of an up or down stroke, the wedge *m* (or $m_1$) is forced beneath the friction wheel, the lever *lcf* is thus turned somewhat, and the rod *lk* drawn upwards a little; this causes the cock *h* to be turned through a quarter circle by the ratchet. When afterwards the piston has gone back through a small portion of the reverse stroke, the lever falls back and the ratchet slips over the next tooth, which it seizes towards the end of the stroke, thus turning the wheel *r* again.

REMARK.—The water-pressure engine in the Alte Mordgrube has a head of 366½ ft., a stroke of 8¼ ft., a cylinder 18½ inches in diameter, and makes four strokes per minute with each piston.

§ **315.—Water-Pressure Engine at Huelgoat.**—One of the finest and most complete water-pressure engines is that at Huelgoat, in Brittany. It is a single-acting, single-cylinder engine, but beside it stands another machine exactly similar. From Fig. 571 the arrangement and manner of working may be seen. $CC_1$ is the working cylinder, $KK_1$ the driving piston, and $BB_1$ the driving piston rod which passes through the stuffing box *b*. While, in the previously described machine, the packing consists of one broad piece of leather, in this one, as may be seen in the figure, a piece of leather is inserted in the piston and another piece also screwed on. The regulator cylinder *A S G*, at one side, is connected with the working cylinder by the connecting pipe *CD*; the supply pipe enters it at *E*, and the discharge pipe leads out of it at *A*. The regulator piston *S*, shown in the middle position of the down-stroke, is connected with the larger piston *T* by the rod *S T*; the whole apparatus, therefore, is forced upwards by the extra pressure of the motive water on *T*, unless a third force prevents. This third force is obtained by introducing the motive water, above the piston *T*, through the pipe $e_1\,ef$; but in order to necessitate the use of only a small quantity of water during the descent of the mechanism occasioned thereby, the hollow cylinder *GH* is added to the piston *T*; this hollow cylinder goes through the stuffing box at *H*, and renders necessary the use of only enough water to fill the ring-shaped space *g*.

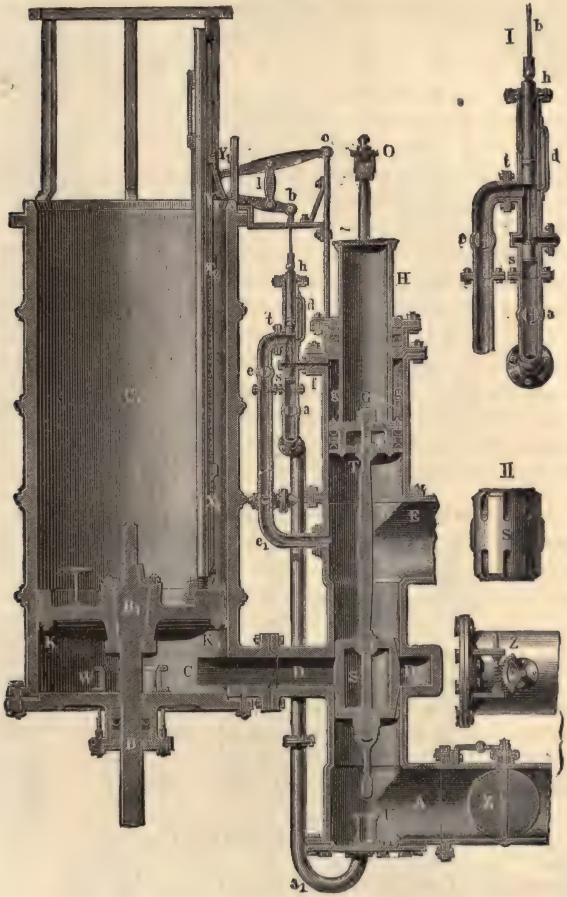
The alternate admission of the motive water to and

FIG. 571.



shutting it off from the space *g*, is effected by an auxiliary regulator, which is altogether similar to the main regulator, and, like this, consists of the regulator piston *s*, the counter piston *t*, and the thick piston rod, passing through the stuffing box *h*. At the position *s t h*, shown in the figure,

FIG. 572.



the motive water can pass unhindered through the passage *ef* into the space *g*, but if *s t h* is raised so that *s* stands over *f*, the connection is interrupted, and, at the same time, a passage $a a_1$ is opened through which the water in *gg* may flow out as the piston *T* ascends. Finally, to connect the auxiliary regulator apparatus *s t h* with the working engine, a rod is affixed to the driving piston $K K_1$; this rod moves in a guide above, and is furnished with a series of holes into which the tappets $X_1$ and $X_2$ may be placed, on opposite sides of the rod. The rod *b h* is attached to two levers, movable about *c* and *o*, and connected with each other by the piece *l*; one of these levers ends in a circular piece which carries two knobs $Y_1$ and $Y_2$. Towards the end of the up-stroke of the driving piston, $X_1$ strikes $Y_1$, and *s t h* is thus carried to its highest position; towards the end of the down-stroke, on the contrary, $X_2$ strikes the knob $Y_2$, and the rod *s t h* is carried back by the lever to its lowest position. The regulation of the machine is thus effected by *S T* and the piston $K K_1$ ascends and descends with regularity.

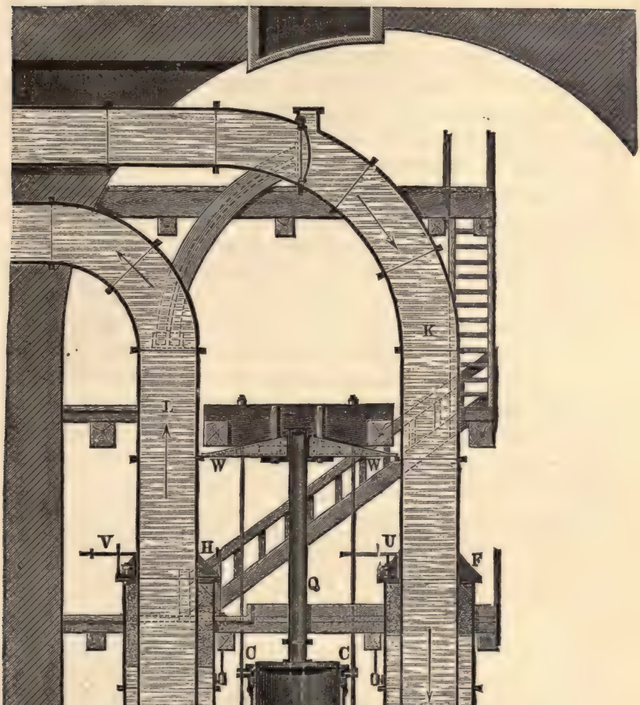
§ **316.—Water-Pressure Engine at the "Centrum Grube."**—Fig. 573 shows the arrangement of a water-pressure engine at the "Centrum" mine at Eschweiler, constructed by Oberbergrath Althaus. This machine has a fall of only 46½ ft., and a water supply of 9·9 cubic ft. per second. The supply pipe, which takes the water from a deep bog, is 32·9 inches in diameter, and has, including a horizontal piece 153 ft. in length, a total length of 234¼ ft. The driving piston has a diameter of 4·12 ft. and makes six strokes 7·2 ft. in length per minute. The mean velocity of the piston is, therefore,

$$v = \frac{7 \cdot 2 \times 6 \times 2}{60} = 1 \cdot 44 \text{ ft.},$$

and that of the water in the supply pipe,

$$v_1 = \left(\frac{49 \cdot 4}{32 \cdot 9}\right)^2 v = (1 \cdot 5)^2 \times 1 \cdot 44 = 3 \cdot 24 \text{ ft.}$$

FIG. 573.



Since the length of the supply pipe is here five times as great as the head, this small velocity is quite proper. The driving cylinder is a so-called *plunger*, which is furnished with a stuffing box, sitting into the working cylinder $BC$. This cylinder is open above and consists below of a large pipe $DD$, 16½ ft. long and 4·1 ft. in diameter, firmly supported at the ends, and of two other cylinders $EF$ and $GH$, 4·1 ft. in diameter and 12⅛ ft. in height, into which opens, on one side, the supply pipe $KF$ and on the other side the discharge pipe $HL$, which has a height of 27 ft. Both pipes are furnished with the necessary flap valves.

The regulator piston $M$ lies vertically beneath the driving piston, and has a height of 11⅛ inches, a diameter of 27·8 inches, and a stroke 16½ inches in length. The regulator cylinder contains a girdle, 5·1 inches wide, of numerous four-sided holes, by which it is connected with the connecting pipe $OP$, leading to the working cylinder. The regulator piston rod is also furnished with the two counter pistons $R$ and $R_1$, likewise 27·8 inches in diameter. To move the regulator piston, the auxiliary engine $ST$ is used, whose piston $S$ has a diameter of 9¼ inches and a length of stroke of 16½ inches, in common with the main regulator piston. The regulator of the auxiliary engine is a slide valve $T$, which is thrown back and forth by tappets screwed on the rod and by levers. The height of the column of back-water is 26·7 ft., the height of the pressing column at the beginning of the up-stroke of the piston is therefore $= 46\frac{1}{8} + 26\frac{1}{8} = 73$ ft., and that at the end of the up-stroke $= 73 - 7 \cdot 2 = 65 \cdot 8$, so that the ratio of the diminution of the force acting upon the piston to the mean force during the whole up-stroke is

$$= \frac{7 \cdot 2}{65 \cdot 8 + 3 \cdot 6} = \frac{7 \cdot 2}{69 \cdot 4} = 0 \cdot 104.$$

The ratio of the increase of resistance to the mean resistance (of the back-water) during the down-stroke, is, on the contrary,

$$= \frac{7 \cdot 2}{19 \cdot 5 + 3 \cdot 6} = \frac{7 \cdot 2}{23 \cdot 1} = 0 \cdot 311.$$

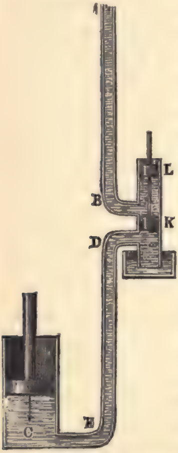
The pipes *E F* and *G H* serve at the same time as air chambers. The air, shut off in the upper spaces of the same by the water itself, receives the shock of the moving water column when this is stopped by the regulator apparatus; an easier motion of the machine is thus attained. The air which, in course of time, is lost through the walls of the pipes, or by mixing with the water, is replaced from time to time by a small air pump. The velocity of the driving piston is regulated by altering the stroke of the regulator piston. This machine serves to raise water by means of pumps whose pistons are fastened to pump rods, moved by the driving piston of the water-pressure engine.

REMARK.—Bauer, in the 4th volume of the "Zeitschrift des deutschen Ingenieurvereins," describes this machine more in detail.

§ **317.—Counterpoise.**—Still other auxiliaries are necessary to regulate the whole working of a water-pressure engine; these will be further explained in what follows. And, first, as regards the ascent and descent of the driving piston; this is regulated by the so-called counterpoise—*i. e.*, by a contrivance which assists the motion of the piston in one direction, but hinders it in the other, so that the piston attains its regular speed without suffering any great change of velocity. If the engine have two cylinders equally loaded, the counterpoise, as we have seen above, becomes a lever with equal arms, which unites the two driving piston rods; but if the machine has only one cylinder, it becomes necessary to apply an outside force to adjust it. According as this force consists of the weight of a solid body or the weight of a column of water, the apparatus is a *mechanical* or an *hydraulic* counterpoise. As these contrivances will be treated at length in the third part of this work, the following general remarks will suffice here.

The mechanical counterpoise or balance consists of a two-armed lever, loaded on one side, and connected with the piston rod, or generally the rods, at the other, so that the weight on the first end acts in opposition to the weight of the rods, etc.; the up-stroke of the piston is thus accelerated, and the down-stroke retarded. The time of the former is made twice the time of the latter. The hydraulic counterpoise or balance, on the contrary, consists of a second stand pipe, which is led upwards from the regulator cylinder, and replaces the simple discharge pipe; the dead water is carried off through this pipe, so that it forms a water column, which nearly holds the weight of the pump rods in equilibrium; these latter, therefore, descend with less velocity. In the machine at Huelgoat, represented in Fig. 572, and also in the one at Clausthal, shown in Fig. 568 in section, an hydraulic balance is used. The discharge pipe in these cases is a stand pipe which causes the water, after it has performed its function, to ascend again a portion of the whole fall.

FIG. 574.



If, in the hydraulic balance, the so-called counter or back-water column *DE* were placed between the working cylinder *C* and the regulator cylinder *KL*, Fig. 574, the double line of pipe would be avoided.

Naturally, neither kind of balance can increase the mechanical effect; that gained by the balance during the ascent of the driving piston being, of course, lost during the descent of the same. The hydraulic balance has the advantage of greater simplicity, and the mechanical balance the advantage that, by laying on or taking off weights, its effect can be increased or diminished.

§ **318.—Stop-Cocks.**—The various *stop-cocks* or *valves* of a water-pressure engine are of material importance, because by them not only the working of the driving engine but also the working of the regulator itself is regulated. All these contrivances, of course, work negatively only—*i. e.*, only a diminution of power can be brought about by them, and not an increase of power; from this point of view, they are by no means acceptable parts of a water-pressure engine. Their action consists only in opposing an obstruction to the flow of the water in a pipe, so that it must necessarily move slower. In order, then, to regulate as well the ascent and descent of the driving piston, and the ascent and descent of the regulator piston, it is necessary to use four cocks or valves, one in the supply pipe, one in the discharge pipe, as at *Z*, Fig. 572, one in the pipe which leads the regulator water above the regulator piston, and one in the pipe by which the regulator water is carried off, as at *e* and *a* in Figs. 569, 570, and 571. If the velocity of the driving or regulating pistons is notably too great in either direction, it may be diminished at once by turning the cock or by closing the valve, since the opposition to the flow of the water column, which is in connection with the piston and moves with it with the same velocity, thus produced, causes a diminution of its velocity. If, on the contrary, the ascent or descent of the one or the other piston is too slow, the reopening of the corresponding cock will increase the velocity of the same; this increase, however, reaches its limit when the cock is fully open, as already seen. The velocity of the driving piston may also be regulated by varying the stroke of the regulator piston, since, by diminishing this, the access to the working cylinder may be diminished at will.

The loss of power caused by stop-cocks or valves, and especially that caused by the contrivance in the supply pipe usually called a "*day pipe*," is brought about in the same way in a water-pressure engine as that caused by the gates of a reaction turbine. Both follow, in this respect, the overshot or breast water-wheels. (Compare § 257 and § 289.)

In order to attain its maximum effect, a water-pressure engine should be so loaded that, with the complete stroke of the regulator piston, the regular action of the engine should take place without having recourse to the cocks. And, again, if the capacity of this machine is greater than what is required, the excess must be destroyed either by turning the regulating cocks or by working it with a shorter stroke. When this last method is sufficient, it should be used, since the necessary diminution of work is brought about by diminishing the water supply, and so alters but very little the percentage of the engine; this method is not, however, applicable with certain loads.

The change of stroke of a water-pressure engine is effected very easily by pins or wedges applied to the driving piston rod; and for this reason the rod $X_1 X_2$, Fig. 572, which follows the motion of the driving piston, should be furnished with a series of holes. The nearer the tappets $X_1$ and $X_2$ are brought to each other, the sooner the regulation is effected, and therefore also the shorter the driving piston stroke.

§ **319.—Work of a Water-Pressure Engine.**—We come now to the theory and calculation of the mechanical effect of water-pressure engines. The following notation will be used: Area of the driving piston $= F$, area of cross-section of supply pipe $= F_1$. diameter of the driving piston $= d$, that of the supply pipe $= d_1$, diameter of the discharge pipe $= d_2$, the fall, measured from the water surface in the supply cistern to the water surface in the discharge cistern, $= h$, the mean pressure on the piston during the up-stroke, or the vertical height from the piston at its middle position to the water surface in the supply cistern, $= h_1$, and the mean pressure during the down-stroke—*i. e.*, the vertical height from the piston at its middle position to the discharge opening, $= h_2$; let, also, $s$ be the stroke of the piston, $l_1$ the length of the supply pipe, $l_2$ that of the discharge pipe, $v$ the mean velocity of the piston, $v_1$ the mean velocity of the water in the supply pipe, and $v_2$ that in the discharge pipe.

Assume that the engine is single-acting, that the piston makes $n$ complete strokes per minute, and uses for this $Q$ cubic feet of water per second.

The mean pressure of the water upon the piston surface $F$ is $P_1 = F h_1 \gamma$ (see Vol. I., § 355), therefore the work done during one stroke, without considering the loss of power by friction, etc., is

$$P_1 s = F s h_1 \gamma,$$

therefore per minute

$$n P_1 s = n F s h_1 \gamma,$$

and, finally, the mean work per second

$$L_1 = \frac{n}{60} P_1 s = \frac{n}{60} F s h_1 \gamma,$$

or, since

$$\frac{n F s}{60} = Q,$$

$$L_1 = Q h_1 \gamma.$$

During the down-stroke of the piston, the mean force

$$P_2 = F h_2 \gamma$$

acts against the movement, and the work

$$P_2 s = F h_2 s \gamma$$

is consumed; the corresponding loss of work per second is, then,

$$L_2 = Q h_2 \gamma;$$

and consequently the remaining useful work is

$$L = L_1 - L_2 = Q (h_1 - h_2) \gamma = Q h \gamma,$$

as in other hydraulic motors.

This formula remains unchanged when the piston does not fill the cylinder completely; when, as is the case with a plunger piston, a space exists between it and the wall of the cylinder, or when the piston, in its lowest position, does not touch the bottom of the cylinder. It also remains the same when the discharge opening is below the piston in its middle position, or when $h_2$ is negative, and therefore $h = h_1 + h_2$. The form of the piston surface also is without influence (see Vol. I., § 361, Remark); by $F$ is always to be understood the area of a section normal to the axis of the piston, or

$$F = \frac{\pi d^2}{4}.$$

It must, however, be understood that, during the descent of the piston, only the quantity of water $Fs$ corresponding to the stroke $s$ should be discharged, and not the whole quantity in the cylinder or in the cylinder and connection and discharge pipes. The last case cannot, indeed, occur if an upright discharge pipe or an hydraulic balance be used; it is quite otherwise, however, when the discharge pipe is directed downward and the mouth is below the lowest position of the piston. In order, in this case, to retain the water in the cylinder to the level of this lowest position of the piston, and to prevent the influx of air, it is necessary to effect the discharge under water.

REMARK.—We see from what precedes that the work of a water-pressure engine depends only upon the total fall $h = h_1 - h_2$, and not upon these latter heights themselves; it must be remembered, however, that the depth of the lower water surface below the piston should never exceed 33·8 ft., since this is the maximum height of a column of water sustained by atmospheric pressure alone.

§ **320.—Piston Friction.**—The friction of the piston is one of the chief causes of loss of power in a water-pressure engine. As there have been no exact experiments made with reference to this, nothing remains but to calculate it from the pressure of the water, using one of the known coefficients of friction. If the packing is hydrostatic, the force with which each element $f$ of its surface is pressed against the inner surface of the cylinder $= f h_1 \gamma$ for the up-stroke and $= f h_2 \gamma$ for the down-stroke; the friction is therefore $= \phi f h_1 \gamma$ and $\phi f h_2 \gamma$ when $\phi$ denotes the coefficient of friction. Although the forces acting upon the different surface elements have very different directions, the various friction elements are parallel to each other and to the axis of the piston, and their resultant, therefore, or the total friction of the piston, is equal to the sum of the friction of all the elements of the packing; it may be found, therefore, by substituting for $f$ in the above expressions the sum of all the surface elements—*i. e.*, the area of the surface of the packing. If we denote by $e$ the breadth of this surface, or when the packing is double the united breadths of the two rings, we can express the area of the same by $\pi d e$, and the piston friction becomes for the two strokes,

$$R_1 = \phi \pi d e h_1 \gamma \qquad \text{and} \qquad R_2 = \phi \pi d e h_2 \gamma.$$

For convenience, these frictions, as well as the remaining losses of power, are usually expressed by the weight of a column of water having a base equal to the cross-section of the driving piston and a height $h_3$ or $h_4$ corresponding to the loss of head due to the friction. Hence we can write

$$R_1 = F h_3 \gamma \qquad \text{and} \qquad R_2 = F h_4 \gamma,$$

and therefore also

$$F h_3 = \phi \pi d e h_1 \qquad \text{and} \qquad F h_4 = \phi \pi d e h_2,$$

or, since $F = \frac{\pi d^2}{4}$,

$$\frac{d h_3}{4} = \phi e h_1 \qquad \text{and} \qquad \frac{d h_4}{4} = \phi e h_2,$$

therefore the losses of head due to the piston friction are

$$h_3 = 4 \phi \frac{e}{d} h_1 \qquad \text{and} \qquad h_4 = 4 \phi \frac{e}{d} h_2.$$

If we take these heights into consideration, we obtain for the mean force during the ascent,

$$P_1 = F(h_1 - h_3)\gamma = \left(1 - 4\phi\frac{e}{d}\right) F h_1 \gamma,$$

and the mean resistance during the descent

$$P_2 = F(h_2 + h_4)\gamma = \left(1 + 4\phi\frac{e}{d}\right) F h_2 \gamma,$$

therefore the resulting mean work is

$$L = \frac{n}{60}(P_1 - P_2)\frac{v}{s} = \frac{n}{60}\left((h_1 - h_2) - 4\phi\frac{e}{d}(h_1 + h_2)\right) F s \gamma$$

$$= \left(h - 4\phi\frac{e}{d}(h_1 + h_2)\right) Q\gamma$$

$$= \left[\left(1 - 4\phi\frac{e}{d}\right) h - 8\phi\frac{e}{d} h_2\right] Q\gamma$$

$$= \left[1 - 4\phi\frac{e}{d}\left(1 + \frac{2h_2}{h}\right)\right] Q h \gamma.$$

If the height $h_2$ is zero or very small, this may be simplified to

$$L = \left(1 - 4\phi\frac{e}{d}\right) Q h \gamma.$$

It is seen also that the loss of effect through friction of the piston packing is greater the greater $\frac{h_2}{h}$ is—*i. e.*, the farther the engine is below the point of discharge or the higher the discharged water rises again.

In order to reduce as much as possible this loss of head, the packing should not be made unnecessarily wide. In existing machines, the ratio $\frac{e}{d}$ lies within the limits 0·1 to 0·2. So long, however, as special experiments are wanting, the mean value of the coefficient of friction $\phi$ must be assumed at 0·25, according to Morin. This admitted, we have

$$4\,\phi\,\frac{e}{d} = 0{\cdot}1 \text{ to } 0{\cdot}2\,;$$

the friction of the piston, therefore, consumes 10 to 20 per cent of the total available work.

§ **321.—Hydraulic Resistances.**—Another loss of work is occasioned by the *friction of the water* in the supply and discharge pipes. According to the theory laid down in Vol. I., § 427, the loss of head corresponding to this friction is

$$h = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g},$$

in which $\zeta$ is the coefficient of friction. Applied to the supply pipe, this becomes

$$y_1 = \zeta \cdot \frac{l_1}{d_1} \cdot \frac{v_1^2}{2g},$$

and to the discharge pipe,

$$z_1 = \zeta \cdot \frac{l_2}{d_2} \cdot \frac{v_2^2}{2g}.$$

But the quantity of water passing through the pipes per second is

$$\frac{\pi\,d_1^2}{4} \cdot v_1 = \frac{\pi\,d_2^2}{4} \cdot v_2 = \frac{\pi\,d^2}{4} \cdot v_2,$$

therefore

$$d_1^2 v_1 = d_2^2 v_2 = d^2 v,$$

or

$$v_1 = \left(\frac{d}{d_1}\right)^2 v$$

and

$$v_2 = \left(\frac{d}{d_2}\right)^2 v;$$

so that we can put

$$y_1 = \zeta \cdot \frac{l_1 d^4}{d_1^5} \cdot \frac{v^2}{2g}$$

and

$$z_1 = \zeta \cdot \frac{l_2 d^4}{d_2^5} \cdot \frac{v^2}{2g};$$

and if the velocities ($v_1$ or $v_2$) are from 5 to 10 ft., we can substitute for $\zeta$ the values 0·022 to 0·020.

In order to diminish this friction, large supply and discharge pipes must be used and the piston moved slowly up and down.

The movement of the water in the supply and discharge pipes of a water-pressure engine is, however, different from the movement in simple conduit pipes. In the former, the velocity is continually changing, now diminishing, now increasing, while in the latter it remains constant. For this reason, the *inertia of the water* is of greater consequence in water-pressure engines than in the movement in simple pipes. In order to communicate the velocity $v$ to a mass $M$, it is well known, the mechanical work

$$\frac{Mv^2}{2}$$

is expended. In order, therefore, to impart to the water column in the supply pipe, whose weight is $F_1 l_1 \gamma$, the velocity $v_1$, it is necessary to apply the mechanical work

$$F_1 l_1 \gamma \frac{v_1^2}{2g}.$$

If the water column were shut off from the driving piston by the regulator piston after a complete stroke of the former, this work would not be lost, for this column would then give back the work to the driving piston during its retardation and gradual stopping; the shutting off of the motive water, however, takes place during the movement of the driving piston (although towards the end of the same), so that the driving piston and the water column come to a stop at the same time. The regulator piston must, therefore, during the first half of its movement, take from the water column all its living force, since, by a gradual closing of the passage, it interposes a continually increasing resistance. It may then be assumed that the greater part of the work of the inertia

$$F_1 l_1 \gamma \frac{v_1^2}{2g}$$

is lost at each stroke.

If, now, we substitute

$$v_1 = \frac{d^2}{d_1^2} v$$

and

$$F_1 = \frac{\pi d_1^2}{4},$$

we obtain for this work the expression

$$\frac{\pi d^2}{4} \times \frac{d^2 l_1}{d_1^2} \gamma \times \frac{v^2}{2g},$$

and for the corresponding mean force during the whole stroke $s$,

$$K = \frac{\pi d^2}{4} \cdot \frac{d^2 l_1}{d_1^2 s} \gamma \cdot \frac{v^2}{2g},$$

and the loss of head corresponding is

$$y_2 = \frac{K}{F\gamma},$$

*i. e.*,

$$y_2 = \frac{d^2 l_1}{d_1^2 s} \cdot \frac{v^2}{2g}.$$

A similar loss of head takes place during the return of the piston, when it becomes necessary to discharge the water with the velocity $v_2$; the living force to be overcome at the commencement of the stroke is lost by this discharge and must be furnished by the machine. The corresponding loss of head is

$$z_2 = \frac{d^2 l_2}{d_2^2 s} \cdot \frac{v^2}{2g}.$$

In order to diminish this loss of work as much as possible, it is necessary to make the supply and discharge pipes large and as short as may be, and also to have a small piston speed and a long stroke.

In order to diminish or remove entirely the disadvantageous effect of the shock of a column of water suddenly arrested, as with the weighted regulator motion, an air vessel (Fr. réservoir à air; Ger. Windkessel)—*i. e.*, a cylindrical vessel filled with compressed air, such as are found on fire-engines (which will he treated of later)—is sometimes appended to the supply pipe. The excess of living force of the water is thus communicated to the confined air by compressing it, and the work of this force is again given out by the expansion of the air at the beginning of the next stroke, the water which is driven out of the air vessel entering the driving cylinder under nearly the hydrostatic pressure. It has been shown that, with great heights of water, the air mixes with the water and is gradually drawn from the air vessel. To prevent this, a piston must be introduced into the vessel to separate the air from the water; or a small air pump must be applied to replace the loss.

§ **322.**—Changes in the form and direction of the various pipes and channels of a water-pressure engine are, again, causes of loss of work in these machines. These losses may be found, partly by the help of the known rules of hydraulics, explained in Vol. I., Section VI., Chaps. 1 and 2, and partly by the help of experiments made for the purpose. (See "Polytechn. Centralblatt," year 1851, Part 4.)

In the supply and discharge pipes, bends are found, by which the direction of the water is changed usually 90°. Let *r* be the radius of the pipe and *a* the radius of curvature of the axis of the bend, the corresponding coefficient of resistance is, according to Vol. I., § 442,

$$\zeta_1 = 0 \cdot 131 + 1 \cdot 847 \left(\frac{a}{r}\right)^{\frac{7}{2}},$$

and the loss of head when the velocity is $v_1$ is

$$\zeta_1 \frac{v_1^2}{2g},$$

therefore, for a curve in the supply pipe,

$$y_3 = \zeta_1 \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g},$$

and for a curve in the discharge pipe,

$$z_3 = \zeta_1 \left(\frac{d}{d_2}\right)^4 \frac{v^2}{2g}.$$

When the water enters and when it is discharged from the regulator cylinder, its direction is suddenly changed 90° by an elbow; a loss of head is then suffered which is, according to Vol. I., § 441,

$$\zeta_2 \frac{v_1^2}{2g} = 0 \cdot 984 \frac{v_1^2}{2g},$$

or very nearly $= \frac{v_1^2}{2g}$; generally, then, the loss of head at the entrance into the valve cylinder is

$$y_4 = \zeta_2 \frac{v_1^2}{2g} = \zeta_2 \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g},$$

and for the discharge from the valve cylinder into the discharge pipe,

$$z_4 = \zeta_2 \cdot \frac{v_2^2}{2g} = \zeta_2 \left(\frac{d}{d_2}\right)^4 \cdot \frac{v^2}{2g}.$$

For the passage of the water from the regulator cylinder to the connecting pipe, the coefficient of resistance $\zeta_3$ may be taken $= 5$, according to the experiments cited above; and for the passage from the connection pipe into the regulator cylinder $\zeta_4 = 34 \cdot 5$. If, now, $d_3$ is the diameter of the regulator cylinder at the piston, we have, for the passage of the water from the regulator cylinder into the connection pipe, the loss of head,

$$y_5 = \zeta_3 \left(\frac{d}{d_3}\right)^4 \cdot \frac{v^2}{2g} = 5 \left(\frac{d}{d_3}\right)^4 \cdot \frac{v^2}{2g},$$

and for the passage of the water from this pipe into the regulator cylinder,

$$z_5 = \zeta_4 \left(\frac{d}{d_3}\right)^4 \cdot \frac{v^2}{2g} = 34 \cdot 5 \left(\frac{d}{d_3}\right)^4 \cdot \frac{v^2}{2g}$$

Finally, for the entrance into the driving cylinder, according to experiments made for this purpose, $\zeta_5 = 31$, and, on the contrary, for the discharge therefrom $\zeta_6 = 26$; therefore, for the loss of head at the entrance, we have

$$y_6 = \zeta_5 \frac{v^2}{2g} = 31 \frac{v^2}{2g},$$

and for that at the discharge,

$$z_6 = \zeta_6 \frac{v^2}{2g} = 26 \frac{v^2}{2g}$$

In order to diminish the loss of head by sudden changes of velocity, the connection pipes and the part of the regulator cylinders through which the motive water passes should have the same area as the supply and discharge pipes; or, at least, the former pipes, etc., should be connected with the latter by a pipe gradually increasing in size.

Various losses of work or head will also be occasioned by the regulating stop-cocks or valves. They are to be calculated by the formula

$$h = \zeta \frac{v^2}{2g},$$

whose coefficient $\zeta = \zeta_7, \zeta_8$ depends upon their angle of position in the pipe, and may be taken from the table in Vol. I., § 443. We have, then, for the ascent of the driving piston,

$$y_7 = \zeta_7 \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g},$$

and for the descent,

$$z_7 = \zeta_8 \left(\frac{d}{d_2}\right)^4 \frac{v^2}{2g}.$$

By varying the position of the regulating apparatus, any desired value between 0 and ∞ can be given to the coefficient of resistance; any excess of force may, therefore, be destroyed, and the velocity of the ascent and descent regulated at will as may be required.

§ **323.—Formula for the Work.**—Neglecting, for the present, the regulator, we can find a formula for the useful work of a water-pressure engine. The mean force during the ascent of the piston is

$$P_1 = [h_1 - h_3 - (y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)] F \gamma,$$

$$= [h_1 - h_3 - \Sigma(y)] F \gamma,$$

and that during the descent is

$$P_2 = (h_2 + h_4 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6 + z_7)\, F\gamma,$$

$$= [h_2 + h_4 + \Sigma\,(z)]\, F\gamma\,;$$

therefore, during a complete stroke, the work is

$$(P_1 - P_2)\, s = [h_1 - (h_2 + h_3 + h_4) - (\,\Sigma\,(y) + \Sigma\,(z)\,)\,]\, Fs\gamma,$$

and the work per second

$$L = [h_1 - (h_2 + h_3 + h_4) - (\,\Sigma\,(y) + \Sigma\,(z))\,] \cdot \frac{n}{60} \cdot Fs\gamma$$

$$= \left(h - 4\phi\frac{e}{d}(h_1 + h_2) - (\Sigma\,(y) + \Sigma\,(z))\right)\frac{n}{60}\, Fs\gamma.$$

If, now, we put

$$\zeta\frac{l_1 d^4}{d_1^5} + \frac{d^2 l_1}{d_1^2 s} + \zeta_1\left(\frac{d}{d_1}\right)^4 + \zeta_2\left(\frac{d}{d_1}\right)^4 + \zeta_3\left(\frac{d}{d_3}\right)^4 + \zeta_5 + \zeta_7\left(\frac{d}{d_1}\right)^4,$$

or

$$\left[\zeta\frac{l_1}{d_1} + \frac{d_1^2 l_1}{d^2 s} + \zeta_1 + \zeta_2 + \zeta_3\left(\frac{d_1}{d_3}\right)^4 + \zeta_5\left(\frac{d_1}{d}\right)^4 + \zeta_7\right]\left(\frac{d}{d_1}\right)^4 = \kappa_1\left(\frac{d}{d_1}\right)^4,$$

and

$$\zeta\frac{l_2 d^4}{d_2^5} + \frac{d^2 l_2}{d_2^2 s} + \zeta_1\left(\frac{d}{d_2}\right)^4 + \zeta_2\left(\frac{d}{d_2}\right)^4 + \zeta_4\left(\frac{d}{d_3}\right)^4 + \zeta_6 + \zeta_8\left(\frac{d}{d_2}\right)^4,$$

or

$$\left[\zeta\frac{l_2}{d_2} + \frac{d_2^2 l_2}{d^2 s} + \zeta_1 + \zeta_2 + \zeta_4\left(\frac{d_2}{d_3}\right)^4 + \zeta_6\left(\frac{d_2}{d}\right)^4 + \zeta_8\right]\left(\frac{d}{d_2}\right)^4 = \kappa_2\left(\frac{d}{d_2}\right),$$

we can express the work simply and briefly by the formula

$$L = \left[h - \left(4\phi\frac{e}{d}(h_1 + h_2) + \left[\kappa_1\left(\frac{d}{d_1}\right)^4 + \kappa_2\left(\frac{d}{d_2}\right)^4\right]\frac{v^2}{2g}\right)\right]\frac{n}{60}\, Fs\gamma.$$

On account of the greater length of the supply pipe, $\kappa_1$ is nearly always greater than $\kappa_2$, and the time $t_1$ for the up-stroke of the piston is, therefore, usually made greater than the time $t_2$ for the down-stroke.

If the time $t_1$ for the up-stroke $= \nu_1 t$, and that $t_2$ for the down-stroke $= \nu_2 t$, in which

$$t = t_1 + t_2 = \frac{60''}{n}$$

denotes the time of a complete stroke, we have for the mean velocity of the complete stroke

$$v = \frac{2\,s}{t} = \frac{2\,n\,s}{60},$$

for the mean velocity of ascent, however,

$$v_1 = \frac{s}{t_1} = \frac{s}{\nu_1\,t} = \frac{1}{\nu_1} \cdot \frac{v}{2},$$

and for the mean velocity of descent,

$$v_2 = \frac{s}{t_2} = \frac{s}{\nu_2\,t} = \frac{1}{\nu_2} \cdot \frac{v}{2}.$$

The general expression for the work now becomes

$$L = \left[h - \left(4\,\phi\,\frac{e}{d}\,(h_1 + h_2) + \left[\kappa_1 \left(\frac{1}{2\,\nu_1}\right)^2 \left(\frac{d}{d_1}\right)^4 + \kappa_2 \left(\frac{1}{2\,\nu_2}\right)^2 \left(\frac{d}{d_2}\right)^4\right] \frac{v^2}{2\,g}\right)\right] \frac{n}{60} \cdot F\,s\,\gamma,$$

or, since

$$\frac{n}{60}\,F\,s = Q,$$

$$L = \left[h - \left(4\,\phi\,\frac{e}{d}\,[h_1 + h_2] + \tfrac{1}{4}\left[\kappa_1 \left(\frac{1}{\nu_1}\right)^2 \left(\frac{d}{d_1}\right)^4 + \kappa_2 \left(\frac{1}{\nu_2}\right)^2 \left(\frac{d}{d_2}\right)^4\right] \frac{v^2}{2\,g}\right)\right] Q\,\gamma,$$

or, again, substituting for $v$ its equivalent

$$\frac{2\,Q}{F} = \frac{8\,Q}{\pi\, d^2},$$

$$L = \left(h - \left[4\,\phi\,\frac{c}{d}\,[h_1 + h_2] + \left(\frac{\kappa_1}{v_1^{\,2}\,d_1^{\,4}} + \frac{\kappa_2}{v_2^{\,2}\,d_2^{\,4}}\right)\cdot\frac{1}{2\,g}\cdot\left(\frac{4\,Q}{\pi}\right)^2\right]\right) Q\,\gamma.$$

For a double-acting engine, this work should of course be doubled.

This formula shows clearly that the useful work of a water-pressure engine is greater the greater $d$, $d_1$ and $d_2$—*i. e.*, the larger the various cylinders and pipes are. It may be shown also by the higher calculus, that the work performed during a given number of strokes is greatest or the resistance least when

$$\frac{\kappa_1}{v_1^{\,3}\,d_1^{\,4}} = \frac{\kappa_2}{v_2^{\,3}\,d_2^{\,4}},$$

*i. e.*, when

$$\frac{v_1}{v_2} = \sqrt[3]{\frac{\kappa_1\,d_2^{\,4}}{\kappa_2\,d_1^{\,4}}}.$$

Since also $v_1 + v_2 = 1$, we have

$$v_1 = \frac{1}{1 + \sqrt[3]{\dfrac{\kappa_2\,d_1^{\,4}}{\kappa_1\,d_2^{\,4}}}},$$

and

$$v_2 = \frac{1}{1 + \sqrt[3]{\dfrac{\kappa_1\,d_2^{\,4}}{\kappa_2\,d_1^{\,4}}}}.$$

If, *e.g.*, $d_2 = d_1$ and $\kappa_1 = 8\,\kappa_2$, then $\frac{v_1}{v_2} = \sqrt[3]{8} = 2$; therefore, the time of the ascent of the piston must be double that of the descent. By the use of a counterpoise attached to the piston, this ratio $\frac{v_1}{v_2}$ of the time of ascent to the time of descent may be easily obtained, by adding or taking off weights. The regulation of the time by stop-cocks in the supply and discharge pipes, on the contrary, entails a loss of power proportional to $\zeta_7$, $\zeta_8$, which is greater the more the cocks are turned.

But if the work to be done is less than the capacity of the engine, the excess of power must be destroyed by such cocks or valves.

### § 324.—The Square of the Velocity.

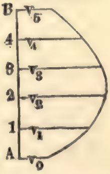
—The question also arises, what value shall be substituted in the last formula for the *mean square of the piston speed* of a water-pressure engine. If the piston moved up and down with an uniform velocity, we must have

$$v^2 = \left(\frac{s}{t_1}\right)^2$$

in which $s$ denotes the path of the piston and $t_1$ the time occupied in passing through the same; since, however, this is not true for either single or double acting engines, it becomes necessary to make a special calculation for $v^2$.

The mean square of the piston velocity is found by adding together the squares of the separate velocities $v_0, v_1, v_2 \ldots$, corresponding to the equal parts of the piston path $s = AB$, Fig. 575, and dividing the sum by the number of these parts. If, now, the motion of the piston were uniformly accelerated or uniformly retarded, the squares of the velocities would be as the spaces travelled; if, therefore, the smallest velocity were $= 0$ and the greatest $= c$, we should have for the velocity squares $v_0^2, v_1^2, v_2^2, v_3^2 \ldots$, corresponding to the spaces

FIG. 575.



$$0, \frac{s}{n}, \frac{2s}{n}, \frac{3s}{n} \ldots,$$

$$0, \frac{1}{n}c^2, \frac{2}{n}c^2, \frac{3}{n}c^2 \ldots,$$

and for the sum of the velocity squares,

$$\frac{c}{n}(1+2+3+\ldots+n) = \frac{c^2}{n} \cdot \frac{n^2}{2} = n\frac{c^2}{2},$$

and, therefore, for the mean value,

$$v^2 = \frac{c^2}{2};$$

or, since $s = \frac{c\,t_1}{2}$,

$$v^2 = \tfrac{1}{2}\left(\frac{2\,s}{t_1}\right)^2 = 2\left(\frac{s}{t_1}\right)^2 = 2\,v_1^2,$$

when the mean velocity $v_1$ of the piston is substituted for the quotient $\frac{s}{t_1}$ of the path of the piston $s$ by the time of movement $t_1$. This formula is also true, of course, when the first part of the piston path is travelled with a uniformly accelerated and the last part with a uniformly retarded motion.

We must, therefore, in this case, put the *mean square* $v^2$ *of the velocity twice as great as the square* $v_1^2$ *of the mean velocity of the piston.*

For a double-acting water-pressure engine with a crank mechanism,

$$v^2 = \frac{\pi^2}{6}\,v_1^2 = 1{\cdot}645\,v_1^2 = 1{\cdot}645\left(\frac{s}{t_1}\right)^2,$$

as will be explained in the article upon "Steam-Engines."

If, therefore, we substitute in the formula for the work (§ 323),

$$L = \Bigg[h - \Bigg(4\phi\frac{e}{d}[h_1 + h_2] + \tfrac{1}{4}\Bigg[\kappa_1\left(\frac{1}{\nu_1}\right)^2\left(\frac{d}{d_1}\right)^4$$

$$+ \kappa_2\left(\frac{1}{\nu_2}\right)^2\left(\frac{d}{d_2}\right)^4\Bigg]\frac{v^2}{2\,g}\Bigg]\,Q\,\gamma,$$

for $v^2$ its equivalent,

$$2 v_1^2 = 2 \left(\frac{2\,Q}{F}\right)^2 = 2 \left(\frac{8\,Q}{\pi\, d^2}\right)^2,$$

we obtain

$$L = \Big[h - \Big(4 \phi \frac{e}{d} [h_1 + h_2] + \tfrac{1}{2} \Big[\frac{\kappa_1}{\nu_1^2\, d_1^4} + \frac{\kappa_2}{\nu_2^2\, d_2^4}\Big] \frac{1}{2g} \Big[\frac{8\,Q}{\pi}\Big]^2\Big)\Big] Q \gamma.$$

Example.—Required the calculation and arrangement of a single-acting, one-cylinder water-pressure engine, for a head $h$ of 350 ft. and a water supply $Q = 1$ cubic ft. per sec. Assuming the mean velocity of the driving piston to be $v = 1$ ft., for both the ascent and descent, we have for its area,

$$F = \frac{2\,Q}{v} = \frac{2 \times 1}{1} = 2 \text{ sq. ft}$$

Assuming also the mean velocity of the water in the supply and discharge pipes $v_1 = v_2 = 5$ ft., we have for the cross-sections of these pipes,

$$F_1 = \frac{2\,Q}{v_1} = \tfrac{2}{5} = 0 \cdot 4 \text{ sq. ft.}$$

The diameter of the driving piston is, consequently,

$$d = \sqrt{\frac{4\,F}{\pi}} = \sqrt{\frac{8}{\pi}} = 1 \cdot 5958 \text{ ft.},$$

and the diameter of the supply and discharge pipes is

$$d_1 = d_2 = \sqrt{\frac{4\,F_1}{\pi}} = \sqrt{\frac{1 \cdot 6}{\pi}} = 0 \cdot 71364 \text{ ft.}$$

For convenience as well as safety, we will use $d = 20$ inches and $d_1 = 9$ inches.

In order to take into account the weight of the piston rod, etc., we will assume the discharge orifice to be 50 ft. above the middle position of the piston, or $h_2 = 50$ ft., so that we have

$$h_1 = h + h_2 = 400 \text{ ft.}$$

Assume also that the length $l_1$ of the supply pipe is 450 ft., and that $l_2$ of the discharge pipe 66 ft. The area of a piston 20 inches in diameter is

$$F = \frac{\pi d^2}{4} = \frac{\pi}{4} \cdot \frac{25}{9} = 2 \cdot 182 \text{ sq. ft.};$$

therefore,

$$v = \frac{2 Q}{F} = \frac{2}{2 \cdot 182} = 0 \cdot 9166 \text{ ft.}$$

Assuming four strokes per minute, the length of stroke will be

$$s = \frac{60 v}{2 n} = \frac{60 \times 0 \cdot 9166}{8} = 6 \cdot 8745 \text{ ft.}$$

If we assume also the breadth $e$ of the packing of the driving piston $= \frac{1}{8} d = 2\frac{1}{2}$ inches, we obtain next the loss of head due to the friction of the piston,

$$4 \phi \frac{e}{d} (h_1 + h_2) = 4 \times 0 \cdot 25 \times \frac{1}{8} (400 + 50) = \frac{450}{8} = 56 \cdot 25 \text{ ft.},$$

and the remaining useful fall or head, after deducting the piston friction, is

$$h - 4 \phi \frac{e}{d} (h_1 + h_2) = 350 - 56 \cdot 25 = 293 \cdot 75 \text{ ft.}$$

In order to find the hydraulic resistances, we must next calculate the coefficients $\kappa_1$ and $\kappa_2$. The one for the supply pipe is

$$\kappa_1 = \zeta \frac{l_1}{d_1} + \frac{d_1^2 l_1}{d^2 s} + \zeta_1 + \zeta_2 + \zeta_3 \left(\frac{d_1}{d_3}\right)^4 + \zeta_6 \left({}^{d_1}\right)^4 + \zeta_7,$$

and the one for the discharge pipe is

$$\kappa_2 = \zeta \frac{l_2}{d_2} + \frac{d_2^2 l_2}{d^2 s} + \zeta_1 + \zeta_2 + \zeta_4 \left(\frac{d_2}{d_3}\right)^4 + \zeta_6 \left(\frac{d_2}{d}\right)^4 + \zeta \gamma.$$

The values to be substituted in this formula are

$$\zeta = 0 \cdot 021, \frac{l_1}{d_1} = \frac{450}{\frac{3}{4}} = 600, \frac{l_2}{a_2} = \frac{66}{\frac{3}{4}} = 88;$$

whence

$$\cdot \frac{l_1}{d_1} = 0 \cdot 021 \times 600 = 12 \cdot 6$$

and

$$\zeta \cdot \frac{l_2}{d} = 0 \cdot 021 \times 88 = 1 \cdot 85;$$

further,

$$\frac{d_1^2 l_1}{d^2 s} = \left(\frac{9}{20}\right)^2 \times \frac{450}{6 \cdot 87} = 13 \cdot 26,$$

and

$$\frac{d_2^2\, l_2}{d^2\, s} = \left(\frac{9}{20}\right)^2 \times \frac{66}{6\cdot 87} = 1\cdot 94.$$

If we assume also a bend in the supply and one in the discharge pipe, whose radius of curvature $a = 4\, r$, whence consequently $\frac{r}{a} = \frac{1}{4}$, we obtain the corresponding coefficient of resistance, § 442, Vol. I.,

$$\zeta_1 = 0\cdot 131 + 1\cdot 847 \times \left(\frac{1}{4}\right)^{\frac{7}{2}} = 0\cdot 15.$$

If we also assume that the supply and discharge pipes are connected with the regulator cylinder by a right-angled elbow, then we must substitute for both these pipes

$$\zeta_2 = 0\cdot 984\,;$$

and if the cross-section of the regulator cylinder is twice as great as that of the supply and discharge pipes, we have

$$d_3^2 = 2\, d_1^2 = 2\, d_2^2,$$

and, therefore,

$$\zeta_3 \left(\frac{d_1}{d_3}\right)^4 = \frac{5}{4} = 1\cdot 25,$$

and also

$$\zeta_4 \left(\frac{d_2}{d_3}\right) = \frac{34\cdot 5}{4} = 8\cdot 62.$$

Finally, also,

$$\zeta_5 \left(\frac{a_1}{d}\right)^4 = 31 \times \left(\frac{9}{20}\right)^4 = 1\cdot 27,$$

and

$$\zeta_6 \left(\frac{d_2}{d}\right)^4 = 26 \times \left(\frac{9}{20}\right)^4 = 1\cdot 07\,;$$

and, moreover, if the stop-cocks in both supply and discharge pipes are entirely open, $\zeta_7$ and $\zeta_8 = 0$, and we have

$$\kappa_1 = \left\{\begin{matrix} 12\cdot 60 \\ 13\cdot 26 \\ 0\cdot 15 \\ 0\cdot 98 \\ 1\cdot 25 \\ 1\cdot 27 \end{matrix}\right\} = 29\cdot 51, \text{and } \kappa_2 = \left\{\begin{matrix} 1\cdot 85 \\ 1\cdot 94 \\ 0\cdot 15 \\ 0\cdot 98 \\ 8\cdot 62 \\ 1\cdot 07 \end{matrix}\right\} = 14\cdot 61\,;$$

and since the ratio of the time of the up-stroke to that of the down-stroke corresponding to the most advantageous movement is

$$\frac{v_1}{v_2} = \sqrt[3]{\frac{\kappa_1}{\kappa_2}} = \sqrt[3]{\frac{29 \cdot 51}{14 \cdot 61}} = 1 \cdot 264,$$

the ratio of the time of the down-stroke to that of a complete stroke is

$$v_2 = \frac{1}{1 + 1 \cdot 264} = \frac{1}{2 \cdot 264} = 0 \cdot 442,$$

and also the ratio of the time of the up-stroke to that of a complete stroke is

$$v_1 = 1 - v_2 = 0 \cdot 558.$$

By substitution of these values, we obtain the effective head

$$h - \left[ 4\,\phi \frac{e}{d} (h_1 + h_2) + \tfrac{1}{2} \left( \frac{\kappa_1}{v_1^2\, d_1^4} + \frac{\kappa_2}{v_2^2\, d_2^4} \right) \frac{1}{2\,g} \left( \frac{8\,Q}{\pi} \right)^2 \right]$$

$$= h - \left[ 4\,\phi \frac{e}{d} (h_1 + h_2) + \tfrac{1}{2} \left( \frac{\kappa_1}{v_1^2} + \frac{\kappa_2}{v_2^2} \right) \cdot \frac{1}{2\,g} \left( \frac{8\,Q}{\pi\, d_1^2} \right)^2 \right]$$

$$= 293 \cdot 75 - \tfrac{1}{2} \left( \frac{29 \cdot 51}{0 \cdot 3114} + \frac{14 \cdot 61}{0 \cdot 1954} \right) \times 0 \cdot 016 \times \left( \frac{8 \times 16}{9\,\pi} \right)^2$$

$$= 293 \cdot 75 - (94 \cdot 7 + 74 \cdot 8) \times \tfrac{1}{2} \times 0 \cdot 016 \times \left( \frac{128}{9\,\pi} \right)^2$$

$$= 293 \cdot 75 - 169 \cdot 5 \times 0 \cdot 008 \left( \frac{128}{9\,\pi} \right)^2 = 293 \cdot 75 - 27 \cdot 86 = 265 \cdot 89.$$

From this it follows that the percentage of this machine, without taking into consideration the power consumed by the regulator, is

$$\eta = \frac{265 \cdot 89}{350} = 0 \cdot 759,$$

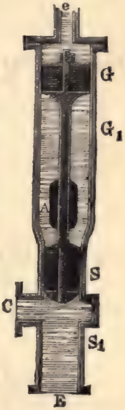
and that the useful work is

$$L = Q \left[ h - \left( 4\,\phi \frac{e}{d} (h_1 + h_2) + etc. \right) \right] \gamma = 265 \cdot 89 \times 1 \times 62 \cdot 5$$

$$= 16618 \text{ ft. lbs.} = 30 \cdot 2 \text{ horse-power.}$$

§ **325.—Calculation of the Regulator.**—The arrangement and calculation of the regulator of a water-pressure engine is also a matter of great importance. Since, in the latest and best machines, only piston regulators are used, in what follows reference will be made to this kind alone. Let us notice, then, next the system of a regulator with double piston as it occurs in some modern machines, and as it is represented in Fig. 576. We will suppose that the regulator piston $S$ is here pressed from beneath by the mean head of water $h_1$, and from above by the mean head $h_2$; if we denote by $k$ the height of the counter piston $G$ above the regulator piston $S$, we will have for the upward pressure on $G$ that due to a height $= h_2 - k$, and for the pressure on $G$ from above, according as the motive water is let on or shut off, that due to the heights $h_1 - k$ or $h_2 - k$. We will assume the diameter of the regulator piston to be $d_1$ and that of the counter piston to be $d_2$, and, further, that the packing is composed of leather rings placed upon each other, and of the same height in the two pistons.

FIG. 576.



If, now, the pistons are in the highest position, as shown in Fig. 576, by admitting the water above $G$ a movement downward is effected; the difference of the water pressures on $S$ and $G$ must then, in connection with the weight $R$ of the regulator and counter pistons and connecting rod, overbalance the friction of both pistons. But the pressure above $G$ is

$$= \frac{\pi d_2^2}{4} (h_1 - k) \gamma,$$

and the counter pressure below $G$ is

$$= \frac{\pi d_2^2}{4} (h_2 - k) \gamma;$$

further, the pressure above

$$S = \frac{\pi d_1^2}{4} h_2 \gamma,$$

and the pressure below

$$S = \frac{\pi d_1^2}{4} h_1 \gamma;$$

the force urging the regulator downward is, therefore,

$$P = \frac{\pi d_2^2}{4} (h_1 - k - h_2 + k) \gamma + \frac{\pi d_1^2}{4} (h_2 - h_1) \gamma + R$$

$$= \frac{\pi}{4} (d_2^2 - d_1^2) (h_1 - h_2) \gamma + R,$$

or, denoting the fall $h_1 - h_2$ by $h$,

$$P = \frac{\pi}{4} (d_2^2 - d_1^2) h \gamma + R.$$

The piston friction, although not hydrostatic, must be looked upon as proportional to its breadth, to the circumference of the piston, and to the difference of pressure on the two sides of the same; it may be expressed by

$$F = \phi \pi d e h \gamma,$$

and, therefore, in the preceding case, by

$$P = \phi \pi e_1 (d_1 (h_1 - h_2) + d_2 [h_1 - k - (h_2 - k)]) \gamma$$

$$= \phi \pi (d_1 + d_2) e_1 h \gamma.$$

We have, then, the equation

$$\frac{\pi}{4} (d_2^2 - d_1^2) h \gamma + R = \phi \pi (d_1 + d_2) e_1 h \gamma,$$

or, reduced,

$$(1)\quad d_2^2 - d_1^2 + \frac{4R}{\pi h \gamma} = 4 \phi e_1 (d_1 + d_2).$$

If, on the contrary, the pistons and connection rod are to move upward from their lowest position after shutting off the water from above $G$, the difference of pressure upon both sides of $S$ must alone be sufficient to overcome the weight of the pistons, etc., and their friction, since the pressures upon the upper and lower faces of $G$ counteract each other. We must have, then,

$$\frac{\pi}{4} d_1^2 (h_1 - h_2) \gamma = R + \phi \pi (d_1 + d_2) e_1 h \gamma,$$

or, simpler,

$$(2) \quad d_1^2 - \frac{4R}{\pi h \gamma} = 4 \phi e_1 (d_1 + d_2).$$

These formulæ can be used to calculate the diameters $d_1$ and $d_2$. Without considering the weight $R$, which, when the head is great, is of very little consequence, we can put

$$d_2^2 - d_1^2 = 4 \phi e_1 (d_1 + d_2)$$

and

$$d_1^2 = 4 \phi e_1 (d_1 + d_2),$$

therefore

$$d_2^2 - d_1^2 = d_1^2$$

or

$$d_2^2 = 2 d_1^2,$$

and consequently the diameter of the counter piston

$$d_2 = d_1 \sqrt{2} = 1 \cdot 414 \, d_1,$$

or, approximately, $\frac{7}{5}$ of the diameter of the regulator piston.

This last is found from the first equation,

$$d_2^2 - d_1^2 = 4 \phi e_1 (d_1 + d_2),$$

or

$$d_2 - d_1 = 4 \phi e_1,$$

by substituting therein the value

$$d_2 = d_1 \sqrt{2}.$$

In this way, we find

$$d_1 = \frac{4\phi e_1}{\sqrt{2} - 1} = (\sqrt{2} + 1)\, 4\phi e_1 = 2 \cdot 414 \times 4\phi e_1,$$

and

$$d_2 = 3 \cdot 414 \times 4\phi e_1.$$

Taking into consideration, however, the weight of the pistons, we have, approximately, but exact enough for practical purposes,

$$d_2 = \sqrt{2\, d_1^2 - \frac{8R}{\pi h \gamma}} = d_1 \sqrt{2} - \frac{4R}{\pi h \gamma d_1 \sqrt{2}}$$

$$= d_1 \sqrt{2} - \frac{(\sqrt{2} - 1)R}{\phi \pi e_1 h \gamma \sqrt{2}},$$

and therefore, from the first equation,

$$d_2 - d_1 = 4\phi e_1 - \frac{4R}{\pi h \gamma (d_1 + d_2)},$$

that is,

$$(\sqrt{2} - 1) d_1 = 4\phi e_1 + \frac{(\sqrt{2} - 1)}{\phi \pi e_1 h \gamma \sqrt{2}} R - \frac{(\sqrt{2} - 1)R}{\phi \pi e_1 h \gamma (1 + \sqrt{2})},$$

therefore

$$d_1 = (\sqrt{2} + 1)\, 4\phi e_1 + \frac{(2 - \sqrt{2})R}{2\phi \pi e_1 h \gamma},$$

and

$$d_2 = (\sqrt{2} + 2)\, 4\phi e_1 + \frac{(3\sqrt{2} - 4)R}{2\phi \pi e_1 h \gamma}.$$

For safety, both diameters are made somewhat larger than these values, and the excess of power, when the regulator piston moves too quickly, is destroyed by means of the previously described regulating cocks. From observation of the best machines, we learn that $4\phi e_1 = 0 \cdot 1$ only, or $\phi e_1 = \frac{1}{40}$. In order that the passage of the motive water through the regulator cylinder may take place with the least possible loss of head, it is usual to make the diameter of this cylinder the same as that of the supply and connecting pipes; and, if the above formulæ lead to a value of $d_1$, smaller than the diameter of the supply pipe, it is to be noticed that there will be an excess of power, which must be nullified by the regulating cocks.

Example.—Required to calculate a regulator with double piston, for a water-pressure engine under 400 ft. head, the weight of pistons, etc., being assumed beforehand at 150 lbs. Without reference to this weight, we have the diameters

$$d_1 = 2 \cdot 414 \times 4 \phi e_1 = 2 \cdot 414 \times 0 \cdot 1 = 0 \cdot 2414 \text{ ft.} = 2 \cdot 897 \text{ inches,}$$

and

$$d_2 = 3 \cdot 414 \times 0 \cdot 1 = 0 \cdot 3414 \text{ ft.} = 4 \cdot 097 \text{ inches ;}$$

but, taking this weight into consideration,

$$d_1 = 0 \cdot 2414 + \frac{0 \cdot 586 \times 150}{0 \cdot 05 \times 400 \times 62 \cdot 5\, \pi} = 0 \cdot 2414 + \frac{0 \cdot 586}{8\frac{1}{3} \cdot \pi} = 0 \cdot 2414 + 0 \cdot 0224$$
$$= 0 \cdot 2638 \text{ ft.} = 3 \cdot 166 \text{ inches ;}$$

and

$$d_2 = 0 \cdot 3414 + \frac{0 \cdot 243 \times 150}{0 \cdot 05 \times 400 \times 62 \cdot 5\, \pi} = 0 \cdot 3414 + 0 \cdot 0093$$
$$= 0 \cdot 3507 \text{ ft.} = 4 \cdot 208 \text{ inches.}$$

If $d_1 = 3\frac{1}{2}$ and $d_2 = 5$ inches are used, it will be safer in practice. For such small pistons, only a small quantity of water is necessary, but the hydraulic resistance opposed to its passage through the regulator cylinder is great. If, on this account, $d_1$ be assumed $= 6$ inches, $d_2$ must be made at least $= d_1 \sqrt{2} = 1 \cdot 414 \times 6 = 8 \cdot 484$ inches, or from $8\frac{3}{4}$ to 9 inches, and the excess of power destroyed by the regulating cocks.

§ **326.**—For a regulator system with three pistons, the calculation is not very different from the foregoing, except that now one of the pistons—the regulator piston proper, for instance—can be given the same diameter that the supply pipe has, which is advantageous. The regulator system of a water-pressure engine with two cylinders, represented in Fig. 569, is to be calculated as follows: Let $d_1$ be the diameter of the lowest or first regulator piston, $d_2$ that of the second and $d_3$ that of the uppermost counter piston. We can write for the down-stroke,

$$(1)\ d_1^2 - d_2^2 + d_3^2 + \frac{4R}{\pi h \gamma} = 4 \phi e_1 (d_1 + d_2 + d_3),$$

and for the up-stroke,

$$(2)\ d_2^2 - d_1^2 - \frac{4R}{\pi h \gamma} = 4 \phi e_1 (d_1 + d_2 + d_3).$$

From these formulæ and the value $d_1$, the other diameters $d_2$ and $d_3$ may be calculated. For safety and on account of the hydraulic resistance, $d_2$ is taken somewhat larger than given by the formulæ. Substituting this value in the formula,

$$2\,(d_1^2 - d_2^2) + d_3^2 + \frac{8\,R}{\pi\,h\,\gamma} = 0,$$

we obtain the diameter of the third piston,

$$d_3 = \sqrt{2\,(d_2^2 - d_1^2) - \frac{8\,R}{\pi\,h\,\gamma}},$$

which, on account of the reasons given above, is to be taken amply large.

For the regulator of the engine, represented in Fig. 572, the following formulæ may be obtained: Let $h_1$ denote the mean height of the motive water, and $h_2$ that of the discharge water; further, $d_1$ the diameter of the regulator piston, $d_2$ that of the counter piston, and $d_3$ that of the third or added piston. The force for the down-stroke is

$$\frac{\pi}{4}[d_1^2\,(h_1 - h_2) + (d_2^2 - d_3^2)\,h_1 - d_2^2\,h_1]\,\gamma + R,$$

and for the up-stroke,

$$\frac{\pi}{4}\,[d_2^2\,h_1 - (d_2^2 - d_3^2)\,h_2 - d_1^2\,(h_2 - h_2)]\,\gamma - R,$$

therefore,

$$(1)\quad d_1^2 - \frac{h_1}{h}\,d_3^2 + \frac{4\,R}{\pi\,h\,\gamma} = 4\,\phi\,e_1\,(d_1 + d_2 + d_3),$$

and

$$(2)\quad d_2^2 - d_1^2 + \frac{h_2}{h}\,d_3^2 - \frac{4\,R}{\pi\,h\,\gamma} = 4\,\phi\,e_1\,(d_1 + d_2 + d_3).$$

Having $d_1$ given, the values of $d_2$ and $d_3$ may be calculated; but in practice $d_2$ should be taken somewhat more and $d_3$ somewhat less, for the reasons already stated. These may, moreover, be calculated more easily from the formulæ

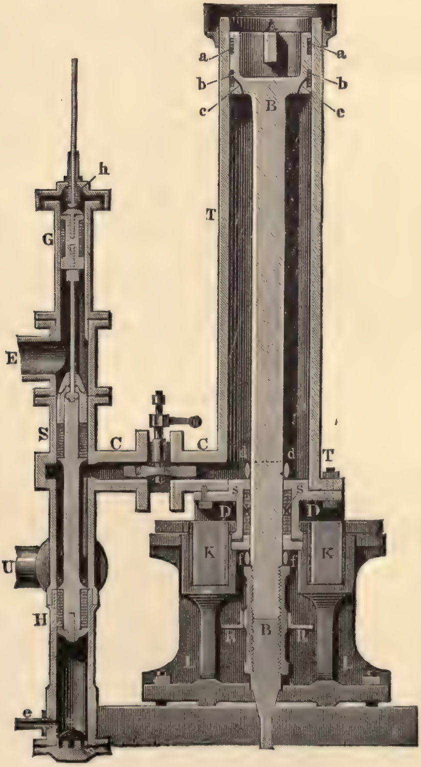
$$(1)\quad d_2^2 - d_3^2 = 8\,\phi\,e_1\,(d_1 + d_2 + d_3)$$

and,

$$(2)\quad d_2^2 + \left(\frac{h_1 + h_2}{h}\right) d_3^2 = 2\,d_1^2 + \frac{8\,R}{\pi\,h\,\gamma}.$$

For the regulator system of the machine at Clausthal, shown in Fig. 577, and already generally described in what

FIG. 577.



precedes, we may finally obtain for the force during the down-stroke,

$$\frac{\pi}{4}\left[d_1^2\,(h_1 - h_2) - d_2^2\,h_1\right]\gamma + R,$$

and for the force during the up-stroke,

$$\frac{\pi}{4}\left[d_3^2\,(h_1 - h_2) - d_1^2\,(h_1 - h_2) + d_2^2 h_1\right]\gamma - R,$$

in which $d_1$ is the diameter of the valve piston, $d_2$ that of the upper or counter piston, and $d_3$ that of the lower or return piston; therefore,

$$(1)\quad d_1^2 - \frac{h_1}{h}\,d_2^2 + \frac{4\,R}{\pi\,h\,\gamma} = 4\,\phi\,e_1\,(d_1 + d_2 + d_3)$$

and

$$(2)\quad d_3^2 - d_1^2 + \frac{h_1}{h}\,d_2^2 - \frac{4\,R}{\pi\,h\,\gamma} = 4\,\phi\,e_1\,(d_1 + d_2 + d_3).$$

EXAMPLE.—If, in the last machine, the head $h_1 = 688$ and $h_2 = 75$, further the weight $R$ of the piston connection be taken $= 170$ lbs., and the diameter of the regulating piston $d_1 = 0{\cdot}5$ ft., the diameters of the other pistons may be obtained as follows:

$$d_3^2 = 8\,\phi\,e_1\,(d_1 + d_2 + d_3) \quad \text{and also} \quad = 2\,d_1^2 - \frac{2\,h_1}{h}\,d_2^2 + \frac{8\,R}{\pi\,h\,\gamma},$$

or numerically,

$$d_3^2 = 0{\cdot}2\,(0{\cdot}5 + d_2 + d_3) \quad \text{and} \quad = 0{\cdot}5 - 2{\cdot}248\,d_2^2 + 0{\cdot}0113.$$

Assuming now $d_2 = 0{\cdot}3$ ft., we obtain once

$$d_3^2 = 0{\cdot}5113 - 0{\cdot}2023 = 0{\cdot}3090 \quad \text{or} \quad d_3 = 0{\cdot}556 \text{ ft.},$$

and, again,

$$d_3^2 = 0{\cdot}2 \times 1{\cdot}356 = 0{\cdot}2712 \quad \text{or} \quad d_3 = 0{\cdot}5208 \text{ ft.};$$

but taking $d_2 = 0{\cdot}33$ ft., we obtain

$$d_3^2 = 0{\cdot}5113 - 0{\cdot}2448 = 0{\cdot}2665 \quad \text{or} \quad d_3 = 0{\cdot}516 \text{ ft.},$$

and, again,

$$d_3^2 = 0 \cdot 2 \times 1 \cdot 346 = 0 \cdot 2692 \qquad \text{or} \qquad d_3 = 0 \cdot 519 \text{ ft.}$$

Therefore, $d_2$ is to be taken $= 0 \cdot 33$ ft., or about 4 inches, and $d_3 = 0 \cdot 52$ ft., or about 6·25 inches. In fact, $d_2$ is 4·26 inches and $d_3$ is 5·98 inches; we conclude therefore that, in this machine, $4 \phi e_1$ is somewhat less than 0·1.

REMARK.—If more exact calculations are required, attention must be paid to the cross-section of the piston rod.

**§ 327.—Quantity of Water Required for the Regulator.**—The water used in operating the regulator occasions an especial loss of useful work or diminishes the percentage of the engine, since all the water thus used is taken from the motive water. This waste then must be lessened as much as possible; therefore, not only the diameter of the counter piston $d_3$, but also the stroke of the regulator piston, should be made as small as possible. This stroke depends upon the height of the regulator piston and upon the height of the connecting pipe, and the former depends again upon the latter; on this account, the vertical dimension of the connecting pipe, which joins the regulator cylinder with the working cylinder, is to be made as small as possible, and the area obtained by increased width. This pipe is, therefore, usually made rectangular in section and of the same width as the working cylinder. Should the section of this pipe be the same as that of the supply pipe, we have

$$a\,d = \frac{\pi d_1^2}{4},$$

from which we get the height of the connecting pipe

$$a = \frac{\pi d_1^2}{4\,d}.$$

In order to close the opening by the regulator piston exactly at the middle of the stroke, this piston is made three times the height of the pipe; taking this height,

$$a_1 = 3\,a,$$

the stroke of the regulator piston is

$$s_1 = a_1 + a = 3\,a + a = 4\,a,$$

and the quantity of water used per stroke is

$$\frac{\pi d_3^2}{4} s_1 = \pi\, a\, d_3^2.$$

If the machine makes $n$ strokes per minute, the quantity of water used per second is

$$Q_1 = \frac{n\, s_1}{60} \times \frac{\pi\, d_3^2}{4} = \frac{n\, a}{60} \pi\, d_3^2,$$

and therefore the corresponding loss of work per second,

$$L_1 = \frac{n\, s_1}{60} \times \frac{\pi\, d_3^2}{4}\, h\, \gamma = \frac{s_1}{s}\left(\frac{d_3}{d}\right)^2 L.$$

This loss will be diminished if the number of strokes is decreased, or if the stroke of the driving piston $s$ is increased.

Finally, as regards the exterior or auxiliary regulator, the force required to move this is so small that it may, without great error, be neglected, or at most it may be included by estimation. The reversion of the motion effected by this apparatus will be treated in detail in the third volume.

EXAMPLE.—When the pressure engine, which was calculated in the example to § 324, has a regulator piston 9 inches diameter, and therefore a counter piston $9\sqrt{2} = 13$ inches in diameter; when, furthermore, the height of the connecting pipe is

$$a = \frac{\pi\, d_1^2}{4\, d} = \frac{9^2\, \pi}{4 \cdot 20} = \frac{81\, \pi}{80} = 3 \cdot 18 \text{ inches},$$

and therefore the height of the regulator piston

$$a_1 = 3\, a = 9 \cdot 54 \text{ inches},$$

and its stroke

$$s_1 = a_1 + a = 12 \cdot 72 \text{ inches} = 1 \cdot 06 \text{ ft.},$$

the quantity of water used by the regulator will be

$$V_1 = \frac{\pi}{4}\left(\frac{13}{12}\right)^2 \times 1\cdot 06 = 0\cdot 977 \text{ cubic ft.},$$

and consequently the corresponding loss of work per second,

$$L_1 = \frac{n}{60} \times 0\cdot 977 \times h\,\gamma = \tfrac{4}{60} \times 0\cdot 977 \times 350 \times 62\cdot 5$$

$$= 1425 \text{ ft. lbs.} = 2\cdot 4 \text{ horse-power.}$$

It would be more economical in this case to have a smaller regulator piston and a less height for the connecting pipe, for although the hydraulic resistance would be greater, the loss of work due thereto would not be so great as the saving of water used for the regulator.

§ **328.—Experimental Results.**—No exhaustive experiments have been made upon the performance of water-pressure engines. These machines are used, generally, only to raise water from mines by the aid of pumps; and the experiments that have been made bear only upon the apparatus of pumps and pressure engines as a whole. Since, however, exact observations of the pumps are also wanting, the efficiency of a water-pressure engine cannot be calculated with great certainty. On the contrary, it is very easy to arrive at an approximation to this efficiency, when it is assumed that the engines and the pumps bear a given ratio to one another; this assumption can furthermore be very well made, for these machines are quite similar in their construction and manner of working. Certainly it will not be to the advantage of the pressure engine, nor be far from the truth, to consider half the loss of work in the whole apparatus due to the pumps and half to the engine. The calculation under this condition becomes very simple. The useful work is

$$L = \frac{n}{60}(F s + F_1 s_1)\, h\, \gamma,$$

when $F_1$ denotes the cross-section and $s_1$ the stroke of the return piston; but the amount of work actually obtained is

$$\frac{n}{60} F_2 h_2 \gamma,$$

when $F_2$ denotes the cross-section of the pump piston, and $h_2$ the height which the water is raised by the pump. The loss of work is, therefore,

$$L_1 = \frac{n}{60} (F s + F_1 s_1) h \gamma - \frac{n s}{6} F_2 h_2 \gamma$$

$$= \frac{n}{60} [(F s + F_1 s_1) h - F_2 s h_2] \gamma,$$

and hence the efficiency of the water-pressure engine

$$\eta = 1 - \frac{1}{2} \frac{(F s + F_1 s_1) h - F_2 s h_2}{(F s + F_1 s_1) h} = \frac{1}{2} + \frac{F_2 s h_2}{2 (F s + F_1 s_1) h}$$

$$= \frac{1}{2} (1 + \eta_1),$$

where $\eta_1$ is the efficiency of the whole machine. Of course it is assumed that no water is lost by leakage; when the apparatus is in good condition, the leakage will be so small that it may be neglected. Among others, Jordan, the builder of the engine at Clausthal, has found that the average loss of water is ¼ per cent for the water-pressure engine, and 2¼ per cent for the pumps. To complete the experiment, the cocks in the supply and discharge pipes are to be wholly opened and the ascent of the pumps increased until the desired number of strokes is attained.

By experiments of this kind, on one of the two twin machines at Clausthal, Jordan found that, with four strokes per minute, $\eta_1$ was equal to 0·6568, and that with three strokes per minute, $\eta_1$ was equal to 0·7055; therefore, in the first instance,

$$\eta = \frac{1{\cdot}6568}{2} = 0{\cdot}8284,$$

and in the second,

$$\eta = \frac{1{\cdot}7055}{2} = 0{\cdot}8527,$$

consequently, the average is

$$\eta = \frac{1 \cdot 6811}{2} = 0 \cdot 84.$$

When it is not practicable to attain the greatest effect of a water-pressure engine by increasing the lift of the pumping machinery, it may be done by diminishing the motive-water column; this, however, is only admissible when the power reserve of the engine is not great, and, therefore, also the back-water column not very high. In such case, the diminution of the water column is brought about by the supply of water to the supply pipe, and the level of the surface in the latter is measured by a float attached to a cord. In this way, for the machine at Alte Mordgrube,

$$\eta_1 = 0 \cdot 684,$$

when the number of strokes per minute is three; and, therefore, the efficiency of the water-pressure engine only is

$$\eta = \frac{1 \cdot 684}{2} = 0 \cdot 84.$$

The most of the data concerning the working of other water-pressure engines is too inaccurate to be of value, since the observations were not made with completely open regulating cocks, and yet the exact positions of these were not measured. If we take, from the table in Vol. I., § 443, the value of the coefficient $\zeta$ corresponding to a known position of these cocks, the head $y$ consumed by this apparatus can be calculated from

$$y = \zeta \cdot \frac{v_1^2}{2g} = \zeta \cdot \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g};$$

the efficiency of the machine, therefore, follows from the equation,

$$\eta = \frac{1}{2}\left[1 + \frac{F_2 s h_2}{F s \left[h - \zeta \left(\frac{d}{d_1}\right)^4 \cdot \frac{v^2}{2g}\right] + F_1 s_1 h}\right].$$

EXAMPLE.—A water-pressure engine uses per stroke 10 cubic ft. of motive and 0·4 cubic ft. of regulating water; the fall of the same is 300 ft., the mean velocity in the supply pipe 6 ft., and the opening of the throttle valve forming the regulating apparatus is 60°. If, now, 3·5 cubic ft. of water are raised through 420 ft. per stroke, what is the efficiency of the engine? According to Vol. I., § 443, for the angle of the valve of 60° $\zeta = 118$, therefore

$$\zeta \cdot \frac{v_1^2}{2g} = 118 \times 0 \cdot 0106 = 6^2 \times 68 \text{ ft.};$$

and, consequently, we may put

$$\eta = \frac{1}{2}\left(1 + \frac{3 \cdot 5 \times 420}{10\,(300 - 68) + 0 \cdot 4 \times 300}\right) = \frac{1}{2}\left(1 + \frac{3 \cdot 5 \times 42}{232 + 12}\right)$$

$$= \frac{1}{2} \times 1 \cdot 6025 = 0 \cdot 81.$$

**§ 329.—Water-Pressure Engines Compared with Water-Wheels.**—Compared with water-wheels, we find that water-pressure engines have many advantages, although, on the contrary, in some respects, the former are to be preferred. Water-wheels have the advantage of simplicity and cheapness; and on this account, where they can be used, or with a fall of about 60 ft., an overshot-wheel, or two overshot-wheels where the fall is 100 ft., give better results than a water-pressure engine. If the fall, however, is more than the height of two of the largest wheels, a water-pressure engine is to be preferred to a system of wheels whose first cost and maintenance will, probably, be more than for a water-pressure engine. With great falls, however, horizontal water-wheels can also be used; it only remains then to explain here the relative merits of these and water-pressure engines. As regards simplicity and cheapness, these wheels have very much the advantage, since with great falls they can be very small, and consequently relatively cheap. It is quite otherwise, however, as regards the duty or the efficiency of the machines. With great heads, the highest percentage attainable with a turbine or reaction-wheel is 0·70, while water-pressure engines give a percentage of 0·80. In this respect, they are ahead of horizontal water-wheels, but must be ranked the same as overshot-wheels. Consequently, where, with a great head, it is necessary to utilize all the power, a water-pressure engine should be used; but where there is no lack of power, and it is desired to economize the cost, the turbines have the advantage.

In this connection, it should be noticed, however, that water-pressure engines give only an up and down, while turbines give a constant rotary motion, which may be changed readily into any sort of motion desired; this cannot be done so easily in the first case. For these reasons, water-pressure engines are only met with rarely, and, indeed, for the great part, are only used to raise water in mining operations.

Water-pressure engines and turbines possess in common the disadvantage that it is necessary to destroy any excess of power or the reserve power by regulating cocks or other apparatus.

REMARK.—The methods of changing the motion of a water-pressure engine into a rotary motion, by means of couplings, bevel gears, etc., will be treated of later.

**§ 330.—Chain-Wheels.**—There are yet other machines which are put in motion by the force of moving water, which cannot be classed with either water-wheels or water-pressure engines. In the following we will offer a few remarks upon these machines.

The chain-pump (Fr. roue à piston; Ger. Kolbenrad) has been again used of late, by Lamolières, as a motor (see "Technologiste," Sept. 1845, or "Polytechn. Centralblatt," Vol. VII., 1846). The main parts of this machine are a wheel *ACB*, Fig. 578, a chain *ADB* working over the same and provided with buckets *E*, *F*, *G*, etc., and a pipe *EG* through which the chain and buckets pass; the latter fitting pretty exactly the bore of the pipe. The water, entering the pipe *EG* from above at *E*, sinks therein and presses upon the buckets *F*, *G*, so that these descend with the water, and, in so doing, put in motion the wheel *A B*, to which a load may have been attached. Lamolières' chain of buckets consists of two chains with 10 or 15

FIG. 578.

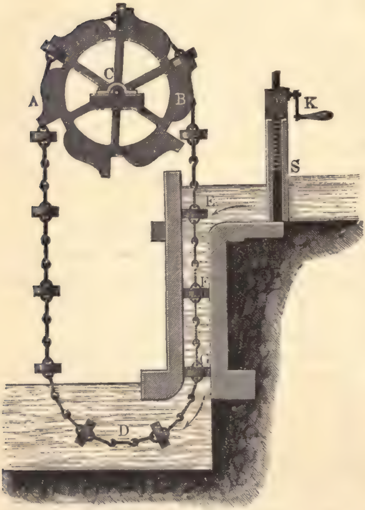
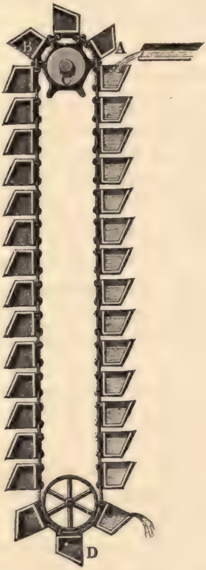


FIG. 579.

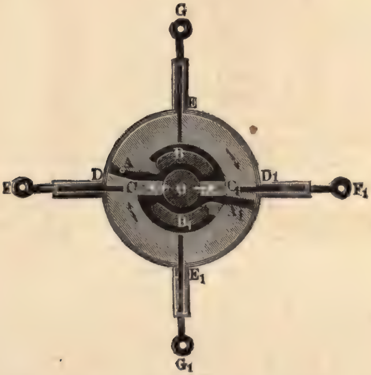


buckets attached, the buckets being furnished with a leather packing; these are also elliptical in form and eight times as wide as long. The wheel consists of two plates with six cuts to receive the buckets. With a head of 6·6 ft., a bucket area of 0·236 square ft., a water supply of 1·1 cubic ft., and 36 to 39 revolutions per minute, the efficiency was 0·71 to 0·72.

A similar apparatus called a chain bucket pump (Fr. noria, chapelet; Ger. Eimerkette) is shown in Fig. 579. The pipe is here done away with by making the buckets to hold the water, which, flowing into them at *A*, causes them to descend by its weight merely, and thus sets the chain and the wheel $A\,C\,B$ in motion. The water is emptied out, of course, at the lowest position of the buckets, and they ascend on the other side empty. This machine should give a large percentage, since it utilizes nearly the whole fall, but it belongs to the more imperfect machines, because of its many moving parts, which are continually being broken and needing repairs.

REMARK 1.—Finally, the so-called rotary pumps, rotary steam-engines, etc., serve to utilize the force of water. Fig. 580 is a section of one of the best of this kind of machines. The author has named this a *water-pressure wheel*, and given a description and theory of the same in the "Polytechn. Centralblatt," year 1840, No. 9. $B\,O\,B_1$ is a strong and exactly-fitted journal, and $A, A_1$ are two arms firmly attached thereto and serving as pistons or buckets. These buckets are enclosed in a strong, close-fitting box $D\,E\,D_1\,E_1$, furnished with four slide valves $D\,F$, $D_1\,F_1$, $E\,G$, and $E_1\,G_1$, which are moved out and in by the machine itself, and perform the function of a regulator for the same. The journal is bored thrice, in the direction of its length, and each bore has a side bore inside the box. The motive water flows through the inside bore $O$, enters, through the side bores $C$ and $C_1$, the otherwise tightly-enclosed space between the journal and the box, presses thus upon the buckets $A, A_1$, and causes the journal to revolve. In order that the slide valves may not interfere to destroy this motion, they must be always drawn back before the buckets reach them; but, in order that then no water

FIG. 580.



pressure shall be exerted on the opposite side of the buckets, they must again slide in and partition off the spaces $ABE$ and $A_1B_1E_1$, so that these shall communicate only with the bores $B$ and $B_1$, through which the water is drawn off after its force is expended.

REMARK 2.—The machine, which has been named by its inventor, L. G. Girard, "Moteur pompe," should also be classed with the bucket machines. (See Delaunay's "Cours de Mécanique," II. Partie.)

FINAL REMARK.—We now give the literature and some of the statistics of water-pressure engines. Belidor describes a water-pressure engine with a horizontal working cylinder in his "Architecture hydraulique," and we also learn from him that Denisard and De la Duaille constructed some such machine as early as 1731. This machine, however, had a head of water of only 9 ft., and raised only about one twentieth of the motive water 32 ft. higher, by a piston. As it appears, Winterschmidt, and soon afterward Höll, first adapted the water-pressure engine to raising water from mines, or, at least, improved it for this purpose. In Busse's "Betrachtung der Winterschmidt- und Höll'schen Wassersäulenmaschine," etc., Freiberg, 1804, more can be learned concerning this invention. A description and explanation of Winterschmidt's machine is to be found in Calvör's "Historisch-chronologischer Nachricht, etc., des Maschinenwesens, etc., auf dem Oberharze," Brunswick, 1763. Delius, in his "Anleitung zur Bergbaukunst," Vienna, 1773, and Poda, "Beschreibung der bei dem Bergbau zu Schemnitz errichteten Maschinen," Prague, 1771, describe Höll's machine.

The engines now in operation are found in Bavaria, Saxony, Hungary, Kärnthen, Brittany, in the Harz, etc. We will treat of the Bavarian machines later, when we come to water-raising; so far, however, there are not at hand complete descriptions of these machines, although much is to be learned about them in Langsdorf's "Maschinenkunde," in Hachette's "Traité élémentaire des Machines," and in Flachat's "Traité élémentaire de Mécanique." A general description of the machine erected by Brendel in Saxony is to be found in Gerstner's "Mechanik," where the machine at Bleiberg in Kärnthen is also quite fully described. Schitko, in his papers on mining in Karsten's "Archiv für Mineralogie," etc., has described the machines in the Schemnitz mining district, while Jordan has described the two machines at Clausthal in the same publication; both, however, have been published separately by Reimer at Berlin. Junker has described in detail the machine at the Huelgoat mine, in Brittany, in the eighth volume of the "Annales des Mines." A description of this machine may also be obtained separately, entitled "Mémoire sur les machines à colonne d'eau de la mine d'Huelgoat," Paris, 1835. But little known is the small water-pressure engine of Althaus at the "*Pfingstwiese*" mines at Ems, also the one in the coal mines at Oberkirchen in Kurhessen, erected by Henschel, and the machines at Sangershausen and at Gerbstädt in Mansfeldischen. All the last-named are also peculiarly constructed machines. The English water-pressure engine (*Darlington's*), described in § 312, is represented and described in Vol. II. of the English translation of this work. The water-pressure engine at Lautenthal in the Harz mountains is described by Jugler in the "Notizblatte des Hannoverschen Architecten- und Ingenieur-Vereins," Vol. III., and a separate copy of the same may be obtained at bookstores. The publication, "Records of Mining and Metallurgy, or Facts and Memoranda for the Use of the Mine Agent and Smelter," by A. Philipps and J. Darlington, London, 1857, contains notices of some English machines. A short discussion of English engines is also found in J. Glynn's "Rudimentary Treatise on the Power of Water," London, 1853, by J. Weale. The water-pressure engine of Lewis is furnished with two air vessels (see "Polytechn. Centralblatt," 1863, No. 17). Many references to the modern machines, used in the Austrian mines, are contained in the publication "Erfahrungen im berg- und hüttenmännischen Maschinenwesen," etc., of Peter Rittinger, in the Vols. for 1854, 1856, 1858, 1860, and 1862. The most peculiar of these machines is the one at the Adelbert shaft at Przibram, described in the last volume. It has a slide-valve regulator, a discharge piston, etc.

The peculiarly constructed water-pressure engine which Bornemann has erected at Schneeberg is described in Vol. II. of "Der Civilingenieur." Water-pressure lifting machines and cranes, as also water-pressure man engines and winches, will be treated of in the third volume.

# CHAPTER VII.

## WIND-WHEELS.

§ **331.—Wind-Wheels.**—The atmospheric air is capable of performing mechanical work, either by its motion or its expansive power. Most commonly it is utilized as wind by means of wheels which communicate to machinery a portion of the force exerted upon them by the air. Such wheels are called wind-wheels (Fr. roues à vent; Ger. Windräder), and the buildings which support them with all their machinery and appurtenances are called wind-mills. A wind-wheel is designed for the reception of wind-power, in a manner similar to that of a water-wheel for the reception of water-power; but the two differ essentially in this, that while the one is wholly immersed in a sea of air, the other encounters a wholly or at least partially circumscribed current of water. A common paddle-wheel presented to the wind cannot revolve, because the force is exerted equally upon the opposite paddles. In order to employ such a wheel, it would be necessary that only the paddles upon one side of it should be exposed to the wind, those upon the opposite side being protected in some manner from it, as by a screen interposed upon the windward side. The screen may evidently be dispensed with if the paddles are pivoted upon their axes in such a manner that, when upon one side of the wheel, they present a broad face to the wind, while, in passing to the other side, they turn upon their pivots so as to feather or present their edges to the wind until they again reach their former positions. In order to avoid directing such a wheel towards the wind, it is placed upon a vertical axis revolving thus in a horizontal plane, whence they are called horizontal wind-wheels (Fr. roues horizontales à vent; Ger. horizontale Windräder).

The so-called sail-wheel possesses advantages over the paddle-wheel. The axis of the sail-wheel is directed to the wind or water, and its arms, which are few in number, support broad vanes or sails which receive the wind at an oblique angle, and, since the direction of the wind is nearly horizontal, it is necessary to place the axis also horizontally, and thus the plane of revolution of the wheel is vertical, and the wheel is accordingly called a vertical wind-wheel.

REMARK.—Horizontal wind-wheels have also been constructed with concave paddles. These are called Panemore wheels. Since the force of the wind is greater against a concave than against a convex surface, and the paddles of these wheels present a concave surface to the wind upon one side of the wheel and a convex surface upon the other, they revolve from this force alone, though with moderate power.

§ **332.—Sail-Wheels.**—Under like conditions of size, weight, etc., sail-wheels are capable of performing a greater amount of work than paddle-wheels. In the case of the paddle-wheel, the propelling action of the wind is exerted only upon one side, and with a force equal to that upon a surface at right angles to the wind, and having the dimensions of the projection of the surface of the exposed paddles in the direction of the wind. The sail-wheel, however, presents the entire surface of each sail constantly to the wind, and although at perhaps a disadvantage with regard to the oblique position of the sails, it is nevertheless able with a like surface to receive and transmit to the machinery a greater amount of power than the paddle-wheel. Much experience has shown that the sail-wheel, under otherwise like conditions, performs at least four times as much work as the paddle-wheel, and even if this were not the case, the greater ease and certainty with which it may be arranged, and especially the slight friction upon its axis, would have secured it a place in practical mechanics.

We shall, therefore, in this chapter speak only of windmills with sail-wheels. The more exact description of the sail-wheel is as follows: It consists, in the first place, of a strong axle called the wind shaft (Fr. l'arbre du volant; Ger. Flügelwelle), which may be of wood, but is preferably of cast-iron. The shaft is inclined to the horizon at an angle of from five to fifteen degrees, in order that the wings may revolve at a proper distance from the building, and that the entire wheel may rest securely in its position. Upon the shaft are the head, the neck, the spur-wheel, and the pivot. The head is the part to which the arms are attached. The neck or journal is the part immediately behind the head where the entire wheel is supported. The spur-wheel serves for the transmission of the motion, or for the union of the wheel with the working machinery, and, finally, the pivot at the rear end of the shaft is necessary for the complete support of the wheel. The loss of work caused by the friction of the shaft in its bed is, on account of its weight, and especially on account of the rapidity of its revolution, very considerable; it is therefore necessary to employ every possible means to lessen the friction. On this account, an iron shaft is preferable to a wooden one, since it may have a considerably smaller neck. While the thickness of the neck of the wooden shaft is from 1½ to 2 ft., that of a cast-iron one is only from ½ to ¾ of a foot. Besides this, the friction upon a wooden shaft is greater than upon an iron one when the size is the same, since, as a rule, the neck of a wooden shaft is not covered with an iron mantle, but only with a series of iron staves, which cause a considerable wear upon the bed.

REMARK.—Reference may be made to the excellent descriptions of Beatson's horizontal and other windmills by the English writers Nicholson, Gregory, etc.; also to the article upon windmills in Rühlmann's "Allgemeiner Maschinenlehre," Vol. I.

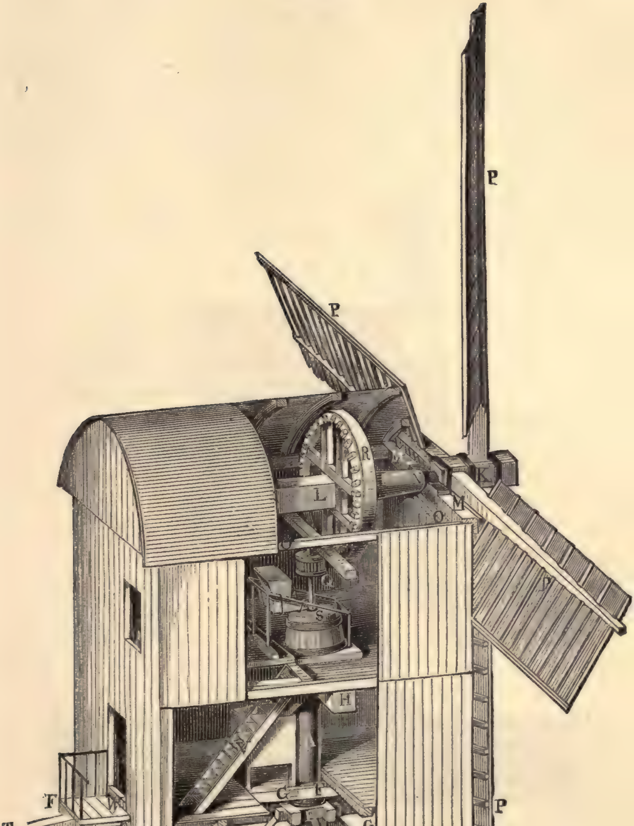
§ **333.—The Windsails.**—The sails consist of the arms, cross-bars, and clothing. The arms radiate from the head of the shaft and are about 30 ft. long, each bearing a sail.

There are commonly four of these arms, as there are generally four sails—sometimes, however, five or six. Near the shaft, these arms are 1 ft. thick and 9 inches wide. At the outer end they are only 6 inches thick and 4½ inches wide. The method of attachment differs very much. If the shaft is of wood, the two arms are passed at right angles through its head, thus forming the four sail arms. The arms may also be fastened by screws to a rosette which constitutes the head of the shaft, like the arms of a water-wheel, this method being employed especially when the shaft is of iron. The bars are of wood and pass at right angles through the arm, which for this purpose is morticed through at distances of 1¼ to 1½ ft. According as the sails are to have a more rectangular or a more trapezoidal form, all the bars are of equal, or, towards the shaft, of decreasing length. The innermost bar is placed $\frac{1}{7}$ to $\frac{1}{6}$ of the length of the arm from the middle of the shaft, and its length is about equal to this distance. The outermost bar is only about $\frac{1}{5}$ or $\frac{1}{4}$ the length of the arm. In most windmills, the arms do not pass through the middle of the sails, but to one side of the middle, so that the part which is directed towards the wind constitutes from $\frac{1}{5}$ to $\frac{2}{5}$ of the entire breadth of the sail. Therefore, the bars project much less upon one side of the arm than upon the other. The narrow part of the sail is usually covered with the so-called wind board, while the broad part is covered with the wind slats or with a covering of sail-cloth.

The sails are made plane, warped, or concave. The warped sails, made slightly concave, are the most serviceable, as will be more exactly explained hereafter. In the case of the plane sails, all the bars have the same angle of inclination, 12° to 18° to the plane of revolution. In the warped sails, the inner bars deviate about 24°, and the outer bars 6° from this plane. The intermediate bars form angles of inclination between those of the inner and outer ones. In order to give the sails a hollow form, bent arms and bars must be used. Although the theory of the impulse of the wind would indicate a gain of power by this form, it is seldom employed, on account of the difficulty of its application. For the complete support of the sail-covering, the outer ends of the bar are united with one another by wooden strips called uplongs; and sometimes, especially if the covering consists of linen, laths are placed between the bars, so that the entire frame-work consists of divisions, of about 2 sq. ft. each. The wooden covering consists of four wind slats, which are composed of thin boards, and are fastened by bolts to the sail frame-work. The sail-cloth covering is attached to the frame-work by cords and hooks.

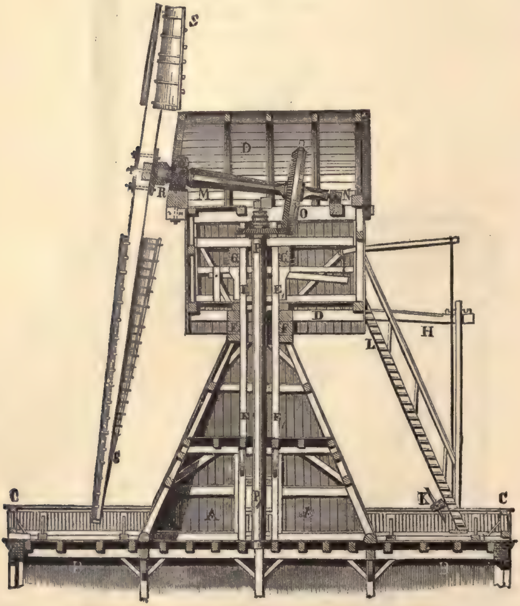
§ **334.—Post-Mills.**—As the direction of the wind is changeable, and the shaft must be directed towards it, the support of the wheel must be movable so that it can revolve about a vertical axis. According to the manner in which this revolution is effected, windmills are divided into two classes: First, the German or post mill (Fr. moulin ordinaire; Ger. Bockmühle), and, second, the tower mill or smock mill (Fr. moulin hollandais; Ger. holländische or Thurmmühle). In the post-mill, the entire building, together with the wheel, revolves about a fixed column or post (Fr. poteau; Ger. Ständer or Hausbaum). In the tower-mill, only the head or cap (Fr. le toit, la calotte; Ger. Haube), with the shaft which it contains, can revolve. Fig. 581 shows a general view of the post-mill, in which $A\ A$ is the upright standard, supported by the cross-timbers $B\ B$ and $B_1\ B_1$, and by the braces $C$ and $D$, all these parts constituting the so-called post. On the head of the post is firmly placed the saddle $E$ composed of four pieces of wood fastened together. The mill house is supported by the two cross-beams $F\ F$ and by two of the six cross-lying floor timbers $G\ G$. It rests also upon the strong cross-timber $H$, which turns, by means of a pivot, upon the head of the post. The neck $N$ of the axle $K\ L$ turns in a metal or stone (basalt) plumber block, which rests upon the strong axle timber $M\ M$, the latter being supported by the roof frame-work $O\ O$.

Fig. 581.



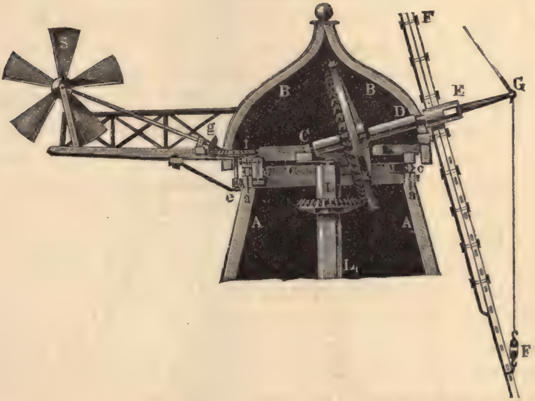
§ **335.—Tower-Mills.**—There are two kinds of tower-mills. In the one, only the extreme upper portion revolves, together with the axis which it contains; in the other, a much larger portion is revolvable. The motion of the sail-wheel is transmitted by means of a pair of cog-wheels to the king-post, which is a strong, upright axis extending through

FIG. 582.



the entire building. In order that the articulation of the cog-wheels may not be changed or displaced by the different positions of the sail-wheel, it is necessary that the axis of the king-post should coincide with the axis of the movable part of the building. In Fig. 582 is shown a section of a tower-mill of the second kind, which is intermediate between a post-mill and a tower-mill of the first kind. *A A* is the fixed tower upon which rests the mill building containing the working machinery, and is surrounded by the gallery *C C*; *D D* is the movable cap which is directly supported by the wooden ring *F F*, and also by the wooden ring *G G* which rests upon the columns *E E* and $E_1 E_1$, which correspond to the standard in the post-mill and around which the cap revolves. The revolution is effected by the capstan *K* which is attached to the stairs *K L*, which are united with the movable part of the building, and especially with the lever *H*.

FIG. 583.

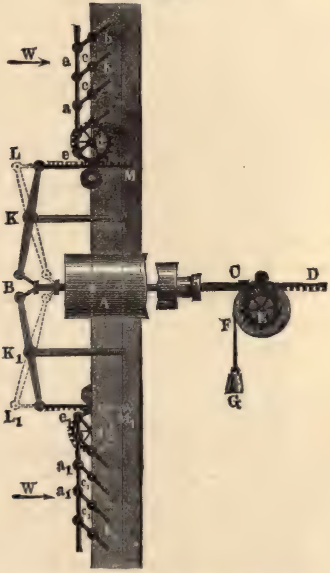


The wind shaft *M N* is of cast-iron, and rests at *M* and *N* in a cast-iron bed, which is lined with common metal. *O* and *P* are iron cog-wheels by which the motion is transferred from the wind shaft to the king-post $P P_1$. The sails *R S*, *R S* . . . are warped and are fastened with screws and an iron cross to the socket *R*, which has upon one side a second cross, and upon the other a cavity which fits over the head of the shaft, to which it is attached by means of wedges.

The upper part of a tower mill of the first kind is represented in Fig. 583. *A A* is the upper part of the fixed tower which is built of wood or stone in the form of a pyramid; $B B$ is the movable cap; $C D E$ is the wind shaft, and $E F$ is an arm made in two parallel sections and protected from bending or breaking under the force of the wind by means of the ropes $F G$ passing over the head of the spindle $E G$, which rests upon the head of the shaft. $K$ and $L$ are two cog-wheels employed to transfer the power from the wind shaft to the vertical axis $L L_1$. The shaft is directed to the wind by means of a lever or by a winch with wheel and driver. This may also be effected by means of a large vane, the plane of which corresponds to that of the axis, or, still better, by a windmill $S$, as shown in the figure. In order that the cap may revolve easily, it is made to rest upon rollers $c\ c\ c$ . . . united by two bands, and acting between two circular rings or tracks, one of which is at the summit and outer margin of the tower, and the other in a corresponding position upon the underside of the cap. Finally, the cap is secured against being blown off by the wind by means of an interior band screwed on at $b$, the friction of which may also be relieved by rollers playing upon the inner surface of $a a$. Where a windmill as $S$ is employed, the external surface of the ring $a a$ is surrounded by a toothed band with which a driver or small cog-wheel $e$ articulates, the latter articulating in turn with the small cog-wheels $f$ and $g$, which are acted upon by the windmill as shown in the figure. In this manner, the cap is made to revolve whenever the direction of the wind varies from the plane of revolution of the windmill $S$.

§ **336.—Regulation of Power.**—The wind is variable not only in its direction, but also in its rapidity or intensity. If, therefore, the resistance occasioned by the machinery were constant, the velocity of revolution would vary with the force of the wind unless regulated by some mechanical means. Such an appliance could evidently act only to moderate, not to increase, the velocity of the wheel. One such means consists of a break or friction ring, which surrounds the upper half of a cog-wheel resting upon the axis, and which is applied by being pressed upon this wheel when the motion is to be moderated or entirely checked. This brake will be more fully described hereafter. Another means of regulating the motion of the wheel consists in varying the covered area of the sails, the effective power of the wheel evidently varying directly with this area. When the covering is of sail-cloth, this method of regulation consists in furling or expanding this covering as may be required. Where the sails are covered with boarding, the same purpose is accomplished by drawing this back or opening it.

FIG. 584.



There are also self-regulating wind-wheels which by automatic action increase their exposed surface as the wind decreases, and lessen it as the wind increases. The best wind-wheel of this kind is that of Cubit, a section of a part of which is shown in Fig. 584. *A* is a hollow wind shaft, through which passes the metallic rod $B\,C$. $C\,D$ is a ratchet so united with $B\,C$ at $C$ that it may move in the direction of the axis $B\,C$, but not take part in the revolution about it. The ratchet articulates with the cog-wheel $E$, which rests upon a common axis with the pulley $F$, around which lies a cord made tense by the weight $G$. The covering of the sails consists only of thin slats of wood or metallic plates $b\,c$, $b_1\,c_1$, etc., which may be revolved about the axes $c$, $c_1$, etc., by means of the arms $a\,c$, $a_1\,c_1$, etc. These arms are united with each other by means of rods $a\,e$, $a_1\,e_1$, etc., and at the same time with the small cog-wheels $d$, $d_1$ by means of the arms $d\,e$, $d_1\,e_1$, so that by the revolution of the cog-wheel the slats may be opened, closed, or placed in any desired position. Finally, the levers $B\,L$, $B\,L_1$ are employed, which may be turned about the axes $K$, $K_1$, and which are united upon one side with $B\,C$, and upon the other with the ratchets $L\,M$, $L_1\,M_1$, articulating with the small wheels $d$, $d_1$. From the drawing, it is easy to see that the action of the wind $W$ tends to open the slats, while the weight $G$, by means of the ratchet $B\,C$ and the levers $B\,L$, $B\,L_1$, etc., tends to close them, and that in this manner the force of the wind against the slats is held in equilibrium by the weight $G$.

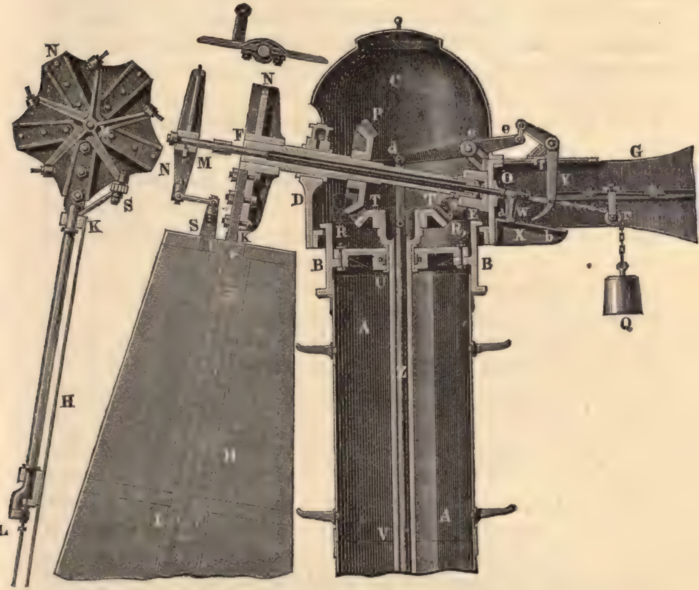
If, now, the velocity of the wind varies, its effective power will nevertheless remain constant in consequence of the variation in the position of the slats, and hence of the area of the exposed surface.

REMARK.—A convenient method of adjusting the covering, when of sailcloth, was devised by Mr. Bywater. In this, the covering is stretched between two rollers which are acted upon by cog-wheels in such a manner as to roll up and unroll the sail with the varying velocity of the wind. This apparatus is described in detail in Barlow's "Treatise on Manufactures and Machinery." A new method of constructing wind-wheels is also described in "Der Ingenieur," Vol. II.

§ **337.**—A form of wind-wheel peculiar in various respects has been constructed by the Engineer Kirchwäger, and is in use at several water stations of the Hanover Railroad. [See a treatise by Chief-Engineer Prüsmann in Vol. 8 (1862) of the "Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover."]

The peculiar construction of this wind-wheel is shown in the vertical section of it, Fig. 585. The sheet-iron tower

FIG. 585.



*A A*, about $1\frac{3}{4}$ ft. wide, rises out of the roof of a brick structure and terminates in a cast-iron head *B B*, upon which rests the cap *C* upon four rollers *R*, $R_1$. The cap supports the bearings *D* and *E* of the axis *E F*, and passes with its cylindrical base over the upper part of the head *B B*, by which means it is secured against being blown off by the wind. The vane *G* (only partly visible) serves, by revolving the cap, to keep the wind-wheel in the face of the wind. The wheel consists of five plates *K H*, which turn about radial arms as *K L*. These arms are screwed to a cast-iron rosette *N N*, which rests upon the head of the wind-shaft. In order to regulate the motion of the wheel or to give the wings a position with regard to the wind corresponding to the required power, the following mechanism is employed: Through the hollow axis of the wind-shaft passes a movable steel rod *M O*, bearing at one end a star from which project five arms which articulate with five arms upon the wings by means of short rods and joints, in such a manner that, when the star is drawn in, the wings are laid flat, and, when it is pushed out, they stand edgewise. The star is drawn inward by the action of the weight *Q*, which is suspended from a chain passing over a guiding pulley *r*, and connected with the ring *W* which encircles the shaft, and is prevented from revolving upon it by means of the arm *a* which projects from it, and is only movable along the fixed track *b*.

The inclination of the wing to the wind is regulated by means of the angle lever *Y*, the long arm of which is connected with the movable rod *M O*. This lever is united with the vertical rod *Z* by means of joints and the lever *d c e*, and is pressed against the end of the rod *M O* by the rod *Z*. When the force of the wind is too great, it is only necessary that the rod *Z* should be drawn down, and, when this power is insufficient, that it should be raised. At the water stations mentioned, where this wind-wheel is employed to drive pump works, the raising and lowering of the rod *Z* is effected by means of floats, which are connected with it by means of lever mechanism. In Fig. 585 there are shown in addition only the two cog-wheels *P T* and *T T*, by means of which the wind-shaft revolves the hollow king-post *U V*, which drives another system of wheel work (not shown in the figure), by which the pump works are operated.

§ **338.—Direction of the Wind.**—The wind, the origin of which is evidently due to an inequality in the expansive power or density of the air (see the formulæ in Vol. I.,

§ 458), varies with regard to direction and intensity or rapidity. As to direction, eight courses are designated—viz.: N, NE, E, SE, S, SW, W, NW—*i. e.*, North, North-east, East, South-east, South, South-west, West, and North-west—being named according to the direction from which they blow. For the purpose of a more exact designation of the direction of the wind, the horizon is divided into sixteen equal parts, or, according to the miner, into twenty-four hours; that of degrees, however, is the more exact. In the course of a year, all these directions occur during a greater or less period. For middle and southern Germany, the mean duration of these winds is, according to Coffin, as follows:

| N. | NNE. | NE. | ENE. | E. | ESE. | SE. | SSE. | S. | SSW. |
|---|---|---|---|---|---|---|---|---|---|
| 23·5 | 2·9 | 35·1 | 3·1 | 41·7 | 3·9 | 30·1 | 2·5 | 23·9 | 3·0 |

| SW. | WSW. | W. | WNW. | NW. | NNW. | CALM. |
|---|---|---|---|---|---|---|
| 63·3 | 3·2 | 77·1 | 4·2 | 42·8 | 0·4 | 0·9 |

The figures denote the number of days in the year.

According to Kämtz, the wind blows in different countries as follows during 1000 days:

| COUNTRY. | N. | NE. | E. | SE. | S. | SW. | W. | NW. |
|---|---|---|---|---|---|---|---|---|
| Germany........... | 84 | 98 | 119 | 87 | 97 | 186 | 198 | 131 |
| England........... | 82 | 111 | 99 | 81 | 111 | 225 | 171 | 120 |
| France............. | 126 | 140 | 84 | 76 | 117 | 192 | 155 | 110 |

It appears from this that in the three countries named the prevailing winds are from the south-west. The changes of direction occur mostly in the order S., SW., W., etc., rarely in the opposite order, S., SE., E., etc. The latter occur, at least, only in slight reactionary forms.

The direction of the wind is determined by the so-called weathervane (Fr. girouette, flouette: Ger. Wind or Wetterfahne). This very simple instrument consists of a metallic plate, which turns about a vertical axis, and is kept parallel with the direction of the wind by the action of the wind itself upon it. In order that it may easily revolve, the friction is reduced by balancing it upon the axis by means of a weight upon the other side, in such a manner that the centre of gravity of the whole may be at the point of support. The weathercock (Fr. coqs à vent; Ger. Wetterhähne) is constructed upon this principle.

§ **339.—Velocity of the Wind.**—The velocity of the wind is, to the miller, a matter of much greater consequence than its direction, since upon it depends the amount of work that he is able to perform by means of it. The following winds are designated with reference to velocity:

Scarcely appreciable wind with a velocity of 1½ ft. per second.

Very feeble wind with a velocity of 3 ft. per second.

Feeble wind (Fr. vent faible; Ger. Schwacher Wind), 6 ft. per second.

Brisk gale (Fr. vent frais; Ger. Lebhafter Wind), 18 ft. per second.

Favorable wind for windmills, 22 ft. per second.

Very brisk wind (Fr. grand frais; Ger. Sehr lebhafter Wind), 30 ft. per second.

High wind (Fr. vent très fort; Ger. Starker Wind), 45 ft. per second.

Very high wind (Fr. vent impétieux; Ger. Sehr starker Wind), 60 ft. per second.

A storm (Fr. tempête; Ger. Sturm) is a violent wind having a velocity of from 70 to 90 ft. A hurricane (Fr. ouragan; Ger. Orkan) is a wind moving at the rate of 100 ft. and over. A wind moving at the rate of 10 ft. is in general not sufficient to move a wind-wheel with power attached, and when it exceeds 35 ft. its power cannot be advantageously applied, since it drives the wheel with too great rapidity. Storms or hurricanes are in the highest degree dangerous to windmills, often blowing them down or dismounting their caps.

In order to determine the rapidity of the wind, anemometers or wind-gauges are employed (Fr. anémomètres; Ger. Windmesser). Although many such instruments have heretofore been used or devised, very few of them are well adapted to the purpose. The most of these instruments are very similar to the hydrometer—indeed, many hydrometers may without change be used as anemometers. The velocity of the wind may be directly determined by means of light bodies carried by it, such as feathers, soap-bubbles, smoke, small air-balls, etc. Since the motion of the wind is, as a rule, not only progressive but circular or whirling, this method is, at least with regard to high velocities, often insufficient. The best object for this purpose is a large air-ball, the interior of which has about the same density as the wind.

Anemometers proper may, like hydrometers, be divided into three classes; the rapidity of the wind is either indicated by a wheel, or by the height of a fluid column which balances the force of the wind, or by the force exerted by the wind upon a level surface directly presented to it. Of these appliances, the most important will be here mentioned.

REMARK.—Anemometers are fully treated by Hülsse in the first volume of the "Allgemeinen Maschinencyclopädie." With regard to the wind, reference may also be made to Rämtz's "Meteorologie" and Gehler's "Physik. Wörterbuch," Volume X., also to the "Lehrbuch der Meteorologie," by E. E. Schmidt, Leipsic, 1860.

§ **340.**—**Anemometers.**—Woltmann's wheel (Vol. I., § 490) may be employed to determine the velocity both of wind and water. The axis of this wheel revolves upon a second axis placed vertically, and is directed to the wind by means of a vane acting also upon the vertical axis. The number of revolutions is then observed, and the velocity of the wind is determined by means of the formula

$$v = v_0 + \alpha u,$$

where $v_0$ is the velocity of the wind at which the wheel begins to stop, and $a$ is the experimental ratio

$$\frac{v - v_0}{u}.$$

If the force of the wind were the same as that of water, and if both increased exactly in proportion to the square of the velocity, then would

$$\alpha = \frac{v - v_0}{u}$$

for both wind and water. But since this is only approximately correct, the coefficients $\alpha$ for the rapidity of wind and water are also only approximately the same. The initial velocity $v_0$ is, in the case of the wind, about $\sqrt{800} = 28 \cdot 3$ times as great as that of water, since the density of the water is about 800 times as great as that of the air; and, therefore, a column of air must, in order to replace a column of water, be 800 times as high, and must have to produce the same impact 28·3 times the velocity of the water. This high value of the constant $v_0$ renders it necessary to make the wheel of the anemometer as light as possible. According to Combes, it should be covered with gold-leaf, and, for the avoidance of friction, it is especially important that it should have a fine steel axis resting upon jewelled bearings.

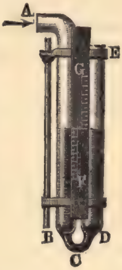
The constants $v_0$ and $\alpha$ are usually determined by the motion or revolution of the instrument in still air, but this method is not exact, because the force of a fluid in motion is not exactly the same as the resistance of the same fluid at rest (see Vol. I., § 511). It is evidently better to determine these constants by observation of the air in motion, as indicated by light floating bodies. For this purpose, a blowing apparatus can be made use of, by placing the instrument within a wide tube through which the blast of air is directed. The calculation of the constants from a number of observed values of $v$ and $u$ are to be made as stated in Vol. I., § 491.

§ **341.**—Pitot's tube may also be very conveniently used as an anemometer, and is then commonly called Lind's anemometer. This instrument is shown in Fig 586. *A B* and *D E* are two upright glass tubes, filled to the height of about $\frac{1}{2}$ inch with water; *B C D* is a bent tube, about $\frac{1}{10}$ inch in diameter, which connects them, and *F G* is a scale upon which the height of the water is reckoned. If, now, the mouth-piece *A* is presented to the wind, the force of the wind depresses the water column in *A B* and correspondingly elevates that in *D E*, the difference of level *h* being read upon the scale, and from it the velocity of the wind is calculated, the formula being

$$v = v_0 + \alpha \sqrt{h},$$

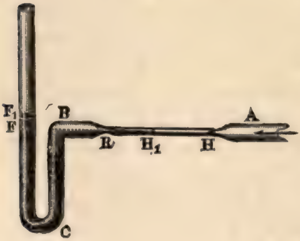
in which $v_0$ and $\alpha$ are coefficients obtained by experiment.

FIG. 586.



This instrument has a very limited application, since the slight elevations of the water column caused by light winds cannot be read with accuracy; for example, a velocity of 20 ft. is indicated upon the scale by a movement of only $\frac{1}{10}$ of an inch. In order to remove this objection and to adapt the instrument to the indications of light winds, the following improvements have been made by Robison and Wollaston. In Robison's anemometer, a narrow horizontal tube *H R*, Fig. 587, is adjusted between the mouth-piece *A* and the perpendicular leg *B C* of the bent tube. Preparatory to use, both legs of the instrument are filled with water to the common level of *F* in the one tube and of *H R* in the other, the water also penetrating the tube *H R* its whole length. The mouth-piece *A* being then directed to the wind, the water in the tube *H R* is driven back into the

FIG. 587.



main tube, where it rises to an extent sufficient to balance the force of the wind exerted through $A$. Let $d$ and $d_1$ represent the breadth and $h$ and $h_1$ the height of the water columns $F F_1$ and $H H_1$. We then have

$$\frac{\pi d^2}{4} h = \frac{\pi d_1^2}{4} h_1,$$

and therefore

$$h = \left(\frac{d_1}{d}\right)^2 h_1,$$

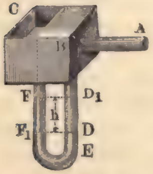
and also

$$h_1 = \left(\frac{d}{d_1}\right)^2 h.$$

Here $h_1$ is always greater than $h$ in the proportion $\left(\frac{d}{d_1}\right)^2$, and can therefore be observed with more accuracy than $h$. If, for example, $\frac{d}{d_1} = 5$, the elevation $F F_1$ is indicated twenty-fivefold in the pipe $H R$.

In the differential anemometer of Wollaston, the velocity of the wind is indicated with still greater exactness. This instrument consists of two vessels $B$ and $C$, Fig. 588, connected underneath by the bent tube $D E F$. One of these vessels is closed at the top, and has a side mouth-piece $A$ which is directed to the wind. The instrument is filled with water and oil. The water fills both legs of the tube about half way; the oil fills the remainder of the tube and a portion of the vessels above. The force of the wind causes the water to rise in one leg of the tube higher than in the other, and this force is balanced by the difference between the pressure upon the water column $F F_1$ and the oil column $D D_1$. If the common height of these fluid columns be indicated by $h$, and the specific gravity of the oil by $\varepsilon$, we have in the last formula, instead of $h$, $h\,(1 - \varepsilon)$, and therefore

FIG. 588.



$$v = v_0 + \alpha \sqrt{(1 - \varepsilon)\, h}.$$

If, for example, linseed-oil is used, the specific gravity of which is 0·94, then

$$v = v_0 + \alpha \sqrt{(1 - 0{\cdot}94)\,h} = v_0 + \alpha \sqrt{0{\cdot}06\,.\,h}$$
$$= v_0 + 0{\cdot}245\ \alpha \sqrt{h}.$$

Here $h$ is $\frac{100}{6} = 16\frac{2}{3}$ times as great as in the case of water alone. By mixing the water with alcohol, its density is brought nearer to that of oil, and therefore $1 - \varepsilon$ is still further reduced—*i. e.*, the difference of water level to be read is increased, and with it the exactness of the instrument.

§ **342.**—Several anemometers have been proposed, which are similar to the stream-quadrant, and constructed upon the same principle, but in which the balls are replaced by discs. A hollow plate ball is, however, better than a disc, since the force of the wind against the ball remains the same with every inclination of the rod to which it is attached, while, in the case of the disc, it changes with the inclination of the rod. In the case of the ball, the formula

$$v = \psi \sqrt{tan.\ \beta}$$

suffices (when $\beta$ indicates the deviation of the rod from the vertical), while with the disc a complicated expression is necessary to denote the velocity.

Finally, it has been sought to measure the velocity of the wind by the direct force which it exerts upon a plane surface presented to it at right angles, and for this purpose anemometers have been employed more or less similar to the hydrometers illustrated and described in Vol. I., § 494. If the law of the impact of the wind were exactly known, the velocity could be exactly determined by such an anemometer. This, however, is not the case. The formulæ in Vol. I., § 510, and the coefficients given in § 512 are only approximately correct.

Employing these, however, for the present, the force of the wind is

$$P = \zeta \cdot \frac{v^2}{2g} F \gamma = 1 \cdot 86 \cdot \frac{v^2}{2g} F \gamma,$$

or, since $\frac{1}{2g} = 0 \cdot 0155$,

$$P = 0 \cdot 028830\, v^2\, F \gamma,$$

or, if we take the density of the wind, $\gamma = \frac{62 \cdot 5}{800} = 0 \cdot 078125$ lbs.,

$$P = 0 \cdot 002252\, v^2\, F;$$

therefore, if the area of the surface is one square foot, the pressure of the wind is

$$P = 0 \cdot 002252\, v^2 \text{ lbs.}$$

Inversely,

$$v = \sqrt{\frac{P}{0 \cdot 002252}} = 21 \cdot 07\, \sqrt{P} \text{ ft.}$$

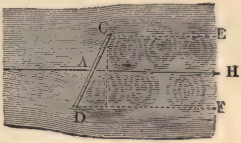
In accordance with this formula, the following table has been computed:

| For the velocities $v =$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 ft. |
|---|---|---|---|---|---|---|---|---|---|
| The impulsive force of the wind per square ft. = | 0·2252 | 0·5067 | 0·9008 | 1·4075 | 2·0268 | 2·7587 | 3·6032 | 4·5603 | 5·6300 |

By multiplying of these factors by the area of an exposed surface, the force of the wind may be calculated.

§ **343.—The Force of the Wind.**—We have now more exactly to study the effective force of the wind as exerted upon the sails of windmills. Let us suppose the entire sail surface divided into narrow parts or elements by parallel planes, and let *CD*, Fig. 589, be one of these elements. On account of the considerable size, and especially on account of the great length of the surface of a sail, we may assume that all the elements of the wind column coming in the direction *A H* and impinging upon the surface *C D* will be reflected in a direction parallel to *C D*. We may then employ the formulæ given in Vol. I., § 502. Let $c$ indicate the velocity of the wind, $v$ the velocity of the sail, and $Q$ the quantity of wind which strikes against *C D* per second; also $\gamma$ the density of the air, and $\alpha$ the angle *C A H* which the direction of the wind makes with *C D;* then, assuming that the surface *C D* moves in the direction of the wind, we have, according to the paragraph quoted, the normal force of the wind against *C D*,

FIG. 589.



$$N = \frac{c - v}{g} \sin. \alpha \,.\, Q \gamma.$$

The quantity of wind $Q$ striking upon *C D* is here, where the transverse diameter of the stream $C N = G$ includes the whole surface, not equal to $G c$, but only to $G (c - v)$, since the surface *C D* yielding to the pressure and moving with a velocity $v$ per second, leaves a space $G v$ behind it which takes from the quantity of wind $G c$ the part $G v$ without a change of direction. The normal force of the wind is therefore.

$$N = \frac{c - v}{g} \sin. \alpha \,.\, (c - v)\, G \gamma = \frac{(c - v)^2}{g} \sin. \alpha \,.\, G \gamma,$$

or if $F$ indicates the area of the element *C D* and $F \sin. \alpha$ be substituted for $G$, we have

$$N = \frac{(c - v)^2}{g} \sin.^2 \alpha \, F \gamma.$$

Besides this force upon the front surface of *C D*, there is also an action upon the back surface of $CD$, since a part of the wind which passes the borders of the surface in the direction $CE$ and $DF$ assumes a whirling motion in filling the space behind $CD$, and thus the pressure corresponding to the relative velocity $(c - v) \sin. \alpha$, viz.:

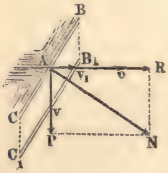
$$\frac{(c - v)^2}{g} \sin.^2 \alpha \,.\, F\gamma$$

is lost. Both these actions being united, we obtain the complete normal force of the wind against the sail element $F$,

$$N = \frac{(c - v)^2}{g} \sin.^2 \alpha\, F\gamma + \frac{(c - v)^2}{2g} \sin.^2 \alpha\, F\gamma$$

$$= 3 \cdot \frac{(c - v)^2}{2g} \sin.^2 \alpha\, F\gamma.$$

FIG. 590.



**§ 344.—Most Advantageous Angle of Impact.**—In applying this formula to wind-wheels, we must consider that the sail $BC$, Fig. 590, revolves, not in the direction of the wind $AR$, but in the direction $AP$ at right angles to it. In the formula

$$N = 3 \frac{(c - v)^2}{2g} \sin.^2 \alpha \,.\, F\gamma,$$

we have, therefore, for the normal impulse, instead of $v$, to substitute $Av_1 = v_1$, this being the rapidity with which the sail moves with reference to the direction of the wind. If $v$ here designates the actual rapidity of revolution $Av$, we have

$$Av_1 = v_1 = v \,.\, \cot.\, A v_1 v = v \cot.\, \alpha,$$

and hence, for the given case,

$$N = 3 \cdot \frac{(c - v \cot.\, \alpha)^2}{2g} \cdot \sin.^2 \alpha\, F\gamma,$$

or

$$N = 3\,\frac{(c\ sin.\ \alpha - v\ cos.\ \alpha)^2}{2\,g}\,F\,\gamma.$$

This normal force is divided into two side forces $P$ and $R$, the one acting in the direction of the revolution, and the other in the direction of the axis of the sail element, and hence

$$P = N\ cos.\ \alpha = 3\,\frac{(c\ sin.\ \alpha - v\ cos.\ \alpha)^2}{2\,g}\,cos.\ \alpha\ F\,\gamma,$$

but, on the contrary,

$$R = N\ sin.\ \alpha = 3\,\frac{(c\ sin.\ \alpha - v\ cos.\ \alpha)^2}{2\,g}\,sin.\ \alpha\ F\,\gamma.$$

Multiplying by the rapidity of revolution $v$, we have from the formula for $P$, the effective force of the wind-wheel,

$$L = P\,v = 3\,\frac{(c\ sin.\ \alpha - v\ cos.\ \alpha)^2}{2\,g}\,v\ cos.\ \alpha \cdot F\,\gamma.$$

The parallel force $R$, on the contrary, performs no work, but tends only to move the wheel in the direction of its axis. It thus forces it backwards against its bearings, causing a considerable friction and consequent loss of working power.

The last formula shows, as is indeed evident without it, that the effective force increases with the velocity of the wind and the area $F$ of the sail surface. The influence of the angle of impulse $\alpha$ upon the value of the force is not so readily seen from it. In order that $L$ may not be reduced to zero, we must have, however, $c\ sin.\ \alpha > v\ cos.\ \alpha$; *i. e.*, $tan.\ \alpha > \frac{v}{c}$ and $cos.\ \alpha > 0$, and therefore $\alpha < 90°$. There must, therefore, be a value of $\alpha$ between the limits $tan.\ \alpha > \frac{v}{c}$ and $\alpha < 90°$, corresponding to a maximum of $L$. In order to determine this value, we employ, instead of $\alpha$, $\alpha \pm x$, where $x$ denotes a very small quantity.

We then obtain

$$\sin. (\alpha \pm x) = \sin. \alpha \cos. x \pm \cos. \alpha \sin. x,$$

or, substituting $\cos. x = 1$ and $\sin. x = x$, we have

$$\sin. (\alpha \pm x) = \sin. \alpha \pm x \cos. \alpha;$$

further,

$$\cos. (\alpha \pm x) = \cos. \alpha \cos. x \mp \sin. \alpha \sin. x = \cos. \alpha \mp x \sin. \alpha,$$

and these values give for the effective force,

$$L = \frac{3\, c^2\, v}{2\, g} F \gamma \left(\sin. \alpha - \frac{v}{c} \cos. \alpha\right)^2 \cos. \alpha,$$

the expression,

$$\begin{aligned} L_1 = {} & \frac{3\, c^2\, v}{2\, g} F \gamma \left[\left(\sin. \alpha \pm x \cos. \alpha - \frac{v}{c} (\cos. \alpha \right.\right. \\ & \left.\left. \mp x \sin. \alpha)\right)^2 (\cos. \alpha \mp x \sin. \alpha)\right] = \frac{3\, c^2\, v}{2\, g} F \gamma \left[\sin. \alpha \right. \\ & \left. - \frac{v}{c} \cos. \alpha \pm \left(\cos. \alpha + \frac{v}{c} \sin. \alpha\right) x\right]^2 (\cos. \alpha \mp x \sin. \alpha) \\ & = \frac{3\, c^2\, v}{2\, g} F \gamma \left(\left(\sin. \alpha - \frac{v}{c} \cos. \alpha\right)^2 \cos. \alpha \pm \left[2 \left(\sin. \alpha \right.\right.\right. \\ & \left. - \frac{v}{c} \cos. \alpha\right) \left(\cos. \alpha + \frac{v}{c} \sin. \alpha\right) \cos. \alpha - \left(\sin. \alpha \right. \\ & \left.\left.\left. - \frac{v}{c} \cos. \alpha\right)^2 \sin. \alpha\right] x + \text{etc.}\right) \\ & = L \pm \frac{3\, c^2\, v}{2 g} F \gamma \left(\left[2 \left(\sin. \alpha - \frac{v}{c} \cos. \alpha\right) \left(\cos. \alpha \right.\right.\right. \\ & \left.\left.\left. + \frac{v}{c} \sin. \alpha\right) \cos. \alpha - \left(\sin. \alpha - \frac{v}{c} \cos. \alpha\right)^2 \sin. \alpha\right] x + \text{etc.}\right) \end{aligned}$$

In order that $\alpha$ may give the maximum value, $L_1$ must be less than $L$, $\alpha$ may be increased or diminished by $x$; *i. e.*, $x$ may be positive or negative. The last formula, however, gives in one case $L_1 > L$, and in the other, $< L$, so long as the second member, $\pm \frac{3\,c^2\,v}{2\,g} F \gamma [\,.\,.\,.\,]\, x$, is real; it is, therefore, necessary, in order to obtain the maximum value, that this second member should be zero, and, therefore, that

$$2\left(\sin.\,\alpha - \frac{v}{c}\cos.\,\alpha\right)\left(\cos.\,\alpha + \frac{v}{c}\sin.\,\alpha\right)\cos.\,\alpha$$

$$-\left(\sin.\,\alpha - \frac{v}{c}\cos.\,\alpha\right)^2 \sin.\,\alpha = 0,$$

or that

$$2\left(\cos.\,\alpha + \frac{v}{c}\sin.\,\alpha\right)\cos.\,\alpha = \left(\sin.\,\alpha - \frac{v}{c}\cos.\,\alpha\right)\sin.\,\alpha,$$

or that

$$\sin.^2\,\alpha - \frac{3\,v}{c}\sin.\,\alpha\cos.\,\alpha = 2\cos.^2\,\alpha.$$

Dividing by $\cos.^2\,\alpha$ and substituting $\frac{\sin.\,\alpha}{\cos.\,\alpha} = \tan.\,\alpha$, we have

$$\tan.^2\,\alpha - \frac{3\,v}{c}\tan.\,\alpha = 2,$$

from which we have, as the angle of maximum effect,

$$\tan.\,\alpha = \frac{3\,v}{2\,c} + \sqrt{\left(\frac{3\,v}{2\,c}\right)^2 + 2}.$$

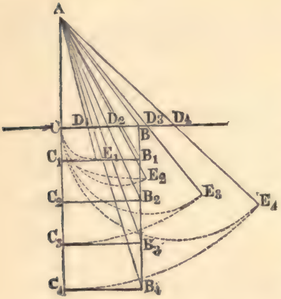
Since, in the case of any sail, the extreme elements have a greater velocity than those next to the axis of revolution, it follows that the extremities must present a greater angle to the wind than the part next the axis in order to obtain a maximum of power. The sails, therefore, must not be flat, but warped (Fr. gauches; Ger.

Windschief), and so placed that the extremities deviate less from the plane of revolution than the part next to the axis.

REMARK.—The most advantageous angle of impulse of a sail may easily be found by the following construction :

Let $CB$, Fig. 591, $= 1$, and place at right angles to it $CA = \sqrt{2} =$ the diagonal of a square on $CB$, and draw $AB$. Then $tan.\ ABC = \sqrt{2}$ and therefore $\angle ABC = 54° 44' 8''$; *i. e.*, the angle of impulse of the sail elements lying close to the axis of revolution. If we now insert $y = 3 \frac{\omega x}{2c}$ for $c$, the velocity of the wind, as well as for $\omega$, the angular velocity, and for $x$, the distances of the slats from the axis of the wind shaft, and lay off these values of $y$, as $CD_1, CD_2, CD_3$, etc., on $CB$ from $C$; if we draw also the hypotheneuses $AD_1$, $AD_2$, $AD_3$, etc., and lengthen them so that $D_1E_1 = CD_1$, $D_2E_2 = CD_2$, $D_3E_3 = CD_3$, etc.; finally, if we lay off $AE_1$, $AE_2$, $AE_3$, etc., in the direction of $AC$, as $AC_1$, $AC_2$, $AC_3$, etc., erect at $C_1$, $C_2$, $C_3$, etc., the perpendiculars $C_1B_1$, $C_2B_2$, $C_3B_3$, etc., $= CB = 1$, and draw $AB_1$, $AB_2$, $AB_3$, etc.; we thus obtain in $AB_1C_1$, $AB_2C_2$, $AB_3C_3$, etc., the angles sought; since

FIG. 591.



$$tan.\ AB_1C_1 = \frac{AC_1}{B_1C_1} = \frac{AE_1}{1} = D_1E_1 + AD_1 = y_1 + \sqrt{y_1^2 + 2},$$

$$tan.\ AB_2C_2 = \frac{AC_2}{B_2C_2} = \frac{AE_2}{1} = D_2E_2 + AD_2 = y_2 + \sqrt{y_2^2 + 2}, \text{ etc.}$$

§ **345.—Mechanical Effect of Windmills.**—The formula for the most suitable angle of impulse may be reversed in order to find the most advantageous velocity of revolution corresponding to a given position of the sail. We have, accordingly,

$$tan.^2\ \alpha - \frac{3v}{c}\ tan.\ \alpha = 2,$$

and, therefore,

$$v = \left(\frac{tan.^2\ \alpha - 2}{tan.\ \alpha}\right) \cdot \frac{c}{3} = (tan.\ \alpha - 2\ cot.\ \alpha)\frac{c}{3}.$$

Employing this value in the formula for the work, we have

$$L = \frac{3\,c^2}{2\,g} F\gamma \cdot \frac{tan.^2\,\alpha - 2}{tan.\,\alpha} \cdot \frac{c}{3} \cdot \left(sin.\,\alpha \right.$$

$$\left. - \frac{tan.^2\,\alpha - 2}{3\,tan.\,\alpha}\,cos.\,\alpha\right)^2 cos.\,\alpha$$

$$= \tfrac{4}{9} \cdot \frac{c^3}{2\,g} F\gamma \cdot \frac{(tan.^2\,\alpha - 2)\,cos.^2\,\alpha}{sin.^3\,\alpha}$$

$$= \tfrac{4}{9}\frac{c^3}{2\,g} F\gamma \cdot \frac{(3\,sin.^2\,\alpha - 2)}{sin.^3\,\alpha}.$$

The theoretical effect of a wind-wheel may from this be calculated for any given velocity of wind or of revolution. From the given number of revolutions $u$ per minute, we have the angular velocity,

$$\omega = \frac{\pi\,u}{30} = 0 \cdot 10472\,.\,u.$$

If now, the entire length of the sail-arm be divided into seven equal parts, the sail beginning with the first part, so that its real length is $\frac{6}{7}\,l$, it is easy, with the help of the formula,

$$tan.\,\alpha = \frac{3}{2}\frac{v}{c} + \sqrt{\left(\frac{3}{2}\frac{v}{c}\right)^2 + 2},$$

to calculate the most advantageous angle of impulse $a_0$, $a_1$, $a_2$, corresponding to each of the seven divisions of the sail—*i. e.*, by introducing successively

$$v^0 = \omega\,.\frac{l}{7},\; v_1 = \omega \cdot \frac{2\,l}{7},\; v_2 = \omega \cdot \frac{3\,l}{7},\; .\;.\;.\; \text{to } v_6 = \omega \cdot \frac{7\,l}{7} \text{ or } \omega\,l.$$

If, now, $b_0$, $b_1$, $b_2$, . . . $b_6$ are the breadths at the several points of division, we may, by applying Simpson's rule, determine an average value $k$ from the formula

$$\left(\frac{3\,sin.^2\,\alpha_0 - 2}{sin.^3\,\alpha_0}\right) b_0,\ \left(\frac{3\,sin.^2\,\alpha_1 - 2}{sin.^3\,\alpha_1}\right) b_1,\ \left(\frac{3\,sin.^2\,\alpha_2 - 2}{sin.^3\,\alpha_2}\right) b_2, \text{ etc.}$$

From this, we obtain the entire mechanical effect of the wheel,

$$L = \tfrac{4}{9}\,k\,\gamma \cdot \tfrac{6}{7}\,l \cdot \frac{c^3}{2\,g},$$

or, more generally, where $l$ denotes the actual length of the sail,

$$L = \tfrac{4}{9}\,\gamma\,k\,l_1\,\frac{c^3}{2\,g}.$$

If the sail were plane, having, therefore, the same angle of impulse $\alpha$ at every point, we should have first, by means of $v_1 = \frac{\omega l}{7}$, $v_2 = \omega \cdot \frac{2\,l}{7}$, etc., to compute first the corresponding values

$$(sin.\,\alpha - \frac{v_0}{c}\,cos.\,\alpha)^2\,\frac{v_0}{c}\,cos.\,\alpha\,.\,b_0,$$

$$(sin.\,\alpha - \frac{v_1}{c}\,cos.\,\alpha)\,\frac{v_1}{c}\,cos.\,\alpha\,.\,b_1, \text{ etc.},$$

and then, by means of Simpson's rule, to obtain from these the mean value $k_1$, and, finally, to introduce this value into the formula,

$$L = 3\,\gamma\,k_1\,.\,l_1 \cdot \frac{c^3}{2\,g}.$$

If $n$ denotes the number of the sails, the last value must, of course, be multiplied by it in order to obtain the entire theoretical effect,

$$L = 3\,n\,\gamma\,k_1\,l_1\,\frac{c^3}{2\,g}.$$

EXAMPLE I.—At what angle of impulse should a sail-wheel be placed which consists of four sails, each 24 ft. long and 6 to 9 ft. broad, and making 16 revolutions per minute, the velocity of the wind being 20 ft. per second? Also, what is the theoretical effect of the wheel?

Here the angular velocity $\omega = 0 \cdot 10472 \times 16 = 1 \cdot 6755$ ft., and the distance of the nearest bars from the axis $= 4$ ft., and hence the entire arm length $l = 24 + 4 = 28$ ft., we have

| For the distances.......... | 4 | 8 | 12 | 16 | 20 | 24 | 28 ft. |
|---|---|---|---|---|---|---|---|
| For the velocities.......... | 6·702 | 13·404 | 20·106 | 26·808 | 33·510 | 40·212 | 46·919 |
| For tangents of the angles of impulse.................. | 2·004 | 2·740 | 3·575 | 4·469 | 5·397 | 6·347 | 7·311 |
| For angles of impulse...... | 63° 29′ | 69° 57′ | 74° 22′ | 77° 23′ | 79° 30′ | 81° 3′ | 82° 13′ |
| Values of $\frac{3\, sin.^2\, \alpha - 2}{sin.^3\, \alpha}$..... | 0·5612 | 0·7810 | 0·8759 | 0·9220 | 0·9472 | 0·9622 | 0·9716 |
| Breadths of wings......... | 6·0 | 6·5 | 7·0 | 7·5 | 8·0 | 8·5 | 9·0 ft. |
| Products of the two last values.................. | 3·367 | 5·076 | 6·131 | 6·915 | 7·578 | 8·179 | 8·744 |

From the last products, we have the average values,

$$k = \frac{3 \cdot 367 + 8 \cdot 744 + 4\,(5 \cdot 076 + 6 \cdot 915 + 8 \cdot 179) + 2\,(6 \cdot 131 + 7 \cdot 578)}{18}$$

$$= \frac{12 \cdot 111 + 80 \cdot 680 + 27 \cdot 418}{18} = \frac{120 \cdot 209}{18} = 6 \cdot 678,$$

and if we now introduce

$$\gamma = \frac{62 \cdot 5}{800} = 0 \cdot 078125 \text{ lbs.}, \ \tfrac{6}{7}\, l = 24, \text{ and } \frac{c^3}{2\,g} = 0 \cdot 0155 \times 20^3 = 124,$$

we have for the mechanical effect,

$$L = 4 \times \tfrac{4}{9} \times 6 \cdot 678 \times 0 \cdot 078125 \times 24 \times 124 = 11 \cdot 872 \times 1 \cdot 85 \times 124$$

$$= 2800 \text{ ft. lbs.} = 5 \cdot 1 \text{ horse-power.}$$

EXAMPLE 2.—What is the mechanical effect of a wind-wheel consisting of four plane sails having an angle of impulse of 75°, and in other respects the same dimensions and conditions as in the preceding example?

We have here

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| The velocity ratio $\frac{v}{c}$....... | 0·3351 | 0·6702 | 1·0053 | 1·3404 | 1·6755 | 2·0106 | 2·3457 |
| The differences $sin.\, \alpha - \frac{v}{c}\, cos.\, \alpha$.... | 0·8792 | 0·7925 | 0·7057 | 0·6190 | 0·5323 | 0·4456 | 0·3588 |
| The breadths $b$............ | 6·0 | 6·5 | 7·0 | 7·5 | 8·0 | 8·5 | 9·0 |
| The products $(sin.\, \alpha - \frac{v}{c}\, cos.\, \alpha)^2 \frac{v}{c}\, cos.\, \alpha\, .\, b$........ | 0·4023 | 0·7081 | 0·9071 | 0·9969 | 0·9830 | 0·8783 | 0·7034 |

From the last products we obtain, by means of Simpson's rule, the average values,

$$k_1 = \tfrac{1}{18}\,[0\cdot4023+0\cdot7034+4\,(0\cdot7081+0\cdot9969+0\cdot8783)+2\,(0\cdot9071+0\cdot9830)]$$

$$= \tfrac{1}{18}\,(1\cdot1057+10\cdot3332+3\cdot7802 = \frac{15\cdot2191}{18} = 0\cdot8455,$$

and hence the mechanical effect sought is

$L = 4 \times 3 \times 0\cdot8455 \times 0\cdot078125 \times 24 \times 124 = 2400$ ft. lbs. $= 4\cdot36$ horse-power,

while in the wheel with warped sails $L = 5\cdot1$ horse-power.

**346.—Loss of Power by Friction.**—A considerable part of the force generated by the wind is lost in consequence of the friction of the axis upon its bearings, especially when, as is indeed usually the case, the axis is very large. We may assume that the entire weight of the wheel is supported by the neck, and may leave the pressure upon the back pivot entirely out of account. Although we thus somewhat overestimate the amount of friction, this error is substantially corrected by the omission of the friction upon the back pivot, resulting from the force exerted by the wind in the direction of the shaft. Since the back pivot is much smaller than the neck or forward bearing, this method of simplifying the problem may the more readily be allowed. With this assumption, then, we obtain from the entire weight of the wheel $G$ the corresponding friction $F = \phi G$, and representing the semi-diameter of the neck of the axis by $r$, and hence the velocity of rotation by $\omega r$, we have as an expression for the amount of this friction,

$$F\omega r = \phi G \omega r = 0\cdot1047\,.\,u \phi G r = \phi G \frac{r}{l} v,$$

where $v$ denotes the velocity of the periphery of the wheel.

This assumed, we have for the useful effect of a wind-wheel with plane sails,

$$L = 3\,n\gamma\,k_1 l_1 \cdot \frac{c^3}{2g} - \phi G \frac{r}{l} v,$$

and for that of such a wheel with warped sails,

$$L = \tfrac{4}{9}\, n\, \gamma\, k\, l_1 \cdot \frac{c^3}{2\,g} - \phi\, G\, \frac{r}{l}\, v.$$

From the formula

$$L = 3\, \frac{(c\ sin.\ \alpha - v\ cos.\ \alpha)^2}{2\,g}\, v\ cos.\ \alpha\ F \gamma$$

for the theoretical effect of a sail element, we may determine the influence of the velocity of the wheel upon the theoretical effect. This influence is found to be at its maximum for

$$v\ cos.\ \alpha = \frac{c\ sin.\ \alpha}{3} \quad \text{(compare § 219),}$$

*i. e.*, where

$$v = \frac{c\ tan.\ \alpha}{3}.$$

Introducing this value into the given formula, we have

$$L = 3 \times \tfrac{4}{27} \times \frac{c^3\ sin.^3\ \alpha}{2\,g}\, F\,\gamma.$$

From this, it appears that the power is greatest where the angle of impulse $\alpha = 90°$, and hence $v = \infty$. But this requirement cannot be met, since, in case of a not very excessive rapidity of revolution, the hindrances, especially the friction upon the neck of the axis, consumes so much force that nothing remains for the effective work. From a great velocity of revolution, we have, therefore, to expect a great effective power, but in any given case it remains to be determined exactly what velocity will yield a maximum of effective power after deducting the loss by friction from the theoretical effect, and this is only to be reached through actual calculation in a series of definite cases, from the results of which the most favorable are to be selected or deduced by interpolation.

EXAMPLE.—If the wheel referred to in the preceding paragraphs weighs 7500 lbs., the semi-diameter of its neck $r = \frac{1}{3}$ ft., and the coefficient of friction $\phi = 0 \cdot 1$, we have for the mechanical effect lost by friction upon the neck,

$$L_1 = 0 \cdot 1 \times 7500\, \omega r = 750 \times \tfrac{1}{3} \times 1 \cdot 6755 = 250 \times 1 \cdot 6755 = 419 \text{ ft. lbs.}$$

There remains, therefore, in the case of a wheel with warped sails, the effective mechanical effect $L = 2811 - 419 = 2392$ ft. lbs.—*i. e.*, about 85 per cent. In the case of wooden shafts, the necks are twice as large, and hence the loss of work by friction is twice as great, and the actual effect is therefore only 70 per cent of the theoretical.

§ **347.—Experiments upon Wind-Wheels.**—Exact observations upon wind-wheels sufficient to thoroughly test the theory, have not yet been made. We have, indeed, statements concerning the work of different windmills, but these are for the most part not sufficient for determining the efficiency of these machines, since they leave the velocity of the wind wholly undetermined, or do not express it with sufficient exactness. The most complete statements are those of Coulomb and Smeaton; but more recent observations of a similar kind are entirely wanting. Coulomb made his observations on one of the numerous windmills in the neighborhood of Lille, from which, however, tolerably exact conclusions may be drawn, since these mills put in motion a stamp work for the preparation of rape-seed oil, the useful effect of which is easily determined. The four sails of this mill were warped, as is customary in Holland, and with an angle of impulse of 63¾° to 81½° each, having about $2 \times 10 = 20$ square metres of surface. The experiments were made under a wind velocity of 2·27 to 9·1 metres, and with a peripheral velocity of from 7 to 22 metres, and agreed, according to the calculations of Coriolis (see his "Calcul de l'effet des machines"), upon an average, tolerably well with the above developed theory, according to which the normal impulse of the wind against a sail element $F$ is

$$N = 3 \cdot \frac{(c \sin. \alpha - v \cos. \alpha)^2}{2g} F \gamma.$$

We may readily see that in the case of better constructions with warped sails, the average value of

$$\frac{3\,sin.^2\,\alpha - 2}{sin.^3\,\alpha}$$

does not vary essentially from that computed in the first example in § 345, which $= 0 \cdot 880$. If we introduce this into the general formula, we obtain the following exceedingly simple expression for the effective power of a wind-wheel,

$$L = \tfrac{4}{9} \times 0 \cdot 88 \times 0 \cdot 078125 \times n\,F\,\frac{c^3}{2\,g} = 0 \cdot 000473\,n\,F\,c^3 \text{ ft. lbs.}$$

The mean from Coulomb's observations gives

$$L = 0 \cdot 026\,n\,F\,c^3 \text{ kilogrammetres,}$$

or, in English measures,

$$L = 0 \cdot 000359\,n\,F\,c^3 \text{ ft. lbs.,}$$

therefore in fair accordance with the theoretical determination. For greater certainty, we may perhaps best employ

$$L = 0 \cdot 00033\,n\,F\,c^3 \text{ ft. lbs.}$$

This formula, however, only gives sufficiently exact results when the peripheral velocity is most favorable—*i. e.*, about 2½ times the velocity of the wind.

Example.—What area of surface must be given to the sails of a wheel in order that they may afford 4 horse-power under a wind velocity of 16 ft. per second? According to the last formula,

$$n\,F = \frac{4 \times 550}{0 \cdot 00033 \times 16^3} = \frac{2200}{1 \cdot 35168} = 1623 \text{ sq. ft.};$$

therefore, in the case of five sails, the area of each $F = 324$ sq. ft. If the length $l_1$ of a sail be taken five times as great as its mean breadth $b$, we have $5\,b^2 = 324$, and consequently the breadth of a sail

$$b = \sqrt{\frac{324}{5}} = \sqrt{64} = 8 \text{ ft.,}$$

and its length,

$$l_1 = 5 \times 8 = 40 \text{ ft.}$$

§ **348.—Smeaton's Rules.**—Smeaton made very complete experiments with miniature wind-wheels. These wheels had arms 21 inches in length, with sails 18 inches long and 5·6 inches wide. They were made to revolve, not by the force of the wind, but by being moved about in a circle in the still air, and hence he observed, not the force of the wind, but the resistance of the air against the wheel, a method which evidently detracts considerably from the value of his results. The motion of the wheel was effected by means of a perpendicular axis bearing a transverse arm 5½ ft. long, to the extremity of which were attached the proper bearings of the wheel. The axis received its motion from the observer himself, by means of a cord, which, previous to each experiment, was wound about the stouter part of the axis as about a top. In order to measure the force of the wind, or rather the resistance of the air, a scale pan with weights was placed directly over the perpendicular axis, suspended by a very fine cord, and the other end of this cord was attached to the axis of the wheel, so that when this axis revolved, the cord was wound about it and the weight at the other end of the cord was raised. The results of these experiments agree very well with the theory already presented—*i. e.*, they show very definitely that the warped sails are more effective than plane ones, and that the angle of impulse found theoretically is in reality the most advantageous. While we, in the above example, § 345, found the seven angles of impulse at equal distances from the shaft outwards to be $63° 29'$; $69° 57'$; $74° 22'$; $77° 23'$; $79° 30'$; $81° 3'$; and $82° 13'$; the experiments of Smeaton give the following six as very advantageous: 72°, 71°, 72°, 74°, 77½°, 83°; the average being about the same in the two cases. Smeaton himself remarks that a difference of two degrees in the angle of impact causes no essential difference in the mechanical effect.

Finally, Smeaton, from his experiments with the wind velocity, or rather wheel velocity, of $4\frac{1}{8}$ to $8\frac{3}{4}$ ft., drew the following conclusions, which accord very well with the theory:

1. With the favorably constructed sail-wheel, the greatest peripheral rapidity is to the most advantageous peripheral rapidity as 3 to 2, and the greatest load is to the most advantageous load as 6 to 5. 2. The greatest peripheral rapidity with an unloaded wheel is about 4 times, and therefore with the most advantageous load $\frac{2}{3} \times 4 = \frac{8}{3}$ times as great as the wind velocity. 3. The load increases with the most advantageous motion—*i. e.*, that which gives the greatest effect, nearly as the square and the effect itself, nearly as the cube of the wind velocity. Twice the wind velocity gave 3·75 times the load and 7·02 times the effect. Various other rules which Smeaton derived from his experiments accord with the theory, and may be equally well derived from it. It is therefore unnecessary to state them further in this place.

According to these experiments, the action of the wind in general upon sail-wheels is greater than is given either by the theory or by Coulomb's experiments.

Other statements concerning the effect of wind-wheels can only be given in the third volume.

CONCLUDING REMARKS.—The complete theory of wind-wheels may be found in the author's "Handbuch der Bergmaschinenmechanik," and in Coriolis's "Traité du calcul de l'effet des machines." In most mechanical text-books, wind-wheels are very briefly treated or entirely unnoticed. Smeaton's experiments are described in the "Philosophical Transactions" of 1759 to 1776. They have also been collected and translated into French by Girard, under the title of "Recherches expérimentales sur l'eau et le vent," Paris, 1827. Extracts from them are found in almost all English works, especially in Barlow's "Treatise on the Manufactures and Machinery of Great Britain." Coulomb's experiments are described in the well-known work, "Théorie des machines simples, par Coulomb." A post windmill, exactly illustrated and described in detail, may be found in Hoffmann's "Sammlung der gebräuchlichsten Maschinen," Heft I., Berlin, 1833. See also Schwahn's "Lehrbuch der prakt. Mühlenbaukunde." It is also described in Vol. VIII. of the "Publication Industrielle, etc., par Armengaud," Paris, 1853.

A tolerably complete treatise upon windmills, by A. Burg, is contained in Vol. VIII. (1826) of the "Jahrbücher des Polytechn. Instituts" of Vienna. See also Rühlmann's "Allgemeine Maschinenlehre," Vol. I.

Mariotte, in his "Grundlehren der Hydrostatik und Hydraulik," treats also of the impulse of the wind. According to him, the impulse

$$P = 1 \cdot 73 \frac{c^2}{2g} F \gamma.$$

Reference may also be made to Borda in the "Mémoires de l'Académie de Paris, 1763," and to Rouse (see the above cited work of Smeaton), and to Hutton and Woltmann. The last-named authors find $P$ much smaller than Mariotte and others, since they calculated not the impulse but the resistance of the air. Hence the coefficient $\zeta = \frac{4}{3}$, *i. e.*, the power

$$P = \frac{4}{3} \cdot \frac{c^2}{2g} F \gamma,$$

as found by Woltmann, is too small, since he did not directly determine the constants of the sails (see his "Theorie und Gebrauch des hydrometrischen Flügels," Hamburg, 1790).

Hutton finds, from his experiments, that the impulse and resistance of the air must be regarded as increasing in the proportion $F^{1 \cdot 1}$ (see his "Philosophical and Mathematical Dictionary," Vol. II.). Assuming, now, that the coefficient $\zeta = 1 \cdot 86$ is correct for a small surface of 1 sq. ft., we must accordingly, for a wind-sail of 200 sq. ft., employ

$$\zeta = 200^{0 \cdot 1} \times 1 \cdot 86 = 1 \cdot 7 \times 1 \cdot 86 = 3 \cdot 162,$$

and this accords very well with the theoretical calculation and with the above discussion where

$$\zeta = 3 \text{ and } P = 3 \cdot \frac{c^3}{2g} F \gamma.$$

Poncelet, in his "Introduction à la mécanique industrielle," gives a very good collection and comparison of experiments concerning impact and resistance of the air. Euler, in his treatise in the Berlin Memoirs, 1756, presents peculiar views regarding the impact of the wind. Also Crelle, in his treatise upon the "Theorie des Windstosses," 1802. An investigation of Coulomb's empirical formula

$$L = 0 \cdot 026\, n\, F c^3$$

may be found in the little work, "Notice sur les moulins à vent à ailes réductibles, par M. Ord. de Lacolange, Besançon, 1856."

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